FIXED STRUCTURE PID CONTROLLER DESIGN FOR STANDARD H_∞ CONTROL PROBLEM

Masami Saeki *

* *Department of Mechanical System Engineering, Hiroshima University, Kagamiyama 1-4-1, Higashi-Hiroshima, 739-8527, Japan*

Abstract: A frequency domain design method of fixed structure multivariable PID controllers that satisfy the performance index of the standard H_∞ control problem with the LMI (linear matrix inequality) constraints of the PID gains is proposed. Frequency dependent bilinear matrix inequalities on the PID gain are derived and they are approximated by LMI's for each frequency. By solving an LMI problem iteratively starting from a stabilizing PID gain, the proposed method gives a convergent sequence of PID gains so that the sequence of H_∞ norm may be non-increasing.

Keywords: H-infinity control, Optimization problems, Frequency domains, PID, MIMO, Decentralized control

1. INTRODUCTION

H_∞ control theory has become a standard design method in the last 15 years, which shows the usefulness of H_∞ norm performance index(Zhou *et al.*, 1996). As the H_∞ controller is centralized and has high McMillan degree, there have been many studies about the design problems of reduced order and/or fixed structure controllers. PID controller is among the fixed structure controllers, and the design problem of multiloop/multivariable PID control systems is still a real challenge to control system engineers (Unar *et al.*, 1996) (Lelic and Gajic, 2000). Therefore, in this paper, we will study the design of multivariable PID controller that satisfies the H_∞ norm bound of the standard H_∞ control problem and control structure constraints such as decentralized control.

In (Zeng *et al.*, 2002), a design method for multivariable PID controllers is developed by transforming the design problem into static output feedback controller design, and, in (Miyamoto and Vinnicombe, 1997), a design method of the multivariable controller with fixed structure, including PID controller, is proposed for the H_∞ loop shaping problem. On the other hand, since a single input single output PID controller has only three parameters, parameter space design approach is suitable and graphical methods of drawing the feasible set of robust PID gains have been developed (Saeki *et al.*, 1998), (Ho *et al.*, 2001).

We have formulated the problem of (Saeki *et al.*, 1998) as an optimization problem in (Saeki and Aimoto, 2000) in order to search for the optimal gain automatically. This frequency domain approach has the next merits; computational complexity is not much affected by the plant degree and the number of the variables are small, namely, the variables are just PID gains. Numerical examples show that the optimal gain can be obtained for several typical plants. Therefore, we consider this frequency domain approach promising, and we will generalize it to multivariable case in this paper.

The notation is standard. M^T, M^*, and $\overline{\sigma}(M)$ are the transpose, the complex conjugate transpose and the maximum singular value of a matrix M, respectively. M^{-1} and M^{-*} are the inverse matrix of M and M^*, respectively. $\|G\|_\infty$ is the L_∞ norm of $G(s)$. Namely, $\|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$ RH_∞ is the set of stable real rational transfer functions. For the matrix M partitioned as $M = [M_{ij}], i, j = 1, 2$, the lower linear fractional transformation $\mathcal{F}_l(M, \bullet)$ is denoted by $\mathcal{F}_l(M, Q) = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$. The next state space realization of a proper transfer function is used.

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$$

2. PROBLEM SETTING

Let us consider the generalized plant described by

$$\dot{x} = Ax + B_1 w + B_2 u \tag{1}$$
$$z = C_1 x + D_{11} w + D_{12} u \tag{2}$$
$$y = C_2 x + D_{21} w + D_{22} u \tag{3}$$

where $x \in R^n$,$z \in R^{p_1}$,$y \in R^{p_2}$,$w \in R^{m_1}$,$u \in R^{m_2}$,$D_{22} = 0$. Assume that the assumptions of the standard H_∞ control problem (Zhou *et al.*, 1996) are satisfied. Namely,

(G1) (A, B_2) is stabilizable and (C_2, A) is detectable.
(G2) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω.
(G3) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full law rank for all ω.

The control law is described by

$$u = K(s)y \tag{4}$$
$$K(s) = K_P + K_I \frac{1}{s} + K_D \frac{s}{1 + \epsilon s} \tag{5}$$

where K_P, K_I, K_D are $m_2 \times p_2$ constant matrices and $\epsilon > 0$. Linear and/or LMI constraints can be given to these matrices. For example, some of the elements can be set zero in order to take the control structure into consideration, and also such an LMI constraint as $K_D + K_D^T > 0$ can be set. This controller class is denoted by $\mathcal{K}_{PID}$.

Let us represent the transfer function of the generalized plant as

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix}$$

where $G_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij}, \quad i, j = 1, 2$, then the closed loop transfer function from w to z is given by

$$\begin{aligned} T_{zw}(s) &= \mathcal{F}_l(G, K) \\ &= G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \end{aligned} \tag{6}$$

Since $G_{22}(\infty) = 0$ and $K(s) \in \mathcal{K}_{PID}$ is proper from $\epsilon > 0$, the feedback system is well-posed, therefore

$$det\{I + G_{22}(\infty)K(\infty)\} \neq 0 \tag{7}$$

The set of all stabilizing controllers such that $\|T_{zw}\|_\infty < 1$ is given by

$$\begin{aligned} \mathcal{K} = \{K(s) | K(s) = \mathcal{F}_l(M, Q), \\ Q(s) \in RH_\infty, \quad \|Q\|_\infty < 1\} \end{aligned} \tag{8}$$

where

$$M(s) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{9}$$

and the orders of $M_{11}(s)$ and $M_{22}(s)$ are $m_2 \times p_2$ and $p_2 \times m_2$, respectively. $M_{12}(s)$ is an $m_2 \times m_2$ nonsingular transfer function matrix, and the inverse system M_{12}^{-1} is stable. Similarly, $M_{21}(s)$ is a $p_2 \times p_2$ nonsingular transfer function matrix and M_{21}^{-1} is stable.

Design problem *For the generalized plant $G(s)$, obtain $K(s) \in \mathcal{K}_{PID}$ that satisfies*

(P1) *$K(s)$ internally stabilizes $G(s)$.*
(P2) *$K(s)$ satisfies the L_∞ norm constraint:*

$$\overline{\sigma}(T_{zw}(j\omega)) < 1, \omega \in R \tag{10}$$

Remark 1 Let $\mathcal{K}_\gamma$ denote the solution set of the H_∞ control problem $\|T_{zw}\|_\infty < \gamma$. Since $\mathcal{K}_\gamma$ agrees with $\mathcal{K}$ of $\|T_{zw}/\gamma\|_\infty < 1$, $\mathcal{K}_\gamma$ can be represented in the form of (8) for the generalized plant:

$$G(s) = \left[\begin{array}{c|cc} A & B_1/\gamma & B_2 \\ \hline C_1 & D_{11}/\gamma & D_{12} \\ C_2 & D_{21}/\gamma & 0 \end{array}\right] \tag{11}$$

The solution of our design problem is given by $\mathcal{K} \cap \mathcal{K}_{PID}$. If $\mathcal{K}$ is empty, the solution does not exist. Therefore, we may assume the existence of $\mathcal{K}$ without loss of generality. Further, if a stabilizing PID gain exists, $\mathcal{K}_\gamma \cap \mathcal{K}_{PID}$ is nonempty for a sufficiently large γ. In the following two chapters, we will represent the conditions P1 and P2 by LMI's, respectively.

3. INTERNAL STABILITY CONDITION

Let $K_1, K_2 \in \mathcal{K}_{PID}$. When $K(s, \beta) = (1 - \beta)K_1(s) + \beta K_2(s)$ has the same number of poles at $s = 0$ for all $\beta (0 \leq \beta \leq 1)$, denote the set of $K_2(s)$ that has this property as $\mathcal{K}_\beta(K_1)$.

Lemma 1 *Represent the integral gains of $K_1(s)$ and $K_2(s)$ as K_{I1} and K_{I2}, respectively. Then, the number of the unstable poles at $s = 0$ of $K(s, \beta)$ is invariant for all $\beta(0 \le \beta \le 1)$, if and only if the rank of $(1-\beta)K_{I1} + \beta K_{I2}$ is invariant for all $\beta(0 \le \beta \le 1)$.*

Lemma 2 *If $A_1 + A_1^T > 0$ and $A_2 + A_2^T > 0$ for real square matrices A_1 and A_2, $A(\beta) = (1-\beta)A_1 + \beta A_2$ is full rank for all $\beta(0 \le \beta \le 1)$.*

When K_I is represented in a more general form: $K_I = block \;\; diag(K_{I11}, 0)$, we may use $K_{I11} + K_{I11}^T > 0$.

Lemma 3 *Assume that $K_1, K_2 \in \mathcal{K}_{PID}$ satisfy*

(1) K_1 stabilizes G and K_2 destabilizes G.
(2) $K_2(s) \in \mathcal{K}_\beta(K_1)$

Then, there exist $s = j\omega_0$ and β $(0 < \beta \le 1)$ that satisfy

$$det\{I + G_{22}(s)((1-\beta)K_1(s) + \beta K_2(s))\} = 0$$

Lemma 4(Zhou *et al.*, 1996) *For the standard H_∞ control problem, if $det\{I+G_{22}(j\omega_0)K(j\omega_0)\} = 0$, then $\overline{\sigma}(T_{zw}(j\omega_0))$ is unbounded.*

Theorem 1 *Assume that $K_1 \in \mathcal{K}_{PID}$ stabilizes $G(s)$ and $K_2(s) \in \mathcal{K}_\beta(K_1)$. Then, if the H_∞ norm $\|T_{zw}\|_\infty$ obtained for $K(s) = (1-\beta)K_1(s) + \beta K_2(s)$ is bounded for all $\beta(0 \le \beta \le 1)$, $K_2(s)$ also stabilizes $G(s)$.*

Proof) By Lemma 3, if $K_2(s)$ destabilizes $G(s)$, the closed-loop system has a pole on the imaginary axis at some value of β. Then, in this case, $\overline{\sigma}(T_{zw}(j\omega))$ is unbounded at some ω from Lemma 4, which contradicts the assumption that $\|T_{zw}\|_\infty$ is bounded for all β. Therefore, $K_2(s)$ is a stabilizing controller. This completes the proof.

4. FREQUENCY DOMAIN CONDITION

Theorem 1 implies that when the matricies of $K(s) \in \mathcal{K}_{PID}$ are continuously deformed from those of a stabilizing controller $K_1(s) \in \mathcal{K}_{PID}$ with the conditions $\|T_{zw}\|_\infty < \infty$ and $K(s) \in \mathcal{K}_\beta(K_1)$ being satisfied, the stability of the closed loop system is guaranteed. In this section, we will give a method of generating a path of PID gain along which $\|T_{zw}\|_\infty$ is at least monotonically nonincreasing.

Let us define the set $\mathcal{K}(\omega)$ by

$$\begin{aligned}&\mathcal{K}(\omega) = \{K(j\omega) | K(j\omega) = M_{11}(j\omega) \\ &+M_{12}(j\omega)Q(j\omega)(I - M_{22}(j\omega)Q(j\omega))^{-1}M_{21}(j\omega), \\ &\overline{\sigma}(Q(j\omega)) < 1\}\end{aligned} \tag{12}$$

Here, $M_{12}^{-1}(j\omega)$ and $M_{21}^{-1}(j\omega)$ exist for all ω from the stability of $M_{12}^{-1}(s)$ and $M_{21}^{-1}(s)$. Since $\mathcal{K}(j\omega)$ is the set of all controllers that satisfy $\overline{\sigma}(T_{zw}(j\omega)) < 1$, our design problem can be restated as follows.

Another expression for design problem *Obtain $K(s) \in \mathcal{K}_{PID}$ which stabilizes $G(s)$ and satisfies $K(j\omega) \in \mathcal{K}(\omega)$ for all $\omega \in R$.*

In the following, ω is dropped for space saving.

Lemma 5 *There exists $Q \in C^{m_2 \times p_2}$ such that*

$$K = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21} \tag{13}$$
$$\overline{\sigma}(Q) < 1$$

, if and only if

$$KPK^* + KL^* + LK^* + R > 0 \tag{14}$$

where

$$P = M_{21}^{-1}(M_{22}M_{22}^* - I)M_{21}^{-*} \tag{15}$$
$$L = -M_{11}M_{21}^{-1}(M_{22}M_{22}^* - I)M_{21}^{-*} + M_{12}M_{22}^*M_{21}^{-*} \tag{16}$$
$$R = M_{11}M_{21}^{-1}(M_{22}M_{22}^* - I)M_{21}^{-*}M_{11}^* - M_{11}M_{21}^{-1}M_{22}M_{12}^* - M_{12}M_{22}^*M_{21}^{-*}M_{11}^* + M_{12}M_{12}^* \tag{17}$$

The condition (14) becomes convex or nonconvex with respect to K depending on the positive definiteness or indefiniteness of the coefficient matrix P. Let us derive LMIs classifying it into three cases.

Case 1 In the case of $P \le 0$, P can be represented as

$$P = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} \tag{18}$$

where $\Lambda_2 \in R^{p_2 \times p_2}$ is a positive definite diagonal matrix and $U = [U_1, \; U_2]$ is a unitary matrix. Then, (14) is expressed as

$$-KU_2\Lambda_2U_2^*K^* + KL^* + LK^* + R > 0 \tag{19}$$

and the Schur complement gives the next equivalent condition.

$$\begin{bmatrix} KL^* + LK^* + R & KU_2 \\ U_2^*K^* & \Lambda_2^{-1} \end{bmatrix} > 0 \tag{20}$$

This is an LMI with respect to K.

Case 2 In the case of $P \ge 0$,

$$P = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} \tag{21}$$

where $\Lambda_1 \in R^{p_2 \times p_2}$ is a positive-definite diagonal matrix and $U = [U_1, \; U_2]$ is a unitary matrix. Then, (14) is expressed as

$$KU_1\Lambda_1U_1^*K^* + KU_1U_1^*L^* \\ + LU_1U_1^*K^* + \tilde{R} > 0 \tag{22}$$
$$\tilde{R} = R + KU_2U_2^*L^* + LU_2U_2^*K^*$$

This constraint is nonconvex with respect to K. We will derive a convex constraint that is also a sufficient condition. Set

$$Z = KU_1\Lambda_1^{1/2} \tag{23}$$
$$\tilde{L} = -LU_1\Lambda^{-1/2} \tag{24}$$

, and represent (22) as

$$(Z-\tilde{L})(Z-\tilde{L})^* > -\tilde{R} + \tilde{L}\tilde{L}^* \tag{25}$$

By using the next inequality, which is satisfied for any Z_q

$$ZZ^* \geq Z_qZ^* + ZZ_q^* - Z_qZ_q^* \tag{26}$$

, we obtain a sufficient condition of (25):

$$(Z_q-\tilde{L})Z^* + Z(Z_q-\tilde{L})^* + \tilde{R} - Z_qZ_q^* > 0 \tag{27}$$

This can be represented as the next LMI.

$$(Z_q\Lambda_1^{1/2}U_1^* + L)K^* + K(Z_q\Lambda_1^{1/2}U_1^* + L)^* \\ +\tilde{R} - Z_qZ_q^* > 0 \tag{28}$$

Though the above is satisfied for any Z_q, Z_q determines the conservativeness of the LMI condition. Therefore, it is crucial to find an appropriate Z_q. Let us examine this problem under the condition that a stabilizing gain $K = K_a$ is given and (25) is satisfied for $Z_a = K_aU\Lambda^{1/2}$, and let us consider a set of Z that satisfies

$$(Z-\tilde{L})(Z-\tilde{L})^* > \hat{R} \tag{29}$$
$$\hat{R} = -\tilde{R}(K_a) + \tilde{L}\tilde{L}^*$$

Note that the right side is fixed at $K = K_a$.

We consider a simple case $\hat{R} > 0$ for the moment. In this case, the set of Z which satisfies this constraint is outside of the hyper sphere with center $\tilde{L}$ and radius $\hat{R}^{0.5}$. Since Z_a satisfies (29), Z_a lies outside the sphere as illustrated in Fig. 1. Let Z_q be the point of intersection of the sphere and the segment which connects Z_a and $\tilde{L}$, then it is expected that the tangent plane at Z_q can be a reasonable convex constraint. The idea of this approximation comes from the study of the SISO case. In this case, the approximation can be visualized on the PI or PD gain plane, and we can see the approximation reasonable and good (Saeki and Aimoto, 2000). Actually, the optimal solution can be obtained for several typical SISO examples.

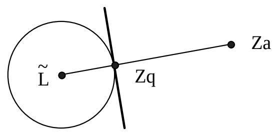

Fig. 1. Approximation of the sphere by the hyperplane at $Z = Z_q$

Z_q is calculated as follows. Substitution of

$$Z_q = (1-q)\tilde{L} + qZ_a, \quad 0 \leq q \leq 1 \tag{30}$$

into (25) gives

$$q^2(Z_a-\tilde{L})(Z_a-\tilde{L})^* > \hat{R} \tag{31}$$

This inequality is satisfied for $q = 1$ from the assumption. Therefore, the infimum of $\alpha = q^2$ always exists in the interval $1 \geq \alpha \geq 0$, and it can be computed by the next Lemma.

Lemma 6 *Suppose that $A = A^* \geq 0$, $B = B^*$, and $A - B > 0$. Then, the infimum of α that satisfies $\alpha A - B > 0$ is given by*

$$\alpha = \max\{eig[H_{11} - H_{12}H_{22}^{-1}H_{21}]\} \tag{32}$$

where

$$A = U\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}U^*, \quad \Sigma_1 > 0 \tag{33}$$
$$H = \begin{bmatrix}\Sigma_1^{-1/2} & 0\\ 0 & I\end{bmatrix}U^*BU\begin{bmatrix}\Sigma_1^{-1/2} & 0\\ 0 & I\end{bmatrix} \\ = \begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix} \tag{34}$$

H_{11} has the size of Σ_1.

Desirable conditions on Z_q are (1) Z_a belongs to the permissible set defined by the LMI, which is necessary to guarantee stability (2) The LMI condition given by Z_q gives a larger permissible set than the LMI condition given by setting $Z_q = Z_a$, which is necessary to guarantee the monotonic decreasing property of γ in the optimization procedure. In the above derivation, Z_q was chosen to satisfy these conditions under the assumption $\hat{R} > 0$. The next lemma shows that these conditions are also satisfied without this assumption.

Lemma 7 *Assume $(Z_q - L)(Z_q - L)^* > \hat{R}$ for $Z_q = qZ_a + (1-q)L, 0 < q \leq 1$, and give two LMI approximations of*

$$(Z-L)(Z-L)^* > \hat{R} \tag{35}$$

by

$$(Z_q-L)(Z-Z_q)^* + (Z-Z_q)(Z_q-L)^* \\ + (Z_q-L)(Z_q-L)^* > \hat{R} \tag{36}$$

$$(Z_a - L)(Z - Z_a)^* + (Z - Z_a)(Z_a - L)^* + (Z_a - L)(Z_a - L)^* > \hat{R} \quad (37)$$

Further, define the sets of Z that satisfies (35), (36), (37) as $\mathcal{Z}$, $\mathcal{Z}_q$, $\mathcal{Z}_a$, respectively. Then, $\mathcal{Z} \supset \mathcal{Z}_q \supset \mathcal{Z}_a \ni Z_a$.

Case 3 In other general case,

$$P = U \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & -\Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^* \quad (38)$$

where $U = [U_1,\ U_2,\ U_3]$ is unitary and Λ_1, Λ_2 are positive definite diagonal matrices. Then, (14) is represented as

$$KU_1\Lambda_1U_1^*K^* - KU_2\Lambda_2U_2^*K^* + KUU^*L^* + LUU^*K^* + R > 0 \quad (39)$$

This is not convex because of the term $KU_1\Lambda_1U_1^*K^*$. By applying the methods of Case 1 and Case 2, Z_q and the next LMI can be obtained. Details are omitted.

$$\begin{bmatrix} \Theta & KU_2 \\ U_2^*K^* & \Lambda_2^{-1} \end{bmatrix} > 0 \quad (40)$$

$$\Theta = K(Z_q\Lambda_1^{1/2}U_1^* + L)^* + (Z_q\Lambda_1^{1/2}U_1^* + L)K^* + R - Z_qZ_q^* \quad (41)$$

Lemma 8 *The set of PID gains that satisfy (20), (28), or (40) for all frequencies is a convex set.*

Theorem 2 *Assume that $K = K_1(s) \in \mathcal{K}_{PID}$ stabilizes $G(s)$ and satisfies $\|T_{zw}\|_\infty < \gamma_1$. Also assume that $K = K_2(s) \in \mathcal{K}_{PID}$ satisfies the LMI conditions (20), (28), or (40) obtained from $\|T_{zw}/\gamma_1\|_\infty < 1$ for all frequencies and $K_2 \in \mathcal{K}_r(K_1)$. Then, $K(s,\beta) = (1-\beta)K_1(s) + \beta K_2(s)$ stabilizes $G(s)$ and satisfies $\|T_{zw}\|_\infty < \gamma_1$ for all β, $(0 \le \beta \le 1)$.*

Proof Since $K = K_1(s) \in \mathcal{K}_{PID}$ satisfies $\|T_{zw}\|_\infty < \gamma_1$, the LMI condition is satisfied for $K = K_1$ and $\gamma = \gamma_1$. Suppose that $K_2(s)$ also satisfies the LMI condition, then $K(s,\beta)$ satisfies the LMI condition for all $\beta (0 \le \beta \le 1)$ from Lemma 8. This implies that $K(s,\beta)$ satisfies the L_∞ condition $\|T_{zw}\|_\infty < \gamma_1$ for all $\beta (0 \le \beta \le 1)$. Therefore, if $K_2 \in \mathcal{K}_r(K_1)$ is satisfied simultaneously, $K(s,\beta)$ stabilizes $G(s)$ for all $\beta (0 \le \beta \le 1)$ from Theorem 1. This completes the proof.

Remark 2 Let $\gamma_2 := \|T_{zw}\|_\infty$ for $K = K_2$, then $\gamma_1 \ge \gamma_2$. Note that strict inequality $\gamma_1 > \gamma_2$ is expected, because K_1 is in the vicinity of the boundary of the feasible set and K_2 is an interior point as shown in Fig. 2. By iterating this procedure, a sequence of the PID controllers $\{K_j(s)\}$ and the monotonically nonincreasing sequence $\{\gamma_i\}$ can be obtained. This gives a path $K(s,\beta) = (1-\beta)K_j(s) + \beta K_{j+1}(s), (0 \le \beta \le 1)$ for $j = 1, 2, \ldots$ along which $G(s)$ is stabilized and $\|T_{zw}\|_\infty < \gamma_j$ is satisfied.

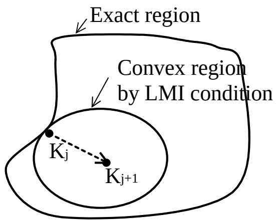

Fig. 2. Exact region and its convex approximation

5. ALGORITHM

In the following algorithm, the infinite number of LMI's will be approximated by a finite number of LMI's by frequency gridding. Note that this gridding poses a slight risk of obtaining a destabilizing PID gain and that this tends to occur when the sampling frequency range is too narrow or N is too small. Therefore, if a destabilizing controller is obtained by this algorithm, it may be avoided by modifying the sampling frequencies.

Step 1 *Set the following: the sampling frequencies $\omega = \omega_i, i = 1, \ldots N$; the iteration number $j = 1$; a sufficiently large positive number γ_0; the controller structure; LMI constraints on the PID gains; the number of iteration M. Also set a stabilizing PID controller K_1 that satisfies $K_{I11} + K_{I11}^T > 0$ and other LMI constraints on the PID gain matrices.*

Step 2 *Set $K = K_j$ and compute $\gamma_j = \|T_{zw}\|_\infty$ for K.*

Step 3 *If $\gamma_{j+1} < 1$, a solution is obtained and stop. Or, if $j > M$, stop.*

Step 4 *Obtain $\mathcal{K}$ for $\|\frac{1}{\gamma_j}T_{zw}\|_\infty < 1$ and make the LMI's from $\|\frac{1}{\gamma_j}T_{zw}\|_\infty < 1$ with $K = K_j$.*

Step 5 *Solve the LMI's of K_P, K_D, K_I with $K_{I11} + K_{I11}^T > 0$ and other LMI constraints. Then, represent the solution as $K_{j+1}(s)$. Set $j = j + 1$ and go to Step 2.*

6. NUMERICAL EXAMPLE

Consider a mixed sensitivity control problem $\|T_{zw}\|_\infty < 1$ shown in Fig. 3. The plant $P(s)$ is given by

$$P(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{0.2}{s+3} \\ \dfrac{0.1}{s+2} & \dfrac{1}{s+1} \end{bmatrix} \quad (42)$$

and the weighting functions are $V(s) = v(s)I_2$, $W(s) = w(s)I_2$, $v(s) = \frac{s+3}{3s+0.3}$, $w(s) = \frac{10s+2}{s+40}$, $a = 0.01$. Let the initial controller be $K_1(s) =$

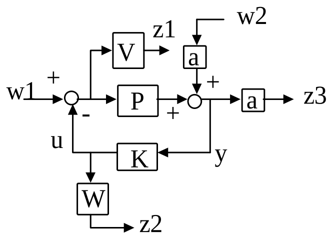

Fig. 3. Mixed sensitivity control problem

$I \times 0.001$. For this controller, the gain plot of $\overline{\sigma}\{T_{zw}(j\omega)\}$ is shown by the number 1 in Fig. 4.

First, let us consider the case that all the PID gains are full matrices. The constraint $K_I + K_I^T > 0$ is used. Note that a destabilizing solution was obtained without this constraint. An LMI constraint $K_D + K_D^T > 0$ is added, though this is dispensable. Now apply our algorithm. The largest singular value plots of $T_{zw}(j\omega)$ are drawn in Fig. 4 where the numbers $1, 5, 10, \ldots$ show the iteration number, and a solution that satisfies $\|T_{zw}\|_\infty < 1$ is obtained in the 48 th iteration. γ decreases as shown by the solid line in Fig. 5. The solution of the 48th iteration is

$$K(s) = \begin{bmatrix} 2.189 & -0.4349 \\ -0.2340 & 2.361 \end{bmatrix} + \begin{bmatrix} 6.417 & 0.2463 \\ 0.05694 & 7.810 \end{bmatrix} \frac{1}{s} + \begin{bmatrix} 9.825 & 2.406 \\ 2.954 & 10.50 \end{bmatrix} \frac{10^{-3}s}{1+0.01s}$$

Next, let us examine a decentralized control case. The PID gains may simply be restricted to diagonal matrices. The convergence is similar to the above case. The graph of γ_j is shown by the dashed line in Fig. 5, and the next feasible solution is obtained in 30 iterations.

$$K(s) = \begin{bmatrix} 2.335 & 0 \\ 0 & 2.391 \end{bmatrix} + \begin{bmatrix} 2.417 & 0 \\ 0 & 2.894 \end{bmatrix} \frac{1}{s} + \begin{bmatrix} 7.347 & 0 \\ 0 & 7.116 \end{bmatrix} \frac{10^{-3}s}{1+0.01s}$$

The minimum γ's attained by the H_∞ control, the multivariabel PID control, and the decentralized PID control are 0.5424, 0.5572, and 0.5882 , respectively.
The programming is on MATLAB where ROBUST CONTROL TOOLBOX, 'SEDUMI for interface', 'SEDUMI solver' are used.

7. CONCLUSION

In this paper, the parametrization of all the solutions of H_∞ control problem is used to obtain BMI constraints on the PID gains for each frequency, then an LMI condition that is a sufficient condition for the BMI is derived for a stabilizing PID gain for each frequency. Starting from a stabilizing PID gain and solving the LMI's iteratively, we can obtain a sequence of PID gains. It is shown that the sequence is convergent and that the corresponding sequence of the H_∞ norm bound is monotonically non-increasing. Numerical examples show the usefulness of our algorithm.

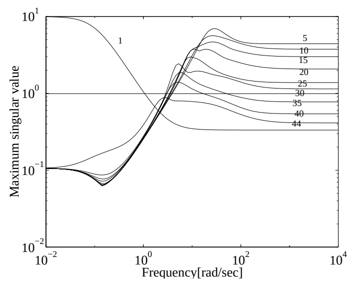

Fig. 4. Graphs of $\sigma_{max}[T_{zw}(j\omega)]$

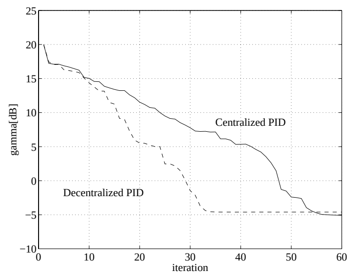

Fig. 5. $\gamma_i,\ i = 1, 2, \ldots, 60$

REFERENCES

Ho, M.-T., A. Datta and S.P. Bhattacharyya (2001). Sufficient conditions for the synthesis of H_∞ fixed-order controllers. *Int. J. of Robust and Nonlinear Control* **11**, 681–708.

Lelic, M. and Z. Gajic (2000). A reference guide to PID controllers in the nineties. *IFAC Digital Control: Past, Present and Future of PID Control.*

Miyamoto, S. and G. Vinnicombe (1997). Fixed structure H_∞ control design for multiple plants-a coprime factorization and LMI approach. *Proc. of 4th ECC, Brussels.*

Saeki, M. and K. Aimoto (2000). PID controller optimization for H_∞ control by linear programming. *Int. J. of Robust and Nonlinear Control* **10**, 83–99.

Saeki, M., J. Kimura, Y. Kurata and S. Tsuji (1998). Parameter space design method of PID controller for H_∞ control problem. *Transactions of the Institute of Systems Control and Information Engineering* **11**, 35–40.

Unar, M.A., D.J. Murray-Smith and S.F.A. Shah (1996). Design and tuning of fixed structured PID controllers a survey. *Technical Report CSC-96016, Glasgow University.*

Zeng, F., Q.-G. Wang and T. H. Lee (2002). On the design of multivariable PID controller via LMI approach. *Automatica* **38**, 517–526.

Zhou, K., J.C. Doyle and K. Glover (1996). *Robust and Optimal Control.* Prentice Hall. London.