OBSERVABILITY OF ROW–FINITE COUNTABLE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract: Infinite–dimensional linear dynamic systems, with output, are studied. They are described by row-finite infinite matrices. Observability of such systems is investigated. It is shown that observability is equivalent to the property that one can calculate the value of each state variable using only finitely many rows of Kalman observability matrix. Different necessary or sufficient conditions of observability of such systems are given.

Keywords: Linear system, system matrices, trajectories, differential equations, dynamics, matrix algebra, observability.

1. INTRODUCTION

Infinite systems of ordinary differential equations appear naturally in many applications, see e.g. (Deimling, 1977; Zautykov, 1965). They involve infinitely many variables that are to be determined, so the state space of the system is infinite-dimensional. Usually it is some Banach space of real sequences and Banach space theory can be applied (Deimling, 1977; Persidski, 1959). In control theory such infinite systems may appear if one considers the infinite extension of a finite-dimensional system (Fliess, *et al*, 1997; Jakubczyk, 1992; Pomet, 1995). The new variables are the derivatives of the control (input). In this case it is not reasonable to impose any restrictions on the values of the variables, so the natural state space is the space of all real sequences. This space is no longer a Banach space, but it is still a Fréchet space; see e.g. (Moszyński and Pokrzywa, 1974) for results concerning differential equations in such spaces.

We study here linear systems that are naturally described by infinite matrices (Cooke, 1950; Wilansky and Zeller, 1955). As the product of such matrices is not always well defined, we restrict our studies to systems that are described by row-finite matrices whose rows contain only finitely many elements different from zero. Such matrices form an algebra over $\mathbb{R}$; in particular the product is associative.

The main problem studied in this paper is observability. Thus we extend the system of differential equations adding the output equations. Observability is easily characterized as injectivity of the operator defined by the Kalman observability matrix (infinite but row-finite again). However checking injectivity is in general not easy. A necessary condition for this is the infinite rank of the Kalman matrix, but this is far from being sufficient. We show that a necessary and sufficient condition for injectivity is existence of a row-finite left inverse of the Kalman matrix. This is equivalent to the fact that any state variable can be expressed as a linear combination of *finitely*

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many outputs and their derivatives. We provide several examples that illustrate the results.

Observability of *discrete-time* systems described by row-finite matrices was studied by Bartosiewicz and Mozyrska (2001).

2. ROW–FINITE MATRICES

Let I, J, K be nonempty countable sets. Consider the countable product $\mathbb{R}^K = \prod_{k\in K} \mathbb{R}$ as the set of all functions $K \to \mathbb{R}$. If $K = \mathbb{N}$, then $\mathbb{R}^{\mathbb{N}}$ is the linear space of all infinite sequences of real numbers represented by infinite columns $x = (x_1, \ldots, x_i, \ldots)^T, x_i \in \mathbb{R}, i \in \mathbb{N}$. Let $K = I \times J$. Then each element $A \in \mathbb{R}^{I\times J}$, $A : I \times J \ni (i,j) \mapsto a_{ij} \in \mathbb{R}$, is called *an $I \times J$ matrix.* We will denote it in the standard way $A = (a_{ij})_{i\in I, j\in J}$. By $E_I = (\delta_{ij})_{i,j\in I}$, where $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$, we denote the identity matrix in $\mathbb{R}^{I\times I}$.

Let us consider a matrix $A = (a_{ij})$ in $\mathbb{R}^{I\times J}$. The transpose of A is a matrix $A^T \in \mathbb{R}^{J\times I}$, $A^T = (b_{ji})_{j\in J, i\in I}$, $b_{ji} = a_{ij}$. The set $\mathbb{R}^{I\times J}$ is a linear space over $\mathbb{R}$ with the standard operations.

Let I be a countable set. Let $\sum_{i\in I} a_i$ be a series of elements of a linear metric space (X, d). We say that the series $\sum_{i\in I} a_i$ *converges to an element* c, called the sum of the series, if for every real number $\varepsilon > 0$ there is a finite set $S_\varepsilon \subset I$ such that for every finite set $S \supset S_\varepsilon : d\left(\sum_{i\in S} a_i, c\right) < \varepsilon$. The series is *convergent* if it converges to some c. Observe that we do not need an order in I and grouping does not change the sum. If $X = \mathbb{R}$ with the standard metric, then a series $\sum_{i\in I} a_i$ is convergent if and only if the series $\sum_{i\in I} |a_i|$ is convergent (Ruiz, 1993).

Now let $A \in \mathbb{R}^{I\times J}, B \in \mathbb{R}^{J\times K}$. Then the product $AB = (c_{ik})_{i\in I, k\in K}$ is well defined if the series $c_{ik} = \sum\limits_{j\in J} a_{ij} b_{jk}$ is convergent for each $(i,k) \in I \times K$.

A matrix $A = (a_{ij})_{i\in I, j\in J}$, is called *row–finite* if for each $i \in I$ the set $S_A(i) := \{j \in J : a_{ij} \neq 0\}$ is finite. Similarly, a matrix whose transpose is row–finite is called *column–finite.* The identity matrix E_I is row–finite and column–finite.

Proposition 2.1.

(1) The set of row–finite matrices from $\mathbb{R}^{I\times I}$ forms an algebra over $\mathbb{R}$ with a unit – the identity matrix E_I.
(2) The set of column–finite matrices from $\mathbb{R}^{I\times I}$ forms an algebra over $\mathbb{R}$ with a unit.

Remark 2.2. The associativity of multiplication is the most essential property for row–finite (column–finite) matrices from $\mathbb{R}^{I\times I}$. It does not hold for all infinite matrices, however. For example, if $I = \mathbb{N}$, $b = (1, 1, 1, \ldots) \in \mathbb{R}^{\{1\}\times\mathbb{N}}$ and $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \ldots \\ 0 & 1 & -1 & 0 & 0 \ldots \\ 0 & 0 & 1 & -1 & 0 \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$, then $(bA)b^T = 1$ and $b(Ab^T) = 0$.

Let I, J be countable sets and let $A \in \mathbb{R}^{I\times J}$, $B \in \mathbb{R}^{J\times I}$. If $AB = E_I$, then B is called *right inverse of* A and is denoted by A'. Similarly, then A is called *left inverse of* B and is denoted by $'B$. If $I = J$ and $AB = BA = E_I$ then B is called *two–sided inverse of* A and is denoted by A^{-1}.

Let $A \in \mathbb{R}^{J\times I}$ be row–finite. Then by $\mathcal{A}$ we denote the mapping $\mathbb{R}^I \to \mathbb{R}^J$, $\forall x \in \mathbb{R}^I : \mathcal{A}(x) = Ax$. Observe that $\mathcal{A}$ is well defined and linear.

Proposition 2.3. If a row–finite matrix A has a row–finite left inverse $'A$, then the mapping $\mathcal{A}$ is injective.

Example 2.4. (Wilansky and Zeller, 1955) Let $I = J = \mathbb{N}$ and $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \cdots \\ 0 & 1 & -1 & 0 & 0 \cdots \\ 0 & 0 & 1 & -1 & 0 \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$. Then $\mathcal{A}$ is not injective, because for $x = (1, 1, \ldots)^T$: $\mathcal{A}(x) = Ax = 0$. However the matrix: $B = \begin{pmatrix} 1 & 1 & 1 & 1 \cdots \\ 0 & 1 & 1 & 1 \cdots \\ 0 & 0 & 1 & 1 \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$ satisfies $AB = BA = E_{\mathbb{N}}$, so it is a two–sided inverse of A. But B is not row–finite.

3. ROW–FINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

Let $x : [0, \infty) \ni t \mapsto x(t) \in \mathbb{R}^I$ be a differentiable function. The system of differential equations $\dot{x}(t) = \frac{dx}{dt}(t) = Ax(t)$, where $A \in \mathbb{R}^{I\times I}$ is a row–finite matrix, is called *a row–finite system.* The function x is its solution. Let us consider the initial value problem

$$\dot{x} = Ax, \; x(0) = x^0 \in \mathbb{R}^I, \tag{3.1}$$

where A is a row–finite matrix.

For $I = \mathbb{N}$ the discussion of existence and uniqueness of solutions of the initial value problem (3.1) can be found, e.g., in (Deimling, 1977). In Proposition 3.1 we extend Theorem 6.2 of (Deimling, 1977) to the general countable case.

Proposition 3.1. Let $A = (a_{ij})_{i,j\in I}$. Assume that there exists a sequence $(S_n)_{n\in\mathbb{N}}$ of finite subsets of I such that for each $n \in \mathbb{N} : S_n \subset S_{n+1}$, $\sum_{i\in\mathbb{N}} S_n = I$ and $S_A(i) \subset S_n$ for $i \in S_n$. Then the initial value problem (3.1) has a unique solution. Otherwise, the problem (3.1) has infinitely many solutions.

If A is row–finite matrix then for each $k \in \mathbb{N}$: $A^k = A^{k-1}A$ is row–finite too. In most cases the exponential matrix is not well defined. But if for example $I = \mathbb{N}$ and A is a lower diagonal matrix then the exponential matrix $e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots$ is well defined (Cooke, 1950). And then for all $x^0 \in \mathbb{R}^{\mathbb{N}}$ the problem (3.1) has the unique solution $x(t) = e^{At}x^0$. If A is row–finite and is not lower diagonal we can lose the uniqueness of solutions and e^{At} may not exist even for $I = \mathbb{N}$.

Example 3.2. Let $I = \mathbb{N} \cup \{0\}$, $\frac{dx_i}{dt} = x_{i+1},\ i \geq 0$ and $x(0) = x^0 = 0 \in \mathbb{R}^I$. Then the solution is produced by an arbitrary smooth function $\varphi = \varphi(t)$ such that $\frac{d^k\varphi}{dt^k}(0) = 0, k = 0, 1, 2, \ldots$, and $x_k(t) = \frac{d^k\varphi}{dt^k}(t),\ k \in I$. Since there are infinitely many such functions (they differ by "flat" functions with all derivatives at $t = 0$ equal 0), we have infinitely many solutions. The corresponding matrix for this system $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$ has well defined exponential matrix $e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \ldots \\ 0 & 1 & t & \frac{t^2}{2!} & \ldots \\ 0 & 0 & 1 & t & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$. In spite of that the product $e^{At}x^0$ may not exists as e^{At} is not row-finite. Observe that the condition of Proposition 3.1 does not hold.

The above discussion shows that an initial value problem (3.1) may have infinitely many smooth solutions. To have uniqueness we consider formal solutions and use formal power series. Let $A \in \mathbb{R}^{I\times I}$ be row–finite and $x \in \mathbb{R}^I$. Then we define a vector of formal power series corresponding to A and x^0 by $\Gamma_{x^0,A} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k x^0$. Observe that for each $k \in \mathbb{N}$ the power A^k is row–finite, hence $A^k x^0 \in \mathbb{R}^I$ exists. Let $(A^k)_i$ denotes the i–th row of the matrix A^k. Then $\Gamma_{x^0,A}$ is a countable family of formal power series of the form $\left\{\sum_{k=0}^{\infty} \frac{t^k}{k!}(A^k)_i x^0\right\}_{i\in I}$.

Proposition 3.3. Let A be a row–finite matrix. Then for all $x^0 \in \mathbb{R}^I$ the initial value problem (3.1) has the unique formal solution $\Gamma_{x^0,A} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k x^0$.

Proof: Observe that the constant term of the series is x^0 and $\frac{d}{dt}\Gamma_{x^0,A} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^{k+1}x^0 = A\Gamma_{x^0,A}$.

□

If $\Gamma_{x^0,A}$ is convergent then we have an analytic solution.

4. OBSERVABILITY CONDITIONS

In this section we are concerned with the system with output:

$$(\Sigma_{I,J}) : \begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t), \end{aligned} \tag{4.1}$$

where $x : [0,\infty) \to \mathbb{R}^I, y : [0,\infty) \to \mathbb{R}^J$, and $A \in \mathbb{R}^{I\times I}$ and $C \in \mathbb{R}^{J\times I}$ are row–finite. Let $x^0 \in \mathbb{R}^I$. Given a formal solution $\Gamma_{x^0,A}$ of the dynamical part of the system and corresponding to the initial condition x^0 we define the formal output: $\mathcal{Y}_{x^0} = C\Gamma_{x^0,A}$. This is a family of formal power series indexed by J.

Definition 4.1. We say that $x^1, x^2 \in \mathbb{R}^I$ are *indistinguishable* (with respect to $\Sigma_{I,J}$) if $\mathcal{Y}_{x^1} = \mathcal{Y}_{x^2}$. Otherwise x^1, x^2 are distinguishable. We say that the system $\Sigma_{I,J}$ is *observable* if every two distinct points are distinguishable.

Proposition 4.2. The points $x^1, x^2 \in \mathbb{R}^I$ are indistinguishable iff for all $k \in \mathbb{N} \cup \{0\} : CA^k x^1 = CA^k x^2$.

Proof: $\mathcal{Y}_{x^1} = \mathcal{Y}_{x^2} \Leftrightarrow \sum_{k=0}^{\infty} \frac{t^k}{k!}CA^k x^1 = \sum_{k=0}^{\infty} \frac{t^k}{k!}CA^k x^2$ $\Leftrightarrow\ \forall k \in \mathbb{N} \cup \{0\} : CA^k x^1 = CA^k x^2$. □

For each $n \in \mathbb{N} \cup \{0\}$ we define the countable set $J_n = J$. Then the disjoint union $K = \bigcup_{n\in\mathbb{N}\cup\{0\}} J_n$ is also countable.

Let $D = (d_{ki})$ be a $K \times I$ matrix whose k-th row is equal $D_k = C_k A^n$ for $k \in J_n$. If C has only finitely many rows, that is when J is finite, the matrix D can be written in the following way: $D = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix}$.

Let $\mathcal{D}(x) = Dx, \mathcal{D} : \mathbb{R}^I \to \mathbb{R}^K$. From Proposition 4.2 we get a similar characterization of observability as in the finite-dimensional case.

Proposition 4.3. $\Sigma_{I,J}$ is observable $\Longleftrightarrow$ $\mathcal{D}$ is injective.

Let $D_{|S}$ denote the matrix obtained from D by choosing only rows with indices in the set $S \subset K$. The following theorem gives an important characterization of observability of row-finite systems.

Theorem 4.4. $\Sigma_{I,J}$ is observable $\iff$ for every $i \in I$ there is a finite $S_i \subset K$ such that $D_{|S_i} x = 0 \Rightarrow x_i = 0$.

The proof of Theorem 4.4 is given in Section 6.

Remark 4.5. Theorem 4.4 says that for an observable row-finite system one can decide that a variable x_i is equal 0 on the basis of only finitely many equations from the infinite system $Dx = 0$. This looks natural, but is not obvious. It is a consequence of row-finiteness of D and does not hold in general. For example, if

$$D = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \frac{1}{2} & 0 & \frac{1}{8} & \frac{1}{16} & \cdots \\ \frac{1}{4} & \frac{1}{8} & 0 & \frac{1}{32} & \cdots \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

then it can be shown (Cross, 1963) that $Dx = 0$ implies $x = 0$. However to decide that $x_1 = 0$ one has to use all the equations of the system.

Let $e_i, i \in I$ be the infinite row with 1 at the i-th position and 0 at other positions and let $\mathcal{E}_i : \mathbb{R}^I \longrightarrow \mathbb{R}$, for $x \in \mathbb{R}^I$: $\mathcal{E}_i(x) = e_i x$. Then we have another characterization of observability.

Proposition 4.6. System $\Sigma_{I,J}$ is observable iff $\forall i \in I \ \exists S_i = \{k_1, \ldots, k_{n_i}\} \subset K \ \exists \{a_1, \ldots, a_{n_i}\} \subset \mathbb{R} : e_i = a_1 D_{k_1} + \cdots + a_{n_i} D_{k_{n_i}}$.

The proof of the above fact relies on Theorem 6.9, shown in Section 6.

Remark 4.7. Since the rows of D correspond to derivatives of the output, one can characterize observability as possibility to compute every state variable as a linear combination of finitely many outputs and their derivatives.

If C has finitely many rows then for all $0 \leq k < \infty$ the rank of the matrix $\left(C \ldots CA^k \right)'$ is finite.

Corollary 4.8. System $\Sigma_{I,J}$ with J finite is observable iff $\forall i \in I \ \exists k_i \in \mathbb{N} \cup \{0\}$:

$$\operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^{k_i} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^{k_i} \\ e_i \end{pmatrix}.$$

The following proposition gives a sufficient condition for observability.

Proposition 4.9. Let $D = (d_{ki})_{k \in K, i \in I}$. Assume that for every $i \in I$ there exist finite $S_i \subset K$ and $T_i \subset I$ such that $i \in T_i$, for every $k \in S_i$: $S_D(k) \subset T_i$ and $\operatorname{rank} (d_{kj})_{k \in S_i, j \in T_i} = \overline{\overline{T_i}}$. Then $\Sigma_{I,J}$ is observable.

Proof: Let for $i \in I$, $S_i \subset K, T_i \subset I$ be as in the assumption. Then for all $x \in \mathbb{R}^I$: $D_{|S_i} x = 0 \Rightarrow \forall (j \in T_i) x_j = 0$. Hence $\mathcal{D}$ is injective and $\Sigma_{I,J}$ is observable. □

Let $D = (d_{ij}) \in \mathbb{R}^{I \times J}$ and let $S_1 \subset I, S_2 \subset J$ be finite sets of the same cardinality k. Then by *a minor of the order* k we will mean the determinant: $|(d_{ij})_{i \in S_1, j \in S_2}|$. To calculate this we need some orders in S_1 and S_2. The minors depend on the orders, but different orders may only change signs of the minors.

It is easy to show the following necessary condition of observability.

Proposition 4.10. If Σ is observable then $\operatorname{rank} D = \infty$.

5. EXAMPLES

Example 5.1. Let us consider the following system, for $I = J = \mathbb{Z}$:

$$\left. \begin{aligned} f_{2k-1}(x) = x_{2k-1} + x_{2k} + x_{2k+1} = 0 \\ f_{2k}(x) = x_{2k} + x_{2k+1} = 0 \end{aligned} \right\} , k \in \mathbb{Z}$$

The above system has a unique solution $x = 0 \in \mathbb{R}^{\mathbb{Z}}$ and we can compute every x_i from finitely many functions f_j : $x_{2k-1} = f_{2k-1} - f_{2k}$ and $x_{2k} = f_{2k} - f_{2k+1} + f_{2k+2}$. For any $k \in \mathbb{N}$ take $S = S_1 = S_2 = \{1, \ldots, k\}$. Then the minor of the matrix of coefficients defined by k is nonzero.

Example 5.2. Let us consider a system in the following form: $(\Sigma_{\mathbb{Z},\mathbb{N}}) : \dot{x}_k = x_{k+1}, k \in \mathbb{Z}, y_n = x_{-n}, n \in \mathbb{N}$. The matrices of this system are: $A = (a_{ij})_{i \in \mathbb{Z}, j \in \mathbb{Z}}$, $a_{i,i+1} = 1$ and for $j \neq i + 1 : a_{ij} = 0$, $C = (c_{ij})_{i \in \mathbb{N}, j \in \mathbb{Z}}$, $c_{i,-i} = 1$, and for $j \neq -i : c_{ij} = 0$. Observe that for $k \in \mathbb{N} \cup \{0\}$: $CA^k x = \begin{pmatrix} x_{k-1} \\ x_{k-2} \\ \vdots \end{pmatrix}$. Hence the map $\mathcal{D}$ for this system is injective and the system is observable.

Now let us change the output so that it is finite-dimensional: $(\Sigma_{\mathbb{Z},S}) : \dot{x}_k = x_{k+1}, k \in \mathbb{Z}, y = \left(x_{s_1}, x_{s_2}, \ldots, x_{s_n} \right)'$, where $S = \{s_1, s_2, \ldots, s_n\} \subset \mathbb{Z}$. In this case for $k \in \mathbb{N} \cup \{0\}$: $CA^k x = \left(x_{s_1+k}, x_{s_2+k}, \ldots, x_{s_n+k} \right)'$. Hence $\Sigma_{\mathbb{Z},S}$ is not observable (e.g. x^1 and x^2 such that $x_i^1 \neq x_i^2$ for some $i < s_1$ are indistinguishable).

Example 5.3. Let the system $\Sigma_{\mathbb{N},\{1\}}$ be in the following form

$$\begin{aligned} \dot{x}_{2n-1} &= x_{2n} - x_{2n+2} - x_{2n+3} \\ \dot{x}_{2n} &= x_{2n+1} + x_{2n+3} + x_{2n+4} + x_{2n+5}\,, \quad n \in \mathbb{N}. \\ y &= x_1 + x_2 + x_3 \end{aligned}$$

Then the matrix

$$D = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Observe that the equation $Dx = 0$ is that of Example 5.1. The system is observable but the condition from Theorem 4.9 is not satisfied.

Example 5.4. Let us consider the partial differential equation describing heat transfer in an infinite rod $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}$. Let us discretize the equation with respect to x variable with the step equal 1. Let $u_k(t) = u(t,k)$ for $k \in \mathbb{Z}$. Then we obtain an infinite system of linear ordinary differential equations parameterized by $\mathbb{Z}$:

$$\dot{u}_k = u_{k-1} - 2u_k + u_{k+1}. \tag{5.1}$$

Let us assume that we observe one of the variables, for instance let the output be $y = u_0$. Then the entries of the $(\mathbb{N}\cup\{0\})\times\mathbb{Z}$ matrix D have the form: $d_{ki} = \binom{2k}{i+k}(-1)^{k+i}$, where $k \geq 0$, $i \in \mathbb{Z}$ and $\binom{n}{j} = 0$ whenever $j < 0$ or $j > n$. The equation $Du = 0$ has infinitely many solutions. Every $u \in \mathbb{R}^{\mathbb{Z}}$ satisfying the condition $u_{-i} = -u_i$ is a solution of this equation, so (5.1) with the output $y = u_0$ is not observable. To get observability one can extend the output to the following: $y_1 = u_0$, $y_2 = u_1$. Then the matrix D satisfies the condition from Proposition 4.9 and the system is observable. In general, we get observability if we observe two subsequent variables u_k and u_{k+1}. On the other hand, observing infinitely many variables does not guarantee observability.

6. PROOF OF THEOREM 4.4

Let us first recall some facts from functional analysis.

The space $\mathbb{R}^I$ with the product topology is metrizable. However there is no norm for this topology, so $\mathbb{R}^I$ is not a Banach space.

Recall that a linear topological space is called *a Fréchet space* if it is metrizable, complete and locally convex. Note that $\mathbb{R}^{\mathbb{N}}$ is a Fréchet space (Banach, 1932). Similarly, for any countable set I the space $\mathbb{R}^I$ is a Fréchet space.

The following fact was proved in (Banach, 1932) for $I = \mathbb{N}$. This can easily be extended for an arbitrary countable set I.

Proposition 6.1. A function $f : \mathbb{R}^I \to \mathbb{R}$ is linear and continuous iff there is a finite set $S \subset I$ and a set of real numbers $\{a_i\}_{i\in S}$ such that for all $x \in \mathbb{R}^I$: $f(x) = \sum_{i\in S} a_i x_i$, where $x_i = x(i)$.

Let I be a countable set. By $\mathbb{R}^{(I)}$ we denote the set of those $x \in \mathbb{R}^I$ that have only finite number of nonzero elements. So, if $x \in \mathbb{R}^{(I)}$ then x is column-finite. The set $\mathbb{R}^{(I)}$ is a linear subspace of $\mathbb{R}^I$.

Let $(\mathbb{R}^I)^* = \{x^* : \mathbb{R}^I \to \mathbb{R} : \ x^* \text{ is linear and continuous}\}$ be the dual space to $\mathbb{R}^I$ (in the functional–analytic sense).

Corollary 6.2. Let I be countable. Then $\left(\mathbb{R}^I\right)^*$ is isomorphic with $\mathbb{R}^{(I)}$.

Remark 6.3. The algebraic dual space to $\mathbb{R}^I$ is much bigger than $\mathbb{R}^{(I)}$ and is not easy to describe. But the algebraic dual to $\mathbb{R}^{(I)}$ is just $\mathbb{R}^I$.

It is easy to verify the following statement.

Proposition 6.4. Let I, J be countable sets. A map $\mathcal{F} : \mathbb{R}^I \to \mathbb{R}^J, \mathcal{F}(x) = (\mathcal{F}_j(x))_{j\in J}$, is continuous iff every $\mathcal{F}_j : \mathbb{R}^I \to \mathbb{R}, j \in J$, is continuous.

Corollary 6.5. A linear map $\mathcal{F} : \mathbb{R}^I \to \mathbb{R}^J$ is continuous iff it is represented by a row–finite matrix $F \in \mathbb{R}^{J\times I}$ (i.e. $\forall x \in \mathbb{R}^I : \mathcal{F}(x) = Fx$).

The following proposition is a direct extension of Toeplitz's theorem proved in (Wilansky and Zeller, 1955) for $I = J = \mathbb{N}$.

Proposition 6.6. Let $D \in \mathbb{R}^{J\times I}$ be a row–finite matrix and $\mathcal{D} : \mathbb{R}^I \to \mathbb{R}^J$, $\mathcal{D}(x) = Dx$. Then $y \in \operatorname{im}\mathcal{D}$ iff

$$\forall v \in \mathbb{R}^{(J)} : v^T D = 0 \Rightarrow v^T y = 0.$$

Proposition 6.7. Let $D \in \mathbb{R}^{J\times I}$ be row–finite. Then $\operatorname{im}\mathcal{D}$ is closed in $\mathbb{R}^J$.

Proof: Let $y^{(n)} \in \operatorname{im}\mathcal{D}$, $\lim_{n\to\infty} y^{(n)} = y$. Then, from Proposition 6.6, $\forall n \in \mathbb{N}\ \forall v \in \mathbb{R}^{(J)} : v^T D = 0 \Rightarrow v^T y^{(n)} = 0$. From Corollary 6.5 the map $\mathbb{R}^J \ni x \mapsto v^T x \in \mathbb{R}$ is continuous, so $v^T y = 0$ as well. Thus $y \in \operatorname{im}\mathcal{D}$. $\square$

Let us observe that a closed linear subspace of a Fréchet space is a Fréchet space too. Moreover we have the following fact about mappings between Fréchet spaces.

Theorem 6.8. (Banach, 1932) Every linear continuous bijection between Fréchet spaces is a homeomorphism.

The following theorem is essential in studying observability.

Theorem 6.9. Let $D \in \mathbb{R}^{J\times I}$ be row–finite and $\mathcal{D}(x) = Dx$ for $x \in \mathbb{R}^I$. The mapping $\mathcal{D}$ is injective iff there exists a row–finite left inverse of D.

Proof: The "if" part is exactly Proposition 2.3.

To prove the converse suppose that $\mathcal{D}$ is injective. Let $Y = \operatorname{im}\mathcal{D} \subset \mathbb{R}^J$. Then Y is closed in $\mathbb{R}^J$. Hence $\mathcal{D} : \mathbb{R}^I \to Y$ is a linear continuous bijection between Fréchet spaces. From Theorem 6.8 we get that $\mathcal{F} = \mathcal{D}^{-1} : Y \to \mathbb{R}^I$ is a linear and continuous mapping. We denote by $\tilde{\mathcal{F}} : \mathbb{R}^J \to \mathbb{R}^I$ a mapping that is linear and continuous and for all $y \in Y$: $\tilde{\mathcal{F}}(y) = \mathcal{F}(y)$. Such a mapping exists by Hahn–Banach Theorem (Taylor and Lay, 1980). Then for all $x \in \mathbb{R}^I$: $\tilde{\mathcal{F}}(\mathcal{D}(x)) = x$. Therefore $\tilde{\mathcal{F}} \circ \mathcal{D} = \mathrm{id}_{\mathbb{R}^I}$. As $\tilde{\mathcal{F}}$ is linear and continuous, then it is represented by a row–finite matrix (from Corollary 6.5). Let $\tilde{\mathcal{F}}(x) = Fx$, where $F \in \mathbb{R}^{I\times J}$ is row–finite. Then $\forall x \in \mathbb{R}^I : (FD)x = F(Dx) = \tilde{\mathcal{F}}(\mathcal{D}(x)) = x$. Hence $FD = E_I$, so $F = {}'\!D$. □

The statement of the following proposition gives in fact Theorem 4.4.

Proposition 6.10. $\mathcal{D} : \mathbb{R}^I \to \mathbb{R}^J$ is injective iff $\forall i \in I\ \exists$ finite $S_i \subset J : \forall x \in \mathbb{R}^I$

$$\left(D_{|S_i}x = 0 \Rightarrow x_i = 0\right). \tag{6.1}$$

Proof: If $\mathcal{D}$ is injective then, from Theorem 6.9, there is a row-finite ${}'\!D \in \mathbb{R}^{I\times J}$ such that for $x \in \mathbb{R}^I$: ${}'\!D(Dx) = x$. Let ${}'\!D = (a_{ij})_{i\in I, j\in J}$. Then for all $i \in I$: ${}'\!D_{|\{i\}} = (a_{ij})_{j\in J}$. Let $S_i = S_{\mathcal{D}}(i)$. Then $x_i = {}'\!D_{|\{i\}}(Dx) = \sum_{j\in S_i} a_{ij} D_{|\{j\}}x = (a_{ij})_{j\in S_i} D_{|S_i}x$. Hence, if $D_{|S_i}x = 0$ then $x_i = 0$.

On the other hand, assume that (6.1) holds and let $Dx = 0$. Take $i \in I$. Then $D_{|S_i}x = 0$, so by (6.1) $x_i = 0$. This gives the injectivity of $\mathcal{D}$. □

Corollary 6.11. Let us consider an infinite countable system of linear equations: $f_j(x) = \sum_{i\in I} a_{ji}x_i = 0$, $(j \in J)$, where for each j only finitely many coefficients $a_{ji} \neq 0$. The above system has exactly one solution $x = 0 \in \mathbb{R}^I$ if and only if for every $i \in I$ there exists finite set $S_i \subset J$ such that $x_i = \sum_{j\in S_i} b_{ij} f_j(x)$ for some $b_{ij} \in \mathbb{R}$.

7. CONCLUSION

We studied here observability of linear systems described by infinitely many differential equations involving infinitely many variables. Such systems may serve as models for many real life phenomena or come from partial discretizations of systems given by partial differential equations.

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