DYNAMIC THRESHOLD GENERATORS FOR FAULT DETECTION IN UNCERTAIN SYSTEMS

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Abstract:
The problem of developing robust thresholds for fault detection is addressed. An inequality for the solution of a linear system with uncertain parameters is provided and is shown to be a valuable tool for developing dynamic threshold generators for fault detection. Such threshold generators are desirable for achieving robustness against model uncertainty in combination with sensitivity to small faults.
The usefulness of the inequality is illustrated by developing an algorithm for detection of clogging in the valves of a flotation process. Simulations with measurement data show that the algorithm detects faults without generating false alarms.

Keywords: Fault detection, thresholds, uncertain linear systems, flotation

1. INTRODUCTION

Technical systems are inherently exposed to faults such as leaking valves, broken bearings, faulty sensors, etc. In most applications it is vital that these faults are detected promptly and accommodated for.

When an analytical process model is available, fault detection methods based on analytical redundancy may be utilized. During the past three decades, extensive research has been carried out in this area and many methods have been developed. All of these consist essentially of two steps, *residual generation* and *residual evaluation*. The purpose of the first step is to generate a signal, the residual, which is supposed to be nonzero in the presence of fault and zero otherwise. This problem has been treated extensively in the literature and solutions based on *e.g.* state observers, parity equations, or on-line identification algorithms have been suggested, see (Frank and Ding 1997).

However, the residual is almost always nonzero due to disturbances and model uncertainty, even if there is no fault. The purpose of the second step of the fault detection algorithm is thus to evaluate the residual and draw conclusions on the presence of a fault. This is done by comparing some function of the residual, the *evaluation signal*, to a threshold and then to declare the presence of a fault if the former exceeds the latter.

Detection thresholds that are robust against frequency domain uncertainty are developed in *e.g.* (Emami-Naeini *et al.* 1988) and (Frank and Ding 1994). However, the thresholds that result from this kind of uncertainty description are generally functions only of some signal norm of the known inputs and are thus essentially constant. In contrast, experience shows that the residual in a real fault detection application is often correlated with the inputs, as a result of model uncertainty. This fact, in combination with the difficulties of extending frequency-domain methods to nonlinear systems, motivates the search for methods to be able to utilize uncertainty descriptions in the time-domain.

In *e.g.* (Zhang *et al.* 2003), unstructured uncertainty in a class of nonlinear state-space systems is treated. Another way of representing the model uncertainty is to assume uncertain parameters. This kind of uncertainty description has been considered in *e.g.* (Johansson and Medvedev 2000) and (Ding *et al.* 2003).

An uncertain parameter in a general, nonlinear system, can clearly affect the system response in many different ways. However, considering a Taylor approximation of a state-space representation of the system motivates distinguishing between additive and multiplicative parameters. In this context, it is clear that parameters entering mutiplicatively with the state constitutes the main difficulty. Therefore, we will here consider systems of the type

$$\dot{x} = Ax + N(\pi \otimes x) + g \quad (1a)$$
$$y = Cx \quad (1b)$$

where $x \in \mathbb{R}^n$ is a state vector and $\pi(t) \in \mathbb{R}^m$ are the parameter uncertainties while $g(t) \in \mathbb{R}^n$ is some input that may also depend on the uncertain parameters.

Applying a Luenberger observer to a linear system with uncertain parameters of the form (1) yields an error system which is also described by (1). As the output of this error system is a residual, it is motivated to search for an upper bound for y in (1). In this paper, an upper bound for the modulus of x in (1) is provided as a dynamic system with g as input. If C has full rank, it may be assumed that $C = [I\ 0]$, since this is possible to achieve with a linear transformation. Thus, the obtained inequality immediately provides an upper bound for the residual.

Application to a flotation process demonstrates that the inequality is a powerful tool in determining fault detection algorithms that are robust against model uncertainty and yet sensitive to faults.

2. PRELIMINARIES

All signals are assumed causal, *i.e.* are defined for $t \geq 0$. The truncation operator $\mathcal{P}_\tau$ is defined as

$$(\mathcal{P}_\tau x)(t) = \begin{cases} x(t) & t \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

and $\|\cdot\|_p$ denotes the usual p-norm or ∞-norm. The space $\mathbb{L}_p^{n\times m}$, $1 \leq p \leq \infty$ is the set of functions from $\mathbb{R}_+$ to $\mathbb{R}^{n\times m}$ such that $\|x\|_p < \infty$ while $\mathbb{L}_{pe}^{n\times m}$, $1 \leq p \leq \infty$ denotes the set of function from $\mathbb{R}_+$ to $\mathbb{R}^{n\times m}$ such that $\|\mathcal{P}_\tau x\|_p < \infty$ for all $\tau \geq 0$.

A star between two functions $F \in \mathbb{L}_{pe}^{n\times m}$ and $G \in \mathbb{L}_{qe}^{m\times k}$ denotes convolution, *i.e.*

$$(F * G)(t) \triangleq \int_0^t F(t-\tau)G(\tau)d\tau$$

The condition $1/p + 1/q = 1$ ensures that the result is finite for all $t \geq 0$ which follows from the Hölder inequality. A linear operator defined by convolution by a weighting function is denoted by the symbol of the weighting function written in bold-face font, thus *e.g.* $\mathbf{F}G \triangleq F * G$. The identity operator is denoted I, *i.e.* $IG \triangleq G$. An induced operator norm is denoted by the same symbol as the signal norm from which it is induced, *i.e.* $\|\mathbf{F}\|_p \triangleq \sup_{\|G\|_p=1} \|\mathbf{F}G\|_p$.

Short-hand notations for differentiation of a matrix with respect to one or two row vectors $x^T \in \mathbb{R}^n$ and $y^T \in \mathbb{R}^m$ are $F'_x \triangleq \partial F/\partial x^T$ and $F''_{xy} \triangleq \partial F'_x/\partial y^T$.

The $n \times m$-matrix with each element equal to zero is denoted $0_{n\times m}$ while the identity matrix of order n is denoted I_n. A column vector with dimension n where each element is equal to 1 is denoted 1_n. If the dimension is clear from context, the index is omitted.

Let $|\cdot|$ denote the matrix modulus function, *i.e.* element-wise absolute value. Inequalities between matrices is also to be interpreted element-wise. The following inequalities for matrix operations are trivial but included in order to increase readability of the proofs in the sequel.

Property 1. Let A, B, and C be matrices of compatible dimension.

(a) If $A \geq 0$ and $B \geq C$, then $AB \geq AC$ and $BA \geq CA$.
(b) $|A + B| \leq |A| + |B|$
(c) $|AC| \leq |A||C|$

The Kronecker product $\otimes$ is used in the sequel to achieve compact notations. Some basic properties of the Kronecker product are the following

Property 2. Let $A \in \mathbb{R}^{n\times m}$, $B, C \in \mathbb{R}^{p\times q}$, $D \in \mathbb{R}^{m\times r}$, $E \in \mathbb{R}^{q\times s}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$ for arbitrary natural numbers m, n, p, q, r, s. Then

(a) If $A \geq 0$ and $B \geq C$, then $A \otimes B \geq A \otimes C$ and $B \otimes A \geq C \otimes A$.
(b) $|A \otimes B| = |A| \otimes |B|$
(c) $(A \otimes B)(D \otimes E) = (AD) \otimes (BE)$
(d) $x \otimes y = (x \otimes I_m)y = (I_n \otimes y)x$

Of the above, (b) and (c) can be found in (Lütkepohl 1996) while (a) is trivial and (d) follows from (c).

For functions, $|\cdot|$ is to be interpreted pointwise, so that $|F|(t) \triangleq |F(t)|$. Inequalities between functions is also intended pointwise, *i.e.* $F \leq G$ means $F(t) \leq G(t)$ for all $t \geq 0$.

3. MAIN RESULT

Before stating the main result, two lemmas to facilitate its proof are given. The first lemma provides some useful inequalities involving the convolution operator.

Lemma 1. Let $F \in \mathbb{L}_{pe}^{n\times m}$ and $G, H \in \mathbb{L}_{qe}^{m\times r}$, $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. Then

(a) If $F(t) \geq 0$ for all t and $H \geq G$ then $F * H \geq F * G$
(b) $|F * G| \leq |F| * |G|$

and all the convolutions above are finite for all $t \geq 0$.

Proof. The proof of (a) is straightforward using the definition of convolution in combination with Property 1 (a). The details are therefore omitted in order to save space. Part (b) is also simple to show using Property 1 (b) and Property 1 (c). □

The second lemma concerns the complementary sensitivity function $\mathbf{T} = (I - \mathbf{G})^{-1} - I$ of a system $\mathbf{G}$ with positive unity feedback. In short, it says that if the impulse response of $\mathbf{G}$ is nonnegative, then the impulse response of $\mathbf{T}$ is also nonnegative.

Lemma 2. Let $G \in \mathbb{L}_p^{n\times n}$, $1 \leq p \leq \infty$ and define the linear operator $\mathbf{G}$ by $\mathbf{G}F \triangleq G * F$. Let $\mathbf{T} \triangleq (I - \mathbf{G})^{-1} - I$ and define T as the function such that $\mathbf{T}F \triangleq T * F$. If $\|\mathbf{G}\|_p < 1$ and $G(t) \geq 0$ for all $t \geq 0$ then $\|\mathbf{T}\|_p < \infty$ and $T(t) \geq 0$ for all $t \geq 0$.

Proof. It is well known (see *e.g.* Theorem 7.3-1 in (Kreyszig 1978)) that if $\|\mathbf{G}\|_p < 1$ then $(I - \mathbf{G})^{-1}$ exists and is bounded and thus also $\mathbf{T}$ is a bounded operator. Define the operator $\mathcal{C}$ by $\mathcal{C}F \triangleq G + G * F$. Clearly, $\mathcal{C}$ is a contraction, since

$$\begin{aligned}\|\mathcal{C}A - \mathcal{C}B\|_p &= \|G + G * A - G - G * B\|_p \\ &= \|\mathbf{G}(A - B)\|_p \leq \|\mathbf{G}\|_p \|A - B\|_p\end{aligned}$$

and $\|\mathbf{G}\|_p < 1$ by assumption. Furthermore, T is a fixed point of $\mathcal{C}$ since, for an arbitrary F,

$$\begin{aligned}(T - \mathcal{C}T) * F &= (T - G - G * T) * F = \\ &= ((I - \mathbf{G})(\mathbf{T} + I) - I)F \\ &= ((I - \mathbf{G})(I - \mathbf{G})^{-1} - I)F = 0\end{aligned}$$

which shows that $T - \mathcal{C}T$ is zero and thus $T = \mathcal{C}T$. Since $\mathcal{C}$ is a contraction with T as fixed point and the space $\mathbb{L}_p^{n\times n}$ is complete, it is concluded from the Banach fixed point theorem that the sequence of functions defined by $T_0 = G$, $T_{i+1} = \mathcal{C}T_i$ converges to T. Furthermore, if $T_i(t) \geq 0$ for all $t \geq 0$ then it follows from Lemma 1 (a) that also $T_{i+1}(t) \geq 0$ for all $t \geq 0$. Since $T_0(t) \geq 0$ for all $t \geq 0$, it follows by induction that $T(t) \geq 0$ for all $t \geq 0$. □

The following theorem is the main result of this study. It provides an upper bound for the modulus of the state vector of a linear system with parametric uncertainty acting multiplicatively on the state vector. The upper bound is time-varying and depends on both the input g and the initial condition x_0 of the system.

Theorem 1. Consider the bilinear differential equation

$$\dot{x} = Ax + N(\pi \otimes x) + g \tag{2a}$$
$$x(0) = x_0 \tag{2b}$$

where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $N \in \mathbb{R}^{n\times nm}$, $\pi \in \mathbb{L}_\infty^m$ and $g \in \mathbb{L}_{pe}^n$, $1 \leq p \leq \infty$. Assume that $|\pi(t)| \leq \Pi \in \mathbb{R}^m$ for all $t \geq 0$ and let $G(t) \triangleq e^{At}$. Let $H \in \mathbb{L}_p^{n\times n}$ be a function that satisfies $H \geq |GN|(\Pi \otimes I_n)$. If $\|\mathbf{H}\|_p < 1$ then $\|(I - \mathbf{H})^{-1}\|_p < \infty$ and

$$|x| \leq (I - \mathbf{H})^{-1} |\mathbf{G}g + Gx_0| \tag{3}$$

Proof. The nominal system, *i.e.* with $\pi(t) \equiv 0$, is $\dot{\xi} = A\xi + g$ with $\xi(0) = x_0$, and has the solution

$$\xi(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} g(\tau) d\tau = (\mathbf{G}g + Gx_0)(t)$$

Similarly, the solution to (2) can be expressed implicitly by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} (N(\pi(\tau) \otimes x(\tau)) + g(\tau)) d\tau$$

or, by utilizing the convolution operator,

$$\begin{aligned} x &= Gx_0 + G * (N(\pi \otimes x) + g) \\ &= \xi + (GN) * ((\pi \otimes I_n)x) \end{aligned} \tag{4}$$

where Property 2 (d) was utilized. An upper bound for the absolute value of the state x can thus be derived as

$$\begin{aligned} |x| &= |\xi + (GN) * ((\pi \otimes I_n)x)| \\ &\leq |\xi| + |(GN) * ((\pi \otimes I_n)x)| \\ &\leq |\xi| + |GN| * |(\pi \otimes I_n)x| \\ &\leq |\xi| + |GN| * (|\pi \otimes I_n||x|) \\ &= |\xi| + |GN| * (|\pi| \otimes |I_n||x|) \\ &\leq |\xi| + |GN| * ((\Pi \otimes I_n)|x|) \\ &= |\xi| + (|GN|(\Pi \otimes I_n)) * |x| \\ &\leq |\xi| + \mathbf{H}|x| \end{aligned}$$

where the first inequality follows from Property 1 (b), the second inequality from Lemma 1 (b), the third inequality from Lemma 1 (a) and Property 1 (c), while the second equality is from Property 2 (b). The fourth inequality follows from Property 1 (a), Property 2 (a), and Lemma 1 (a), while the last inequality is a consequence of the definition of $\mathbf{H}$.

Obviously, the above implies that $(I - \mathbf{H})|x| \triangleq \zeta \leq |\xi|$. Furthermore, the definition $\mathbf{T} \triangleq (I - \mathbf{H})^{-1} - I$ yields $|x| = (I - \mathbf{H})^{-1}\zeta = \mathbf{T}\zeta + \zeta$. By Lemma 2, it is clear that $\mathbf{T}$ is bounded and that $T(t) \geq 0$ for all $t \geq 0$. Finally, from Lemma 1 (a) it follows that $|x| = T * \zeta + \zeta \leq T * |\xi| + |\xi| = ((I - \mathbf{H})^{-1} - I)|\xi| + |\xi| = (I - \mathbf{H})^{-1}|\xi|$. □

Remark 1. An interesting question is when the inequality (3) in Theorem 1 is tight, *i.e.* under what circumstances there exists a $\pi(t)$ satisfying $|\pi(t)| \leq \Pi$

such that (3) is an equality. A simple result in this matter is that if GN and the nominal solution $\mathbf{G}g + Gx_0$ are both nonnegative for all $t \geq 0$ then H may be chosen so that the inequality (3) is tight. This can be seen by choosing $H = |GN|(\Pi \otimes I_n)$ and assuming that $\pi(t) \equiv \Pi$. Then (4) is equivalent to $x = \xi + H * x$ and therefore $x = (I - \mathbf{H})^{-1}\xi$ and the result follows.

Theorem 1 will be instrumental in developing the fault detection algorithm in the sequel. A major issue that remains to be solved is, however, how to find a system $\mathbf{H}$ with impulse response $H \geq |GN|(\Pi \otimes I_n)$, *i.e.* finding a realizable upper bound for an impulse response. An exact solution, *i.e.* $H = |GN|(\Pi \otimes I_n)$ requires, in general, a system of infinite order and thus approximate solutions are of interest. The following lemma gives a solution for a special case of A.

Lemma 3. Let $T(t) = Ce^{At}B$ where A has only real eigenvalues and is diagonalizable *i.e.* $A = SDS^{-1}$ where D is a diagonal matrix with the eigenvalues of A on the diagonal. Then

$$|T(t)| \leq U(t) = |CS|e^{Dt}|S^{-1}B|$$

Proof. The modulus of the impulse response matrix is $|T(t)| = |Ce^{At}B| = |CSe^{Dt}S^{-1}B| \leq |CS||e^{Dt}||S^{-1}B| = |CS|e^{Dt}|S^{-1}B|$ where the inequality follows from Property 1 (c). □

Typically, the matrix A in the lemma above comes from the error system of a state observer and therefore the conditions on A can always be satisfied in the design of this observer. The problem of finding a realizable upper bound of a general impulse response, *e.g.* when A has multiple or complex eigenvalues, is nonetheless a topic of future research.

4. APPLICATION TO CLOGGING DETECTION IN A FLOTATION PROCESS

Froth flotation is an important and versatile mineral-processing technique in which valuable minerals are separated from the rocky material. It is important that the flotation tank levels are controlled, which is performed with valves on the outflow of each tank. To ensure the controller is able to fulfill its requirements, a fault detection algorithm is needed to detect clogging of the valves at an early stage.

4.1 *Flotation process model*

The flotation process at Boliden Area Concentrator, Sweden, consists of four cascade coupled tanks with control valves after each tank for the purpose of controlling the levels in the tanks. The input signals to the process are the valve control signals $v(t) \in \mathbb{R}^4$ and the external inflow to the tanks is denoted $q(t)$. The level in the tanks are denoted $h(t) \in \mathbb{R}^4$. All four tank levels are measured but in this example only the level in Tank 4 is utilized.

The continuous time model of the tank levels $h(t)$ can be described as a system of first order differential equations.

$$\dot{h}(t) = F(h(t), v(t), \phi(t)) + Dq(t) \tag{5}$$

where $\phi(t) \in \mathbb{R}^4$ is the fault signal, *i.e.* the clogging of each control valve, see (Bask and Johansson 2003) for more details on the model. The time argument t is dropped in the sequel to enhance readability. A Taylor expansion of the right hand side of equation (5) with respect to h, v, ϕ, q around a working point $h_0, v_0, 0, q_0$ gives the linearized model

$$\dot{x} = Ax + Dw + Bu + E\phi + Q \tag{6a}$$
$$y = Cx + \pi_n \tag{6b}$$

where π_n is additive measurement noise and x, u, and w are defined by $x \triangleq h - h_0$, $u \triangleq v - v_0$, and $w \triangleq q - q_0$ while $A = F_h'(h_0, v_0, 0)$, $B = F_v'(h_0, v_0, 0)$, $E = F_\phi'(h_0, v_0, 0)$, and $Q = F(h_0, v_0, 0) + Dq_0$. The matrix Q will be equal to zero if neglecting uncertainties but nonzero otherwise and thus affect the process and is therefore retained.

4.2 *Sensitivity analysis*

In the function F, there are two parameter vectors $K, c \in \mathbb{R}^4$. These parameters are uncertain which causes uncertainty in the working point h_0, q_0 and v_0. The measurement w and the control signal u are also assumed to be uncertain. Uncertainties in the measurement, y can be described as an uncertainty in C. In summary

$$\begin{array}{ll} q_0 = \hat{q}_0(1 + \pi_{q_0}) & h_0 = \hat{h}_0 \circ (1_4 + \pi_{h_0}) \\ v_0 = \hat{v}_0 \circ (1_4 + \pi_{v_0}) & C = \hat{C}(1 + \pi_y) \\ K = \hat{K} \circ (1_4 + \pi_K) & u = \hat{u} \circ (1_4 + \pi_u) \\ c = \hat{c} \circ (1_4 + \pi_c) & w = \hat{w}(1 + \pi_w) \end{array} \tag{7}$$

where hat signifies nominal value and $\circ$ denotes the Hadamard product, *i.e.* element-wise product between two matrices of equal dimension.

The matrices A, B, and Q in equation (6) depend nonlinearly on the uncertainties in (7),

$$\pi = \left[\pi_{h_0}^T \; \pi_{q_0} \; \pi_{v_0}^T \; \pi_w \; \pi_u^T \; \pi_K^T \; \pi_c^T \; \pi_y \; \pi_n\right]^T \in \mathbb{R}^{24}$$

and can be approximated by first order Taylor expansions as

$$\begin{aligned} A(\pi) &\approx A(0) + A_\pi'(0)(\pi \otimes I_4) \triangleq \hat{A} + \tilde{A}(\pi \otimes I_4) \\ B(\pi) &\approx B(0) + B_\pi'(0)(\pi \otimes I_4) \triangleq \hat{B} + \tilde{B}(\pi \otimes I_4) \\ Q(\pi) &\approx Q(0) + Q_\pi'(0)\pi \triangleq \hat{Q} + \tilde{Q}\pi \end{aligned} \tag{8}$$

Explicit expressions for these dependencies are, however, left out in order to save space. The uncertainties are assumed to be bounded by $|\pi| < \Pi \in \mathbb{R}^{24}$.

4.3 *Residual generation*

To generate a residual, a linear observer extended with an integrator, $\imath(t)$, is employed. The feedback will be the difference between the measured level in Tank 4, y and the estimated level $\hat{y}$ and therefore,

$$\begin{aligned}\dot{\hat{x}} &= \hat{A}\hat{x} + D\hat{w} + \hat{B}\hat{u} + \hat{Q} + E_\imath \imath + L_h(y - \hat{y}) \\ \dot{\imath} &= L_\imath(y-\hat{y}) \\ \hat{y} &= C\hat{x}\end{aligned} \tag{9}$$

where $L_h \in \mathbb{R}^4$, $L_\imath \in \mathbb{R}$ are the feedback matrices and $E_\imath$ determines how the integral action is connected to the observer.

The dynamics of the estimation error, $\tilde{x} = x - \hat{x}$ can be calculated by combining the observer (9) with the process model (6) and the uncertainty description (7) and (8). Neglecting products between uncertainties yields, after some rearrangement,

$$\begin{aligned}\dot{\tilde{x}} = &(\hat{A} - L_h\hat{C})\tilde{x} + \tilde{A}(\pi \otimes I_4)x - L_h\hat{C}\pi_y x \\ &+\tilde{B}(\pi \otimes I_4)\hat{u} + \tilde{Q}\pi + D\hat{w}\pi_w \\ &+\hat{B}(\hat{u} \circ \pi_u) - L_h\pi_n + E\phi - E_\imath \imath\end{aligned}$$

In the above, x represents the true state vector of the process and is thus not known but it can be expressed as $x = \tilde{x} + \hat{x}$ and thus

$$\begin{aligned}\dot{\tilde{x}} = &(\hat{A} - L_h\hat{C})\tilde{x} + \tilde{A}(\pi \otimes I_4)\tilde{x} - L_h\hat{C}\pi_y\tilde{x} \\ &+\tilde{A}(\pi \otimes I_4)\hat{x} + \tilde{B}(\pi \otimes I_4)\hat{u} + \tilde{Q}\pi - L_h\hat{C}\hat{x}\pi_y \\ &+D\hat{w}\pi_w + \hat{B}\mathrm{diag}(\hat{u})\pi_u - L_h\pi_n + E\phi - E_\imath\imath\end{aligned}$$

where it was also utilized that $x \circ y = \mathrm{diag}(x)y$ for two vectors x and y of the same dimension. From Property 2 (c) it follows that $\pi_y\tilde{x} = ([0_{1\times 22}\ 1\ 0]\pi) \otimes (I_4\tilde{x}) = [0_{4\times 88}\ I_4\ 0_{4\times 4}](\pi \otimes \tilde{x})$ which, in combination with the definition $H_h \triangleq [0_{4\times 9}\ D\hat{w}\ \hat{B}\mathrm{diag}(\hat{u})\ 0_{4\times 8} - L_h\hat{C}\hat{x}\ \ {-L_h}]$ yields

$$\begin{aligned}\dot{\tilde{x}} = &(\hat{A} - L_h\hat{C})\tilde{x} + \tilde{A}(\pi \otimes \tilde{x}) \\ &-L_h\hat{C}[0_{4\times 88}\ I_4\ 0_{4\times 4}](\pi \otimes \tilde{x}) + \tilde{A}(\pi \otimes I_4)\hat{x} \\ &+\tilde{B}(\pi \otimes I_4)\hat{u} + \tilde{Q}\pi + H_h\pi + E\phi - E_\imath\imath \\ = &(\hat{A} - L_h\hat{C})\tilde{x} + (\tilde{A} - L_h\hat{C}[0_{4\times 88}\ I_4\ 0_{4\times 4}])(\pi \otimes \tilde{x}) \\ &+\tilde{A}(I_{24} \otimes \hat{x})\pi + \tilde{B}(I_{24} \otimes \hat{u})\pi + \tilde{Q}\pi \\ &+H_h\pi + E\phi - E_\imath\imath\end{aligned}$$

where Property 2 (d) was used. Similarly,

$$\begin{aligned}\dot{\imath} &= L_\imath(y - \hat{y}) = L_\imath(\hat{C}\tilde{x} + \hat{C}\pi_y x + \pi_n) \\ &= L_\imath(\hat{C}\tilde{x} + \hat{C}\pi_y\tilde{x} + \hat{C}\pi_y\hat{x} + \pi_n) \\ &= L_\imath\hat{C}\tilde{x} + L_\imath\hat{C}[0_{4\times 88}\ I_4\ 0_{4\times 4}](\pi \otimes \tilde{x}) + H_\imath\pi\end{aligned}$$

where $H_\imath \triangleq [0_{4\times 22}\ L_\imath\hat{C}\hat{x}\ L_\imath]$. Defining the new state vector $z \triangleq [\tilde{x}^T\ \imath]^T$ and noting that $\tilde{x} = [I_4\ 0_{4\times 1}]z$ yields, using Property 2 (c), $\pi \otimes \tilde{x} = (I_{24}\pi) \otimes ([I_4\ 0_{4\times 1}]z) = (I_{24} \otimes [I_4\ 0_{4\times 1}])(\pi \otimes z)$. In summary, choosing the integrator $\imath$ as residual, the dynamics of the error system can be written as

$$\begin{aligned}\dot{z} &= A_z z + N(\pi \otimes z) + E_\pi\pi + E_\phi\phi \\ r &= C_z z\end{aligned}$$

where

$$\begin{aligned}A_z &= \begin{bmatrix} \hat{A} - L_h\hat{C} & -E_\imath \\ L_\imath\hat{C} & 0 \end{bmatrix} \\ N &= \begin{bmatrix} \tilde{A} - L_h\hat{C}[0_{4\times 88}\ I_4\ 0_{4\times 4}] \\ L_\imath\hat{C}[0_{4\times 88}\ I_4\ 0_{4\times 4}] \end{bmatrix}(I_{24} \otimes [I_4\ 0_{4\times 1}]) \\ E_\pi &= \begin{bmatrix} \tilde{A}(I_{24} \otimes \hat{x}) + \tilde{B}(I_{24} \otimes \hat{u}) + \tilde{Q} + H_h \\ H_\imath \end{bmatrix} \\ E_\phi &= \begin{bmatrix} E \\ 0_{1\times 4} \end{bmatrix} \\ C_z &= [0_{1\times 4} 1]\end{aligned}$$

Note that the matrix E_π is a function of time but depends only on measured signals and known parameters.

4.4 *Dynamic threshold generator*

The evaluation signal is chosen as the absolute value of the residual, *i.e.* $s(t) \triangleq |\imath(t)| = C_z|z(t)|$. The threshold function $\sigma(t)$ should thus satisfy

$$\sigma(t) \geq \sup_{|\pi|<\Pi, \phi\equiv 0} |\imath|(t)$$

An upper bound of $|z|$, by using Theorem 1, is

$$\begin{aligned}|z| &\leq (I - \mathbf{H})^{-1}|\mathbf{G}(E_\pi\pi) + Gz(0)| \\ &\leq (I - \mathbf{H})^{-1}|G * (E_\pi\pi)| + (I - \mathbf{H})^{-1}|Gz(0)| \\ &\leq (I - \mathbf{H})^{-1}|G| * |E_\pi\pi| + (I - \mathbf{H})^{-1}|Gz(0)| \\ &\leq (I - \mathbf{H})^{-1}(|G| * |E_\pi|)\Pi + (I - \mathbf{H})^{-1}|Gz(0)|\end{aligned}$$

where the second inequality follows from Property 1 (b), the third inequality from Lemma 1 (b) while the last inequality is from Property 1 (c) and Property 1 (a). Lemma 1 (a) was also utilized in all inequalities. It is assumed that the observer has converged before the fault detection algorithm is employed and therefore $z(0) = 0$. An upper bound of the evaluation signal $s = |\imath|$ is thus obtained as

$$s \leq C_z(I - \mathbf{H})^{-1}\mathbf{\Gamma}|E_\pi|\Pi \triangleq \sigma$$

In the above threshold generator, upper bounds $H(t) \geq |G(t)N|(\Pi \otimes I_4)$ and $\Gamma(t) \geq |G(t)|$ may be determined using Lemma 3.

4.5 *Experimental results*

Experiments have been carried out on data from Boliden's flotation series at the Boliden Area Concentrator, Sweden. The bounds Π of the uncertainties have been tuned manually so that the threshold should be as close as possible to the residual but still larger at all times. Since Π has 24 elements, an automatic way to determine these bounds should be developed in the future, which is likely to improve the thresholds in Fig. 1 even further.

Fig. 1 (b) shows the evaluation signal and the corresponding threshold for a data set without clogging. Note that the residual is smaller than the threshold and no alarm is raised. Also, the threshold imitates the bumps in the evaluation signal caused by oscillations in the control signal to Valve 4 (Fig. 1 (a), solid line).

The results of an experiment with a simulated clogging is shown in Fig. 1 (c). A clogging means that the actual valve opening is less than expected and can thus be simulated by adding a positive quantity to the logged control signal which gives total control of the fault. In this example, a ramp signal starting at $t = 2000$ and ending at $t = 3000$ with final value 0.05 is added to the measured control signal of Valve 4 to simulate clogging (Fig. 1 (a), dashed line). Note that the clogging is detected at $t = 2881$ as the evaluation signal rises above the threshold.

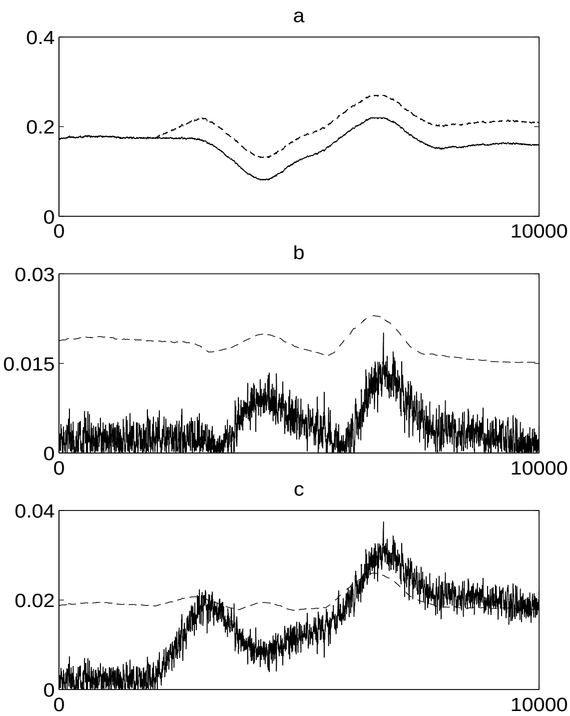

Fig. 1. (a) Control signal, Valve 4. Without clogging (solid line), with simulated clogging (dashed line). The evaluation signal $s(t)$ (solid line) and detection threshold $\sigma(t)$ (dashed line) is shown in (b) (No clogging) and (c) (Simulated clogging).

5. CONCLUSIONS

An inequality for the solution of a linear system with uncertain parameters was developed. This inequality is expected to be a valuable tool for providing the time-varying fault detection thresholds that are desired to achieve robustness against parametric uncertainty in combination with sensitivity to small faults.

The usefulness of the inequality was illustrated by developing an algorithm for detecting clogging in the valves of a flotation process. The proposed method consists of a Luenberger observer with integral action. A robust detection threshold was calculated under the assumption of parametric uncertainty in the process model. Successful simulations with measurement data show that the obtained detection thresholds exhibit the desired behaviour, *i.e.* imitate the behavior of the evaluation signal when no fault is present.

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