IDENTIFICATION OF IIR NONLINEAR SYSTEMS WITHOUT STRUCTURAL INFORMATION

E.W. Bai * **R. Tempo** ** **Y. Liu** *

* *Dept of ECE, University of Iowa, Iowa City, Iowa, USA*
** *IEIIT-CNR, Politecnico di Torino, Torino, Italy*

Abstract: In this paper, we propose an algorithm for identification of dynamic IIR nonlinear systems without prior structural information. The algorithm is based upon the well-known kernel method, which is generally used for probability density function estimation. Asymptotic convergence properties (in probability) are rigorously established for identification of IIR nonlinear systems. The performance of the algorithm is tested on three real world applications and one simulated example, thus showing the efficiency of the method.

Keywords: identification, nonlinear systems, estimation

1. INTRODUCTION

Identification of nonlinear systems is an important problem which received much attention in recent years. Unfortunately, despite of various progresses made regarding specific methods, see e.g. (Haber and Unbehauen, 1990; Juditsky *et al.*, 1995; Sjoberg *et al.*, 2000), the problem remains mostly intractable if the structure is unknown and/or the input is non-Gaussian.

In this paper, we study identification of a time-invariant nonlinear IIR system of the form

$$y(k)=f(y(k-1),\ldots,y(k-n),u(k-1),\ldots,u(k-n))+v(k), \quad (1)$$

where $u(\cdot)$ is an iid random input over the interval $[\underline{u},\bar{u}]$ with an unknown probability distribution, $v(\cdot)$ is a bounded iid random noise of zero mean and unknown variance σ_v^2 and $y(\cdot)$ is the output. Obviously, the output variable $y(k)$ is a random variable. The order n of the system is assumed to be known. However, no a priori information on the structure of f is available or used. We also assume that the system is exponential input-to-output stable, see e.g. (Sontag, 1963) and references therein.

In this paper, we propose to use the kernel approach, see e.g. (Parzen, 1962; Nadaraya, 1989) for identification of nonlinear IIR systems. The kernel approach falls in the class of nonparametric estimation and is frequently used for estimating density functions, or other static nonlinear functions. The main technical contribution of the paper is therefore to extend existing asymptotic convergence results (in probability) of the kernel method to identification of dynamic nonlinear IIR systems. Because of the page limit, the proofs are condensed. For complete derivations, the interested readers may refer to the full version of the paper (Bai *et al.*, 2004) .

2. PRELIMINARIES

For given initial time k_0, initial conditions $\{y(k_0),\ldots,y(k_0-n+1)\}$, input and noise sequences $\{u(\cdot)\}_{k_0-n}^{k-1}$ and $\{v(\cdot)\}_{k_0}^{k}$, let the solution of (1) at time k be denoted by

$$y(k)=\xi(k,\{y(k_0),\ldots,y(k_0-n+1)\},\{u(i)\}_{k_0-n}^{k-1},\{v(i)\}_{k_0}^{k}).$$

Assumption 2.1. (1) The nonlinear system (1) is assumed to be exponentially input-to-output stable (Sontag, 1963), i.e.,

- For any $k > k_0$ and any initial conditions $\{y(k_0), \dots, y(k_0 - n + 1)\}$,

$$| \xi(k, \{y(k_0), \dots, y(k_0 - n + 1)\}, \{u(i)\}_{k_0-n+1}^{k-1},$$
$$\{v(i)\}_{k_0+1}^{k}) | \le M_1(y(k_0),\dots,y(k_0 - n + 1))\lambda^{k-k_0}$$
$$+\gamma(\max_{k_0 \le i \le k} \{|u(i)|, |v(i)|\}),$$

for some $0 \le \lambda < 1$ and some bounded positive functions M_1 and γ.

- Consider two solutions started at the initial time k_0 with different initial conditions but the same input and noise sequences. Then,

$$|\xi(k,\{y(k_0),\dots,y(k_0 - n + 1)\},\{u(i)\}_{k_0-n+1}^{k-1},$$
$$\{v(i)\}_{k_0+1}^{k}) - \xi(k, \{\hat{y}(k_0), \dots, \hat{y}(k_0 - n + 1)\},$$
$$\{u(i)\}_{k_0-n+1}^{k-1}, \{v(i)\}_{k_0+1}^{k}) |$$
$$\le M_2(y(k_0), ..., y(k_0 - n + 1), \hat{y}(k_0), ...,$$
$$\hat{y}(k_0 - n + 1))\lambda^{k-k_0},$$

for some bounded positive function M_2 and $0 \le \lambda < 1$. In other words, the contribution of the initial condition is forgotten exponentially if the input and the noise for the two solutions are the same after the initial time k_0.

(2) The function $f(y_1, \dots, y_n, u_1, \dots, u_n)$ is locally Lipschitz.

(3) For all k, the probability density function of the random output $y(k)$ and the joint probability density function of $y(k - 1), \dots, y(k - n)$ exists and are continuous. Moreover, the joint probability density function of the random variables $y(k - 1), \dots, y(k-n), u(k-1), \dots, u(k-n)$, denoted by $q_k(y_1, \dots, y_n, u_1, \dots, u_n)$ is locally Lipschitz in $y_1, \dots, y_n$, i.e., for sufficiently small $\Delta y_1, \dots, \Delta y_n$, we have

$$| q_k(y_1 + \Delta y_1, \dots, y_n + \Delta y_n, u_1, \dots, u_n)$$
$$-q_k(y_1, \dots, y_n, u_1, \dots, u_n) | \le M_3 \max_{1 \le i \le n} |\Delta y_i|,$$

for some bounded positive function M_3.

Lemma 2.1. Consider the system (1) with initial time $k_0 = 0$ under Assumption 2.1. Then, for any x, the probability density functions $p_{k,\{y(0),\dots,y(-n+1)\}}(\cdot)$ of $y(k)$ and $p_{l,\{\hat{y}(0),\dots,\hat{y}(-n+1)\}}(\cdot)$ of $y(l)$ satisfy

$$|p_{k,\{y(0),\dots,y(-n+1)\}}(x) - p_{l,\{\hat{y}(0),\dots,\hat{y}(-n+1)\}}(x)|$$
$$\to 0, \quad as \ \min\{k, l\} \to \infty,$$

for any $y(0), \dots, y(-n + 1), \hat{y}(0), \dots, \hat{y}(-n + 1)$ provided that

$$\max\{|y(0)|, .., |y(-n+1)|, |\hat{y}(0)|, \dots, |\hat{y}(-n+1)|\} < M < \infty.$$

Proof: W.l.o.g., we may assume that $l-k = j$ for some $j \ge 0$. We now make three observations.

(1) Consider the system (1). For $j - m \ge 1$, let $\hat{y}(j - m)$ be the solution with the initial conditions $\{\hat{y}(0), \dots, \hat{y}(-n + 1)\}$,

$$\hat{y}(j-m) = \xi(j-m, \{\hat{y}(0), \dots, \hat{y}(-n+1)\}, \{u(i)\}_{-n+1}^{j-m-1},$$
$$\{v(i)\}_1^{j-m}).$$

We may now write $y(l)$ in terms of the initial time j and initial conditions $\{\hat{y}(j), \dots, \hat{y}(j - n + 1)\}$,

$$y(l) = \xi(l, \{\hat{y}(0), \dots, \hat{y}(-n + 1)\}, \{u(i)\}_{-n+1}^{l-1}, \{v(i)\}_1^{l})$$
$$= \xi(k+j, \{\hat{y}(j), .., \hat{y}(j-n+1)\}, \{u(i)\}_{j-n+1}^{k+j-1}, \{v(i)\}_{j+1}^{k+j}).$$

The system (1) is time invariant. By shifting the time by j units, it follows that

$$y(l) = \xi(k, \{\bar{y}(0), \dots, \bar{y}(-n+1)\}, \{\bar{u}(i)\}_{-n+1}^{k-1}, \{\bar{v}(i)\}_1^{k}),$$

where $\bar{y}(0) = \hat{y}(j), \dots, \bar{y}(-n + 1) = \hat{y}(j - n + 1), \bar{u}_i = u_{i+j}, \bar{v}_i = v_{i+j}$.

(2) From the input to output exponential stability and the above observation, we have

$$|y(l) - \xi(k, \{0, \dots, 0\}, \{\bar{u}(i)\}_{-n+1}^{k-1}, \{\bar{v}(i)\}_1^{k})|$$
$$= |\xi(k, \{\bar{y}(0), \dots, \bar{y}(-n + 1)\}, \{\bar{u}(i)\}_{-n+1}^{k-1}, \{\bar{v}(i)\}_1^{k})$$
$$-\xi(k, \{0, \dots, 0\}, \{\bar{u}(i)\}_{-n+1}^{k-1}, \{\bar{v}(i)\}_1^{k})|$$
$$= |\delta_1(k, \bar{y}(0), \dots, \bar{y}(-n + 1))| \le M_1\lambda^k \to 0, \ as \ k \to \infty,$$

and

$$|y(k) - \xi(k, \{0, \dots, 0\}, \{u(i)\}_{-n+1}^{k-1}, \{v(i)\}_1^{k})|$$
$$= |\xi(k, \{y(0), \dots, y(-n + 1)\}, \{u(i)\}_{-n+1}^{k-1}, \{v(i)\}_1^{k})-$$
$$\xi(k, \{0, \dots, 0\}, \{u(i)\}_{-n+1}^{k-1}, \{v(i)\}_1^{k})|$$
$$= |\delta_2(k, y(0), .., y(-n + 1))| \le M_1\lambda^k \to 0, \ as \ k \to \infty.$$

(3) $\{u_i\}$ and $\{\bar{u}_i\} = \{u_{i+j}\}$ are two iid random sequences with the identical distributions. Further, $\{v_i\}$ and $\{\bar{v}_i\} = \{v_{i+j}\}$ are two iid noise sequences with the identical distributions. Thus, the solutions

$$\xi(k, \{0, \dots, 0\}, \{u(i)\}_{-n+1}^{k-1}, \{v(i)\}_1^{k})$$

and

$$\xi(k, \{0, \dots, 0\}, \{\bar{u}(i)\}_{-n+1}^{k-1}, \{\bar{v}(i)\}_1^{k})$$

must have the identical probability density function, say $p_{k,\{0,\dots,0\}}(\cdot)$.

From the above three observations, we have

$$p_{l,\{\hat{y}(0),\dots,\hat{y}(-n+1)\}}(x) = \frac{d}{dx} \ Prob\{y(l) \le x\}$$
$$= \frac{d}{dx} \ Prob\{\xi(k, \{0, \dots, 0\}, \{\bar{u}(i)\}_{-n+1}^{k-1}, \{\bar{v}(i)\}_1^{k}) \le x + \delta_1\}$$
$$= \frac{d}{dx} \int_{-\infty}^{x+\delta_1} p_{k,\{0,\dots,0\}}(s)ds = p_{k,\{0,\dots,0\}}(x + \delta_1).$$

Similarly,

$$p_{k,\{y(0),\dots,y(-n+1)\}}(x) = \frac{d}{dx} \ Prob\{y(k) \le x\}$$
$$= \frac{d}{dx} \ Prob\{\xi(k, \{0, \dots, 0\}, \{u(i)\}_{-n+1}^{k-1}, \{v(i)\}_1^{k}) \le x + \delta_2\}$$
$$= \frac{d}{dx} \int_{-\infty}^{x+\delta_2} p_{k,\{0,\dots,0\}}(s)ds = p_{k,\{0,\dots,0\}}(x + \delta_2).$$

Since the density function $p_{k,\{0,\dots,0\}}(\cdot)$ is assumed to be continuous and δ_1 and $\delta_2 \to 0$ as $k \to \infty$, the conclusion follows.

The result implies that the probability density function of $y(\cdot)$ does not depend on time k if k is large enough. In other words, in the steady state or the initial time $k_0 = -\infty$, the contribution due to the initial condition vanishes. To avoid unnecessary complications, in the rest of the paper, we assume that the steady state has been reached or the initial time starts at $-\infty$ so that the probability density functions do not dependent on k.

3. NONLINEAR SYSTEMS IDENTIFICATION

The goal of this paper is to estimate the nonlinear function $f(y_1,\ldots,y_n,u_1,\ldots,u_n)$ for bounded $y_i \in [\underline{y},\bar{y}]$ and $u_i \in [\underline{u},\bar{u}]$ based on the input-output measurements $y(k),u(k) \in [\underline{y},\bar{y}] \times [\underline{u},\bar{u}]$.

Lemma 3.1. Let $q(y_1,\ldots,y_n,u_1,\ldots,u_n)$ be the joint probability density function of $y(k-1),\ldots,y(k-n),u(k-1),\ldots,u(k-n)$ and

$$q(y_1,\ldots,y_n,u_1,\ldots,u_n \mid y(i-1),\ldots,y(i-n), u(i-1),\ldots,u(i-n))$$

be the conditional probability density function of $y(k-1),\ldots,y(k-n),u(k-1),\ldots,u(k-n)$ given $y(i-1),\ldots,y(i-n),u(i-1),\ldots,u(i-n)$. Then, there exists $M_4,M_5 < \infty$ and $0 \le \lambda < 1$ so that for $k-i \ge M_5$,

$$|q(y_1,\ldots,y_n,u_1,\ldots,u_n \mid y(i-1),\ldots,y(i-n), u(i-1),\ldots,u(i-n)) - q(y_1,\ldots,y_n,u_1,\ldots,u_n)| \le M_4\lambda^{k-i}.$$

Proof: Let $y(k)$ be the solution of (1) at time k for given $y(i-1),\ldots,y(i-n),u(i-1),\ldots,u(i-n)$. When $k-i>n$, $y(k)$ may be written as

$$\begin{aligned} y(k) &= \xi(k,\{y(i-1),\ldots,y(i-n)\},\{u(l)\}_{i-n}^{k-1},\{v(l)\}_i^k) \\ &= \xi(k,\{y(i+n-1),\ldots,y(i)\},\{u(l)\}_i^{k-1},\{v(l)\}_{i+n}^k). \end{aligned}$$

Let $\bar{y}(k)$ be a solution for arbitrary initial conditions $\{\bar{y}(i-1),\ldots,\bar{y}(i-n)\}$,

$$\begin{aligned} \bar{y}(k) &= \xi(k,\{\bar{y}(i-1),\ldots,\bar{y}(i-n)\},\{u(l)\}_{i-n}^{k-1},\{v(l)\}_i^k) \\ &= \xi(k,\{\bar{y}(i+n-1),\ldots,\bar{y}(i)\},\{u(l)\}_i^{k-1},\{v(l)\}_{i+n}^k), \end{aligned}$$

for some $\bar{y}(i+n-1),\ldots,\bar{y}(i)$. By the exponential input-to-output stability,

$$|y(k)-\bar{y}(k)| \le M_2\lambda^{k-(i+n)} = \frac{M_2}{\lambda^n}\lambda^{k-i} = M_6\lambda^{k-i}.$$

In other words,

$$\bar{y}(k) = y(k)+\Delta y(k), \quad |\Delta y(k)| \le M_6\lambda^{k-i}.$$

Hence, these exists $M_7>0$ and for $k-i \ge M_7$, we have

$$\begin{aligned} & q(y_1,\ldots,y_n,u_1,\ldots,u_n \mid y(i-1),\ldots,u(i-n)) \\ &= \frac{d^{2n}}{dy_1..dy_n du_1..du_n}\int_{\underline{y}}^{y_1}\cdots\int_{\underline{y}}^{y_n}\int_{\underline{u}}^{u_1}\cdots\int_{\underline{u}}^{u_n} q(s_1,..,s_n, \\ & w_1,\ldots,w_n \mid y(i-1),\ldots,y(i-n),u(i-1),\ldots,u(i-n))ds_1\cdots \\ & ds_n dw_1\cdots dw_n = \frac{d^{2n}}{dy_1\cdots dy_n du_1\cdots du_n}\mathrm{Prob}\{y(k-1)\le y_1, \\ & \ldots,y(k-n)\le y_n, u(k-1)\le u_1,\ldots,u(k-n)\le u_n \\ & \mid y(i-1),\ldots,y(i-n),u(i-1),\ldots,u(i-n)\} \\ &= \frac{d^{2n}}{dy_1\cdots dy_n du_1\cdots du_n}\mathrm{Prob}\{\bar{y}(k-1)\le y_1+\Delta y(k-1),.., \\ & \bar{y}(k-n)\le y_n+\Delta y(k-n), u(k-1)\le u_1,\ldots,u(k-n)\le u_n\} \\ &= q(y_1+\Delta y(k-1),\ldots,y_n+\Delta y(k-n),u_1,\ldots,u_n). \end{aligned}$$

From the assumption that the joint density function is locally Lipschitz, it follows that there exist $M_4,M_5<\infty$ and $0\le\lambda<1$ such that for $k-i \ge M_5$,

$$\begin{aligned} & |q(y(k-1),\ldots,y(k-n),u(k-1),\ldots,u(k-n) \mid y(i-1),\ldots, \\ & u(i-n)) - q(y(k-1),\ldots,y(k-n),u(k-1),\ldots,u(k-n))| \\ &= |q(y(k-1)+\Delta y(k-1),\ldots,y(k-n)+\Delta y(k-n),u(k-1),\ldots, \\ & u(k!-n)) - q(y(k-1),..,y(k-n),u(k-1),..,u(k-n))| \le M_4\lambda^{k-i}. \end{aligned}$$

This completes the proof.

Let the kernel function $K(y_1,\ldots,y_n,u_1,\ldots,u_n)$ be a bounded and continuous function satisfying

$$K(y_1,\ldots,y_n,u_1,\ldots,u_n) = \begin{cases} >0, & y_i \in (\underline{y},\bar{y}) \text{ and } u_i \in (\underline{u},\bar{u}), i=1,\ldots,n, \\ 0, & y_i \notin [\underline{y},\bar{y}] \text{ or } u_i \notin [\underline{u},\bar{u}] \text{ for some } i, \end{cases}$$

and

$$\int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\cdot\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}} K(y_1,\ldots,y_n,u_1,\ldots,u_n) dy_1\cdots dy_n du_1\cdots du_n = 1.$$

Given $\{y(k),u(k)\}_1^N$, we now define the estimate $\hat{f}_N(y_1,..,y_n,u_1,..,u_n)$ of $f(y_1,..,y_n,u_1,..,u_n)$ as

$$\hat{f}_N(y_1,\ldots,y_n,u_1,\ldots,u_n) = \frac{\sum_{j=1}^N K(\frac{y_1-y(j-1)}{r},..,\frac{y_n-y(j-n)}{r},\frac{u_1-u(j-1)}{r},..,\frac{u_n-u(j-n)}{r})y(j)}{\sum_{j=1}^N K(\frac{y_1-y(j-1)}{r},..,\frac{y_n-y(j-n)}{r},\frac{u_1-u(j-1)}{r},..,\frac{u_n-u(j-n)}{r})} \quad (2)$$

for some sufficiently small $r>0$. The kernel estimate (2) can be computed recursively by setting

$$\begin{aligned} & \hat{f}_0(y_1,\ldots,y_n,u_1,\ldots,u_n) = g_0(y_1,\ldots,y_n,u_1,\ldots,u_n) = 0, \\ & g_i(y_1,\ldots,y_n,u_1,\ldots,u_n) = g_{i-1}(y_1,\ldots,y_n,u_1,\ldots,u_n) + \\ & K\left(\frac{y_1-y(i-!1)}{r},..,\frac{y_n-y(i-n)}{r},\frac{u_1-u(i-1)}{r},..,\frac{u_n-u(i-!n)}{r}\right), \\ & \hat{f}_i(y_1,\ldots,y_n,u_1,\ldots,u_n) = \hat{f}_{i-1}(y_1,\ldots,y_n,u_1,\ldots,u_n) + \\ & g_i{}^{-1}(y_1,\ldots,y_n,u_1,\ldots,u_n)\cdot[y(i)-\hat{f}_{i-1}(y_1,\ldots,y_n,u_1,\ldots,u_n)] \\ & \cdot K\left(\frac{y_1-y(i-1)}{r},..,\frac{y_n-y(i-n)}{r},\frac{u_1-u(i-1)}{r},..,\frac{u_n-u(i-n)}{r}\right). \end{aligned}$$

We now show the convergence of the estimate (2).

Theorem 3.1. Consider the system (1). Then, for every $(y_1,\ldots,y_n,u_1,\ldots,u_n) \in (\underline{y},\bar{y})^n \times (\underline{u},\bar{u})^n$ so that the probability density function $q(y_1,\ldots,y_n,u_1,\ldots,u_n) \ne 0$, we have

$$\hat{f}_N(y_1,..,y_n,u_1,..,u_n) \to f(y_1,..,y_n,u_1,..,u_n)$$

in probability as $N\to\infty$, provided that $r\to 0$, $r^{2n}N\to\infty$ as $N\to\infty$.

Proof: Since $q(y_1,\ldots,y_n,u_1,\ldots,u_n)\ne 0$, it suffices to show that in probability

$$\begin{aligned} nu_N &= \frac{1}{r^{2n}N}\sum_{j=1}^N K(\frac{y_1-y(j-1)}{r},..,\frac{u_n-u(j-n)}{r})y(j) \\ & \to q(y_1,\ldots,y_n,u_1,\ldots,u_n)f(y_1,\ldots,y_n,u_1,\ldots,u_n), \quad (3) \\ de_N &= \frac{1}{r^{2n}N}\sum_{j=1}^N K(\frac{y_1-y(j-1)}{r},\ldots,\frac{u_n-u(j-n)}{r}) \\ & \to q(y_1,\ldots,y_n,u_1,\ldots,u_n). \quad (4) \end{aligned}$$

To simplify notation, we define

$$K(l) = K(\frac{y_1-y(l-1)}{r},\ldots,\frac{u_n-u(l-n)}{r}),$$

$$f(l) = f(y(l-1),\dots,y(l-n),u(l-1),\dots,u(l-n)),$$
$$q(l) = q(y(l-1),\dots,y(l-n),u(l-1),\dots,u(l-n)),$$
$$q(i \mid j) = q(y(i-1),\dots,y(i-n),u(i-1),\dots,u(i-n) \mid y(j-1),\dots,y(j-n),u(j-1),\dots,u(j-n)),$$
$$d\boldsymbol{y}_l = dy(l-1)dy(l-2)\cdots dy(l-n),$$
$$d\boldsymbol{u}_l = du(l-1)du(l-2)\cdots du(l-n),$$

where $l = i, j$. Now, to show (3), we write

$$\mathbf{E}|nu_N - f(y_1,..,y_n,u_1,..,u_n)q(y_1,..,y_n,u_1,\dots,u_n)|^2 = [\mathbf{E}(nu_N)^2 - (\mathbf{E}nu_N)^2] + [\mathbf{E}nu_N - f(y_1,..,y_n,u_1,\dots,u_n)q(y_1,..,y_n,u_1,\dots,u_n)]^2 \quad (5)$$

,where $\mathbf{E}$ stands for the expectation operator. First, we have

$$\mathbf{E}(nu_N)^2 = \mathbf{E}[\frac{1}{r^{2n}N}\sum_{i=1}^{N} K(i)(f(i)+v(i))]^2$$
$$= \mathbf{E}\frac{1}{r^{2n}N}\sum_{i=1}^{N} K(i)f(i)\cdot\frac{1}{r^{2n}N}\sum_{j=1}^{N} K(j)f(j) +$$
$$\mathbf{E}\frac{1}{r^{2n}N}\sum_{i=1}^{N} K(i)v(i)\cdot\frac{1}{r^{2n}N}\sum_{j=1}^{N} K(j)v(j) +$$
$$2\mathbf{E}\frac{1}{r^{2n}N}\sum_{i=1}^{N} K(i)f(i)\cdot\frac{1}{r^{2n}N}\sum_{j=1}^{N} K(j)v(j). \quad (6)$$

Clearly, the last term of (6) is zero. Also, since $\mathbf{E}v(i)v(j) = 0$ for $i \neq j$, we have for all $r > 0$

$$\underline{y} \le \frac{y_j - y(i-j)}{r} \le \bar{y} \iff y_j - r\bar{y} \le y(i-j) \le y_j - r\underline{y},$$
$$\underline{u} \le \frac{u_j - u(i-j)}{r} \le \bar{u} \iff u_j - r\bar{u} \le u(i-j) \le u_j - r\underline{u},$$

where $j = 1,\dots,n$ and

$$\mathbf{E}\frac{1}{r^{2n}}K^2(i) = \frac{1}{r^{2n}}\int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}} K^2(\frac{y_1-y(i-1)}{r}, .., \frac{y_n-y(i-n)}{r}, \frac{u_1-u(i-1)}{r}, .., \frac{u_n-u(j-n)}{r})d\boldsymbol{y}_i d\boldsymbol{u}_i.$$

Next, define two new variables s_j and w_j

$$y_j - y(i-j) = rs_j,\ u_j - u(i-j) = rw_j,\ j = 1,2,\dots,n.$$

It follows that

$$\mathbf{E}\frac{1}{r^{2n}}K^2(i) = \int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\cdot\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}} K^2(s_1,\dots,s_n, w_1,\dots,w_n)ds_1\cdots ds_n dw_1\cdots dw_n$$

which is bounded. Thus, the middle term of (6) is bounded by

$$\left|\frac{\sigma_v^2}{r^{2n}N^2}\sum_{j=1}^{N}\mathbf{E}\frac{1}{r^{2n}}K^2(j)\right| = O(\frac{1}{r^{2n}N}).$$

In turn, this implies that

$$\mathbf{E}(nu_N)^2 - (\mathbf{E}nu_N)^2 = O(\frac{1}{r^{2n}N}) + \frac{1}{r^{4n}N^2} \cdot\sum_{i=1}^{N}\sum_{j=1}^{N}\{\mathbf{E}K(i)K(j)f(i)f(j) - \mathbf{E}K(i)f(i)\mathbf{E}K(j)f(j)\}$$
$$= O(\frac{1}{r^{2n}N}) + \frac{1}{r^{4n}N^2}\cdot$$
$$\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}} K(i)f(i)K(j)f(j)$$
$$\cdot[q(y(i-1),..,y(i-n),u(i-1),..,u(i-n),y(j-1),..,y(j-n), u(j-1),...,u(j-n)) - q(y(i-1),...,y(i-n),u(i-1),...,u(i-n)) \cdot q(y(j-1),\dots,y(j-n),u(j-1),\dots,u(j-n))]d\boldsymbol{y}_i d\boldsymbol{u}_i d\boldsymbol{y}_j d\boldsymbol{u}_j.$$

Then, we write

$$q(y(i-1),\dots,y(i-n),u(i-1),\dots,u(i-n), y(j-1),\dots,y(j-n),u(j-1),\dots,u(j-n)) = q(y(i-1),\dots,y(i-n),u(i-1),\dots,u(i-n) \mid y(j-1),\dots,y(j-n),u(j-1),\dots,u(j-n)) \cdot q(y(j-1),...,y(j-n),u(j-1),...,u(j-n)).$$

From the definitions of $q(i)$ and $q(i \mid j)$, it follows that

$$\mathbf{E}(nu_N)^2 - (\mathbf{E}nu_N)^2 = O(\frac{1}{r^{2n}N}) + \frac{1}{N^2}$$
$$\cdot\sum_{i=1}^{N}\sum_{j=1}^{N}\{\int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}}\frac{1}{r^{2n}}K(i)f(i)\frac{1}{r^{2n}}K(j)f(j) \cdot[q(i \mid j) - q(i)]q(j)\}d\boldsymbol{y}_i d\boldsymbol{u}_i d\boldsymbol{y}_j d\boldsymbol{u}_j.$$

We make two observations here.

- The integral within the parenthesis in the above equation is bounded by some constant C for all i and j.
- Because of exponential input-to-output stability, there exist some M_4, M_5 and $0 \le \lambda < 1$, and for $|i-j| \ge M_5$
$$|q(i \mid j) - q(i)| \le M_4\lambda^{|i-j|}.$$

It follows that

$$|\mathbf{E}(nu_N)^2 - (\mathbf{E}nu_N)^2| \le O(\frac{1}{r^{2n}N}) + \frac{1}{N^2}\sum_{i,j=1,\dots,N,|i-j|<M_5} C + \frac{1}{N^2}\sum_{i,j=1,\dots,N,|i-j|\ge M_5} M_4\lambda^{|i-j|}$$
$$= O(\frac{1}{r^{2n}N}) + \frac{C}{N^2}(N^2 - (N-M_5)^2) + \frac{M_4}{N^2}\sum_{i=1}^{N} 2(\lambda^{M_5} + \lambda^{M_5} + \cdots + \lambda^{N-i})$$
$$\le O(\frac{1}{r^{2n}N}) + O(\frac{1}{N}) + \frac{M_4}{N}\frac{2\lambda^{M_5}}{1-\lambda} = O(\frac{1}{r^{2n}N}) + O(\frac{1}{N}).$$

This bounds the first term in (5). Similar to the derivation of (6), we now consider the second term in (5).

$$\mathbf{E}nu_N = \mathbf{E}\frac{1}{r^{2n}N}\sum_{i=1}^{N} K(\frac{y_1-y(i-1)}{r},\dots,\frac{y_n-y(i-n)}{r}, \frac{u_1-u(i-1)}{r},\dots,\frac{u_n-u(i-n)}{r}) \cdot(f(y(i-1),\dots,y(i-n),u(i-1),\dots,u(i-n)) + v(i))$$
$$= \frac{1}{r^{2n}N}\sum_{i=1}^{N}\int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\cdot\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}} K(i)f(i)q(i)d\boldsymbol{y}_i d\boldsymbol{u}_i$$
$$= \int_{\underline{y}}^{\bar{y}}\cdots\int_{\underline{y}}^{\bar{y}}\int_{\underline{u}}^{\bar{u}}\cdots\int_{\underline{u}}^{\bar{u}} K(s_1,\dots,s_n,w_1,\dots,w_n) \cdot f(y_1-rs_1,\dots,y_n-rs_n,u_1-rw_1,\dots,u_n-rw_n) q(y_1-rs_1,..,y_n-rs_n,u_1-rw_1,\dots,u_n-rw_n)ds_1\cdots dw_n$$
$$= f(y_1,\dots,y_n,u_1,\dots,u_n)q(y_1,\dots,y_n,u_1,\dots,u_n)$$

$$\cdot \int_{\underline{y}}^{\bar{y}} \cdots \int_{\underline{y}}^{\bar{y}} \int_{\underline{u}}^{\bar{u}} \cdots \int_{\underline{u}}^{\bar{u}} K(s_1,\ldots,s_n,w_1,\ldots,w_n) ds_1 \cdots dw_n$$
$$+ \int_{\underline{y}}^{\bar{y}} \cdots \int_{\underline{y}}^{\bar{y}} \cdot \int_{\underline{u}}^{\bar{u}} \cdots \int_{\underline{u}}^{\bar{u}} K(s_1,\ldots,s_n,w_1,\ldots,w_n)$$
$$O(r) ds_1 \cdots dw_n$$
$$= f(y_1,\ldots,y_n,u_1,\ldots,u_n) q(y_1,\ldots,y_n,u_1,\ldots,u_n) + O(r),$$

for sufficiently small r. This implies that the second term in (5) is given by

$$[\mathbf{E} nu_N - f(y_1,\ldots,y_n,u_1,\ldots,u_n) q(y_1,\ldots,y_n,u_1,\ldots,u_n)]^2$$
$$= O(r).$$

Combining the first and the second term in (5), we have

$$\mathbf{E}|(nu_N) - f(y_1,\ldots,y_n,u_1,..,u_n) q(y_1,..,y_n,u_1,..,u_n)|^2$$
$$\le O(\frac{1}{r^{2n} N}) + O(\frac{1}{N}) + O(r),$$

and this implies

$$nu_N \to q(y_1,\ldots,y_n,u_1,\ldots,u_n) f(y_1,\ldots,y_n,u_1,\ldots,u_n).$$

Convergence of (4) can be similarly established. This finishes the proof.

4. APPLICATION EXAMPLES

To show the efficacy of the proposed algorithm without a priori structural information, we present identification results for 4 examples in this section.

The first two data sets are obtained from Identification Database DaISy

www.east.kuleuven.ac.be/ tokka/daisydata.html

and the third one is from the University of Iowa Hospital and Clinic. Advantages using the real system data are obvious. A disadvantage is that no information on the actual nonlinear function f is available and thus, it is not easy to compare the actual f with the estimated $\hat{f}$. To be able to compare f and $\hat{f}$ directly, we also include a computer simulation example, where f is known exactly, though no information was used in simulation.

Example 1: This is a computer generated example. Let the unknown nonlinear system be

$$\begin{aligned} y(k) &= f(y(k-1), u(k-1)) + v(k) \\ &= 0.2y(k-1) - 0.5y(k-1)^2 u(k-1) + u(k-1)^3 + v(k) \end{aligned}$$

where the inputs $u(k)$'s are iid. uniformly in $[-1,1]$ and the noise is iid uniformly in $[-0.05, 0.05]$. For simulation purposes, we take $N = 20000$ and $r = 0.05$. Figure 1 shows $f(y,u)$ and its estimate $\hat{f}(y,u)$ which is very close to the true but unknown f, as expected. To further test the obtained estimate $\hat{f}$, we generate a new data set $k = 20001,\ldots,20100$ and define the predicted output as

$$\hat{y}(k) = \hat{f}(y(k-1), u(k-1))$$

where $\hat{f}$ was estimated from the previous data set $k = 1,\ldots,20000$. Figure 2 shows the actual output $y(k)$ (solid) and its estimate (dash-dotted) $\hat{y}(k)$. Again, the actual output and its estimate almost coincide.

Example 2: This is an input-output record of a continuous stirring tank reactor, where the input is the coolant flow (l/min) and the output is the concentration (mol/l). The data set consists of 7500 samples. No a priori knowledge on the actual model, including the structure and the order, is available. We model this tank reactor by a first order IIR nonlinear system

$$y(k) = f(y(k-1), u(k-1)).$$

The first 6000 data points were used to obtain the estimate $\hat{f}$ with $r = 0.1$. Then, the output estimates, $k = 1,\ldots,7500$

$$\hat{y}(k) = \hat{f}(y(k-1), u(k-1))$$

were calculated and compared to the actual output $y(k)$. The results are given in Figure 3. The top figure shows the whole range and the bottom one focuses on $y(k)$ (solid) and $\hat{y}(k)$ (dash-dotted) for $k = 6001,\ldots,7500$. Note that $y(k)$'s, $k = 6001,\ldots,7500$ were not used in identification and their estimates $\hat{y}(k)$ do predict $y(k)$'s very well. This validates the identification method proposed in the paper.

Example 3: This data set is input-output samples of a liquid-saturated steam heat exchanger, the input is the liquid flow rate and the output is the outlet liquid temperature. The data set contains 4000 samples and no a priori knowledge on the actual model is available. We model this tank reactor by a first order IIR nonlinear system

$$y(k) = f(y(k-1), u(k-1)).$$

The first 3500 data points were used to obtain $\hat{f}$ with $r = 0.1$. Then, the output estimates, $k = 3501,\ldots,4000$

$$\hat{y}(k) = \hat{f}(y(k-1), u(k-1))$$

were calculated and compared to the actual output $y(k)$. The results are shown in Figure 4. The top figure shows the whole range and the bottom one focuses on $y(k)$ (solid) and $\hat{y}(k)$ (dash-dotted) for $k = 3501,\ldots,4000$.

Example 4: This is a recorded patient EEG data and no actual model of EEG signal is available. In fact, no input signal is available. We model the EEG signal $y(k)$ by a first order nonlinear equation

$$y(k) = f(y(k-1), u(k-1))$$

where $y(k-1)$ is the recorded EEG value at time $k-1$ and $u(k)$ is iid in $[-1,1]$. The form of f is unknown. The given data set contains 4000 samples. We use the first 3000 samples, $k = 1,\ldots,3000$, to obtain the estimate $\hat{f}$ and then, to predict $\hat{y}(k) = \hat{f}(y(k-1), u(k-1))$ for $k = 3001,\ldots,4000$. The actual and predicted EEG signals $y(k)$(solid) and $\hat{y}(k)$(dash-dotted) are shown in Figure 5. The top graph is the whole range $k \in [3000, 4000]$ and the bottom is a zoomed-in graph for $k = 3900,\ldots,4000$. Clearly, $\hat{y}(k)$ predicts $y(k)$ satisfactorily, which demonstrates the efficacy of the identification method introduced in this paper.

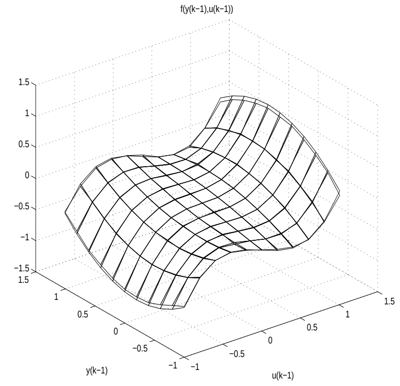

Fig. 1. $f(y, u)$ and its estimate $\hat{f}(y, u)$.

5. CONCLUDING REMARKS

In this paper, identification of IIR nonlinear systems is studied, and asymptotic convergence properties of the kernel method are shown. To test the efficacy of the method proposed, three real world examples are used together with a simulation example. The numerical results obtained are very promising.

REFERENCES

Bai, E.W., R. Temp and Y. Liu (2004). Recursive identification of iir nonlinear systems without structral infromation. Technical report. ECE Dept., the Univ. of Iowa.

Haber, R. and H. Unbehauen (1990). Structure identification of nonlinear dynamic systems-a survey on input/output approaches. *Automatica* **31**, 651–677.

Juditsky, A., H. Hjalmarsson, A. Benveniste, B. Delyon, L. Ljung, J Sjoberg and Q. Zhang (1995). Nonlinear block-box models in system identification: mathematical foundations. *Automatica* **31**, 125–247.

Nadaraya, E.A. (1989). *Nonparametric estimation of probability densities and regression curves*. Kluwer Academic Pub. Dordrecht, The Netherlands.

Parzen, E. (1962). An estimation of a probability density function and mode. *Annals Mathematical Statistics* **33**, 1065–1076.

Sjoberg, J., Q. Zhang, L. Ljung, A. Benveniste, P-Y. Glorennec B. Delyon, H. Hjalmarsson and A. Juditsky (2000). Nonlinear black-box modeling in system identification: a unified overview. *Automatica* **31**, 1691–1724.

Sontag, E. (1963). The iss philosophy as a unifying framework for stability-like behavior. In: *Nonlinear Control in the Year 2000* (F. Lamnabhi-Lagarrigue A. Isidori and W. Respondek, Eds.). Vol. 2. pp. 443–468. Springer-Verlag. Berlin.

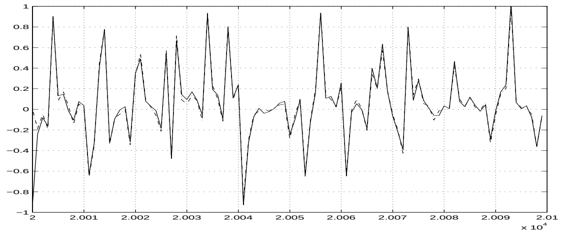

Fig. 2. Actual output $y(k)$(solid) and the predicted output $\hat{y}(k)$(dash-dotted) based on the estimate $\hat{f}(y, u)$.

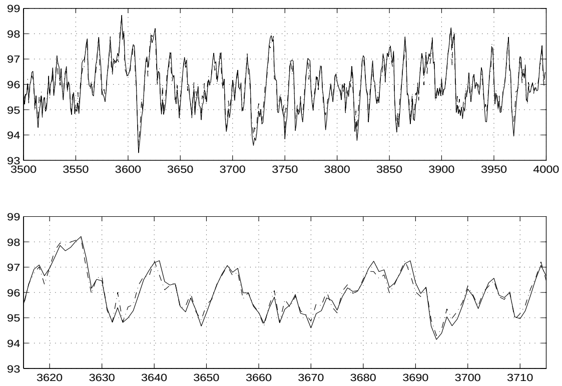

Fig. 3. Actual concentration (solid) and the predicted concentration (dash-dotted).

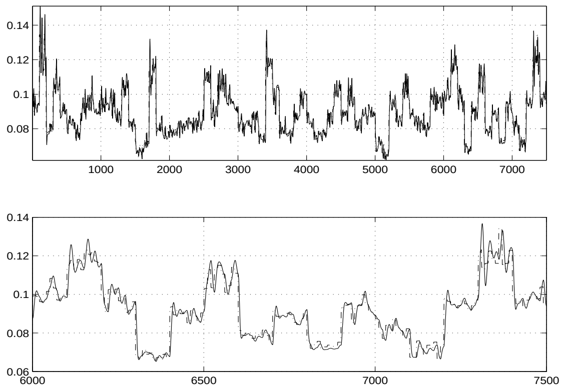

Fig. 4. Actual temperature(solid) and the predicted temperature (dash-dotted).

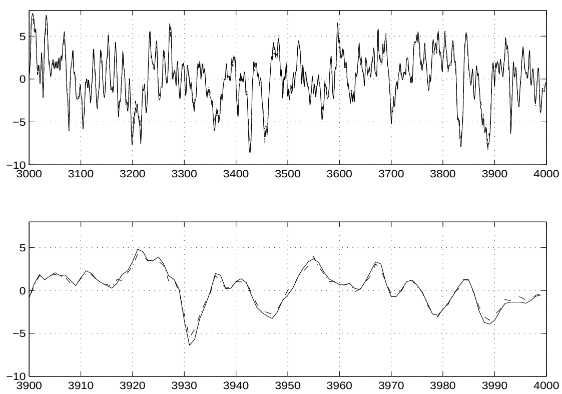

Fig. 5. Actual EEG signal(solid) and the predicted EEG signal(dash-dotted).