REDUCTION OF SECOND ORDER SYSTEMS USING SECOND ORDER KRYLOV SUBSPACES

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Abstract: By introducing the second order Krylov subspace, a method for the reduction of second order systems is proposed leading to a reduced system of the same structure. This generalization of Krylov subspace involves two matrices and some starting vectors and the reduced order model is found by applying a projection directly to the second order model without any conversion to state space. A numerical algorithm called second order Arnoldi is used to calculate the projection matrix. A sufficient condition for stability of the reduced model is given and finally, the method is applied to an electrostatically actuated beam.

Keywords: Model reduction, Second-order systems, Large scale systems, Reduced-order models.

1. INTRODUCTION

Nowadays, with the help of powerful computers and applying advanced modelling techniques like Finite Element Methods (FEM), complex behaviors can be modelled leading to differential equations of high orders. Such a modelling in Electrical Circuits and Micro-Electro-Mechanical Systems, may lead to a large set of differential equations of *second* order form (Ramaswamy and White 2001, Bai *et al.* 2001, Sheehan 1999). In reduced order modelling of second order models, it is mostly desirable to construct a reduced system which preserves the second-order structure (Bastian and Haase 2003, Su and Craig Jr. 1989, Salimbahrami and Lohmann 2004*b*, Lohmann and Salimbahrami 2004, Salimbahrami and Lohmann 2004*a*).

A leading method in reducing large scale systems is moment matching by means of Krylov subspaces (Freund 2003). In this paper, a generalization of Krylov subspace methods for second order systems using a *second order Krylov subspace* is presented which was first introduced in (Salimbahrami and Lohmann 2003, Salimbahrami and Lohmann 2004*a*) and here is extended to be useful in reduction of multi-input multi-output (MIMO) systems including some further discussions on passivity, stability and matching the moments around different points as an extension of the work in (Grimme 1997) to the second order systems.

Second order Krylov subspaces modify the method presented in (Su and Craig Jr. 1989) to match the moments around different points and to increase the number of matching moments when using two-sided methods. Because the method is based on applying the projection directly to the original second order system, all calculations can be done in half dimension without using a state space equation.

2. SECOND ORDER KRYLOV SUBSPACE

The high-order models considered in this paper are assumed to be given in the form

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{G}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{L}\mathbf{z}(t), \end{cases} \tag{1}$$

[1] This work is partially supported by German Research Council (DFG).

with n second order differential equations, m inputs and p outputs. Equivalently, the model (1) can be written in state space with $N = 2n$ first order differential equations as follows,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \ddot{\mathbf{z}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u} \\ \mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}, \end{cases} \quad (2)$$

where $\mathbf{F} \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. For simplicity, $\mathbf{F} = \mathbf{I}$ can be chosen or for the case that $\mathbf{K}$ is nonsingular $\mathbf{F} = \mathbf{K}$.

The i-th moment (around zero) of the system (1) can easily be calculated using (2):

$$\mathbf{m}_i = \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \right)^i \times \begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{D} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix}. \quad (3)$$

To calculate the moments by a recursive procedure, the second order Krylov subspace is defined as follows.

Definition 1. The *second order Krylov subspace* is defined as,

$$\mathcal{K}_q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{G}_1) = colspan\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{q-1}\}, \quad (4)$$

where

$$\begin{cases} \mathbf{P}_0 = \mathbf{G}_1 \ , \ \mathbf{P}_1 = \mathbf{A}_1 \mathbf{P}_1 \\ \mathbf{P}_i = \mathbf{A}_1 \mathbf{P}_{i-1} + \mathbf{A}_2 \mathbf{P}_{i-2}, \ i = 2, 3, \cdots \end{cases} \quad (5)$$

and $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{n\times n}, \mathbf{G}_1 \in \mathbb{R}^{n\times m}$ are constant matrices. The columns of $\mathbf{G}_1$ are called the starting vectors and the matrices $\mathbf{P}_i$ are called basic blocks.

Definition 2. The second order Krylov subspaces $\mathcal{K}_{q_1}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ and $\mathcal{K}_{q_2}(-\mathbf{K}^{-T}\mathbf{D}^T, -\mathbf{K}^{-T}\mathbf{M}^T, -\mathbf{K}^{-T}\mathbf{L}^T)$ are called the input and output second order Krylov subspaces for system (1), respectively.

3. REDUCTION THEOREMS

The idea of the reduction method is to find the projection matrices $\mathbf{V}$ and $\mathbf{W}$ that can directly be applied to the second order model (1) and match some of the first moments. To do so, the connection between the second order Krylov subspaces and the moments of the second order systems is used as formulated in the following lemma.

Lemma 3. Consider the input and output second order Krylov subspaces for system (1) with corresponding basic blocks $\mathbf{P}_i$ and $\tilde{\mathbf{P}}_i$, respectively. Then,

$$\mathbf{m}_i = \mathbf{L}\mathbf{P}_i = \tilde{\mathbf{P}}_i^T \mathbf{G}, i = 0, 1, \cdots .$$

The proof of this lemma is straightforward and comes from the definition of the second order Krylov subspace and equation (3). Now, consider a projection as follows,

$$\mathbf{z} = \mathbf{V}\mathbf{z}_r, \mathbf{V} \in \mathbb{R}^{n\times q}, \mathbf{z} \in \mathbb{R}^r, \mathbf{z}_r \in \mathbb{R}^q, \quad (6)$$

where $q < n$. By applying this projection to system (1) and then multiplying the state equation by the transpose of a matrix $\mathbf{W} \in \mathbb{R}^{n\times q}$, a reduced model of order $Q = 2q$ is found,

$$\begin{cases} \overbrace{\mathbf{W}^T\mathbf{M}\mathbf{V}}^{\mathbf{M}_r} \ddot{\mathbf{z}}_r + \overbrace{\mathbf{W}^T\mathbf{D}\mathbf{V}}^{\mathbf{D}_r} \dot{\mathbf{z}}_r + \overbrace{\mathbf{W}^T\mathbf{K}\mathbf{V}}^{\mathbf{K}_r} \mathbf{z}_r = \underbrace{\mathbf{W}^T\mathbf{G}}_{\mathbf{G}_r} \mathbf{u}, \\ \mathbf{y} = \underbrace{\mathbf{L}\mathbf{V}}_{\mathbf{L}_r} \mathbf{z}_r. \end{cases} \quad (7)$$

This reduced model is in the desired form (1) and thereby preserves the second-order character of the original model! For the calculation of $\mathbf{V}$ and $\mathbf{W}$ the second order Krylov subspaces are used, as described by the following theorems.

Theorem 4. If the columns of $\mathbf{V}$ used in (7), form a basis for the input second order Krylov subspace and the matrix $\mathbf{W}$ is chosen such that $\mathbf{K}_r$ is nonsingular, then the first q_1 moments (the moments from $\mathbf{m}_0$ to $\mathbf{m}_{q_1-1}$) of the original and reduced models match, , assuming that $\mathbf{K}$ is invertible.

PROOF. Consider the matrices

$$\begin{cases} \mathbf{P}_{r0} = -\mathbf{K}_r^{-1}\mathbf{G}_r \ , \mathbf{P}_{r1} = \mathbf{K}_r^{-1}\mathbf{D}_r\mathbf{K}_r^{-1}\mathbf{G}_r \\ \mathbf{P}_{ri} = -\mathbf{K}_r^{-1}\mathbf{D}_r\mathbf{P}_{r(i-1)} - \mathbf{K}_r^{-1}\mathbf{M}_r\mathbf{P}_{r(i-2)} \end{cases} \quad (8)$$

By using lemma 3, we just prove that $\mathbf{P}_i = \mathbf{V}\mathbf{P}_{ri}$ for $i = 0, \cdots, q_1 - 1$ where $\mathbf{P}_i$ and $\mathbf{P}_{ri}$ are the i-th basic blocks of the input second order Krylov subspace for the original and reduced order models, respectively. For the first basic vector we have,

$$\begin{aligned} \mathbf{V}\mathbf{P}_{r0} &= -\mathbf{V}\mathbf{K}_r^{-1}\mathbf{G}_r = -\mathbf{V}(\mathbf{W}^T\mathbf{K}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{G} \\ &= \mathbf{V}(\mathbf{W}^T\mathbf{K}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{K}(-\mathbf{K}^{-1}\mathbf{G}) \\ &= \mathbf{V}(\mathbf{W}^T\mathbf{K}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{K}\mathbf{P}_0. \end{aligned}$$

The matrix $\mathbf{P}_0$ is in the second order Krylov subspace and there exists $\mathbf{R}_0 \in \mathbb{R}^{q\times m}$ such that $\mathbf{P}_0 = \mathbf{V}\mathbf{R}_0$,

$$\begin{aligned} \mathbf{V}\mathbf{P}_{r0} &= \mathbf{V}(\mathbf{W}^T\mathbf{K}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{K}\mathbf{V}\mathbf{R}_0 \\ &= \mathbf{V}\mathbf{R}_0 = \mathbf{P}_0. \end{aligned} \quad (9)$$

For the next moment, the result in equation (9) is used,

$$\begin{aligned}\mathbf{VP}_{r1} &= -\mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{DVP}_{r0}\\ &= -\mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{DP}_0\\ &= \mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{K}(-\mathbf{K}^{-1}\mathbf{DP}_0)\\ &= \mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{KP}_1.\end{aligned}$$

$\mathbf{P}_1$ is in the Second Order Krylov Subspace and there exists $\mathbf{R}_1 \in \mathbb{R}^{q\times m}$ such that $\mathbf{P}_1 = \mathbf{VR}_1$. Therefore,

$$\begin{aligned}\mathbf{VP}_{r1} &= \mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{KVR}_1\\ &= \mathbf{VR}_1 = \mathbf{P}_1. \end{aligned}\tag{10}$$

Now consider that the statement is true until $i = j-1$, i.e. $\mathbf{P}_i = \mathbf{VP}_{ri}$ for $i = 0, \cdots, j-1$. By using the results for $i = j-2$ and $i = j-1$, for $i = j$ we have,

$$\begin{aligned}\mathbf{VP}_{rj} = \mathbf{V}\,[&-(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{DVP}_{r(j-1)}\\ &-(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{MVP}_{r(j-2)}]\\ = \mathbf{V}\,[&-(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{DP}_{j-1}\\ &-(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{MP}_{j-2}]\\ = \mathbf{V}\,[&-(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{KK}^{-1}\mathbf{DP}_{j-1}\\ &-(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{KK}^{-1}\mathbf{MP}_{j-2}]\\ = \mathbf{V}&(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{K}\,[-\mathbf{K}^{-1}\mathbf{DP}_{j-1}\\ -\mathbf{K}^{-1}&\mathbf{MP}_{j-2}] = \mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{KP}_j.\end{aligned}$$

$\mathbf{P}_j$ is in the second order Krylov subspace and can be written as $\mathbf{P}_j = \mathbf{VR}_j$ for $\mathbf{R}_j \in \mathbb{R}^{q\times m}$. Thus,

$$\begin{aligned}\mathbf{VP}_{rj} &= \mathbf{V}(\mathbf{W}^T\mathbf{KV})^{-1}\mathbf{W}^T\mathbf{KVR}_j\\ &= \mathbf{VR}_j = \mathbf{P}_j, \end{aligned}\tag{11}$$

and by induction, it is proved that $\mathbf{P}_i = \mathbf{VP}_{ri}$ for $i = 0, \cdots, q_1 - 1$. For $i = q_1$, because the matrix $\mathbf{P}_{q_1}$ is not in the second order Krylov subspace, the proof fails, and q_1 moments match.

In the SISO case, theorem 4 has some similarities to the results in (Su and Craig Jr. 1989) but it is independent of the output of the system. This fact is important for doubling the number of matching moments (by theorem 5) compared to (Su and Craig Jr. 1989). Also, theorem 4 is more straightforward and only uses the state equations, similar to the standard Krylov subspace methods in state space (Freund 2003).

By using both, input and output second order Krylov subspaces, it is possible to match more moments and to find better approximations for the original large scale system:

Theorem 5. If the columns of $\mathbf{V}$ and $\mathbf{W}$ used in (7), form bases for the input and output second order Krylov subspaces, respectively, both with rank q, then the first $q_1 + q_2$ moments of the original and reduced models match, assuming that $\mathbf{K}$ and $\mathbf{K}_r$ are invertible.

PROOF. To prove this theorem, we use the definition (3) of the moments in state space. It can be shown that applying the projection to the second order system (1) to find the reduced order model (7) is equivalent to applying the projection

$$\begin{bmatrix}\mathbf{z}\\ \dot{\mathbf{z}}\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{V} & \mathbf{0}\\ \mathbf{0} & \mathbf{V}\end{bmatrix}}_{\tilde{\mathbf{V}}}\begin{bmatrix}\mathbf{z}_r\\ \dot{\mathbf{z}}_r\end{bmatrix}, \tag{12}$$

to the system (2) with $\mathbf{F} = \mathbf{K}$ and multiplying the state equation by,

$$\tilde{\mathbf{W}}^T = \begin{bmatrix}\mathbf{W}^T & \mathbf{0}\\ \mathbf{0} & \mathbf{W}^T\end{bmatrix}. \tag{13}$$

According to theorem 4, independent of the definition of the output equation, the first q_1 moments match,

$$(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B} = \tilde{\mathbf{V}}(\mathbf{A}_r^{-1}\mathbf{E}_r)^i\mathbf{A}_r^{-1}\mathbf{B}_r, \tag{14}$$

for $i = 0 \cdots q_1 - 1$ and dual to this fact, we have

$$\mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1} = \mathbf{C}_r(\mathbf{A}_r^{-1}\mathbf{E}_r)^i\mathbf{A}_r^{-1}\tilde{\mathbf{W}}^T, \tag{15}$$

for $i = 0 \cdots q_2 - 1$. The matrices $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{C}$ are defined as in system (2) and the matrices $\mathbf{A}_r$, $\mathbf{E}_r$ and $\mathbf{C}_r$ are defined by converting the reduced system (7) into state space form. We factorize the moments of the original model into two parts,

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^{i-q_1}\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{E}^{q_1-1}\mathbf{A}^{-1}\mathbf{B},$$

for $i > q_1 - 1$. By using the equations (14) and (15), for $i = q_1, \cdots, q_1 + q_2 - 1$ we have,

$$\begin{aligned}\mathbf{m}_i = \mathbf{C}_r(\mathbf{A}_r^{-1}\mathbf{E}_r)^{i-q_1}\mathbf{A}_r^{-1}\tilde{\mathbf{W}}^T \times \mathbf{E} \times\\ \tilde{\mathbf{V}}(\mathbf{A}_r^{-1}\mathbf{E}_r)^{q_1-1}\mathbf{A}_r^{-1}\mathbf{B}_r.\end{aligned}$$

$\tilde{\mathbf{W}}^T\mathbf{E}\tilde{\mathbf{V}} = \mathbf{E}_r$ and then $\mathbf{m}_i = \mathbf{m}_{ri}$ where $i = 0, \cdots, q_1 + q_2 - 1$.

4. SECOND ORDER ARNOLDI ALGORITHM

In this section, the Arnoldi algorithm is extended to find a basis for a given second order Krylov subspace. Consider the second order Krylov subspace $\mathcal{K}_q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{G}_1)$ with m starting vectors. The algorithm 1 finds an *orthonormal* basis for this subspace, i.e. $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, and the columns of the matrix $\mathbf{V}$ form a basis for the given subspace.

To show that the algorithm 1 produces the required basis for a given subspace, just consider the algorithm is applied to the input second order Krylov subspace of system (1). Then, the vectors $\mathbf{v}_i$ and $\mathbf{l}_i$ are the upper and lower parts of the basis vectors of the Krylov subspace $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ in system (2), respectively. Using the results in (3), it is not difficult to show that the algorithm 1 produces a basis for the corresponding second order Krylov subspace. In

Algorithm 1. Second order Arnoldi algorithm

0. (a) Delete all linearly dependent starting vectors to get m_1 vectors.
   (b) Set $\mathbf{v}_1 = \frac{\mathbf{g}_1}{\|\mathbf{g}_1\|_2}$. where $\mathbf{g}_1$ is the first starting vector after deleting the dependent starting vectors and set $\mathbf{l}_1 = \mathbf{0}$ for $\mathbf{l}_1 \in \mathbb{R}^n$.

(1) For $i = 2, 3, \cdots$, do,
   (a) Calculating the next vector: if $i \leq m_1$ then set $\mathbf{v}_i$ as the i-th starting vector and $\mathbf{l}_i = \mathbf{0}$. Otherwise, set

$$\hat{\mathbf{v}}_i = \breve{\mathbf{D}}\mathbf{v}_{i-m_1} + \breve{\mathbf{M}}\mathbf{l}_{i-m_1}\ ,\ \hat{\mathbf{l}}_i = \mathbf{v}_{i-m_1}.$$

   (b) Orthogonalization: For j=1 to i-1 do,

$$h = \hat{\mathbf{v}}_i^T\mathbf{v}_j\ ,\ \hat{\mathbf{v}}_i = \hat{\mathbf{v}}_i - h\mathbf{v}_j\ ,\ \hat{\mathbf{l}}_i = \hat{\mathbf{l}}_i - h\mathbf{l}_j.$$

   (c) Deflation: If $\hat{\mathbf{v}}_i \neq \mathbf{0}$ then go to 1d.
   Else, if $\hat{\mathbf{l}}_i \neq \mathbf{0}$ then $\mathbf{v}_i = \mathbf{0}$ and go to 1e.
   Else, $m_1 = m_1 - 1$ and go to 1a (but go to step 2 if $m_1 = 0$).
   (d) Normalization: $\mathbf{v}_i = \frac{\hat{\mathbf{v}}_i}{\|\hat{\mathbf{v}}_i\|_2}$ and $\mathbf{l}_i = \frac{\hat{\mathbf{l}}_i}{\|\hat{\mathbf{v}}_i\|_2}$.
   (e) Increase i and go to step 1a.

(2) Delete the zero columns of the matrix $\mathbf{V}$ produced by deflation process.

two-sided methods, the algorithm 1 is used twice, first for the input second order Krylov subspace and then for the output second order Krylov subspace, and the matrices $\mathbf{V}$ and $\mathbf{W}$ are found. The resulting reduction scheme can be called a *two-sided second order Arnoldi* method.

For deflation, in step 1c, it is checked if the new vector is a linear combination of the previous ones. If only $\hat{\mathbf{v}}_i$ is expanded by $\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}$ (it is identified by $\hat{\mathbf{v}}_i = \mathbf{0}$), then $\mathbf{l}_i$ should not be deleted to be used in the next iteration and $\mathbf{v}_i$ is substituted by zero (which is deleted at the end). If both vectors $\hat{\mathbf{v}}_i$ and $\hat{\mathbf{l}}_i$ are expanded by $\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}$ and $\mathbf{l}_1, \cdots, \mathbf{l}_{i-1}$, respectively then, the algorithm deletes both vectors. In practice, $\hat{\mathbf{v}}_i = 0$ and $\hat{\mathbf{l}}_i = 0$ should be substituted with $\|\hat{\mathbf{v}}_i\|_2 < \epsilon$ and $\|\hat{\mathbf{l}}_i\|_2 < \epsilon$, where ϵ is a small positive number.

The moments of the MIMO system (1) are $p \times m$ matrices, where each column is related to an input and each row is related to an output of the system. In algorithm 1, the order of the reduced system is independent of the number of inputs and outputs which is an advantage specially when the system is not square with high number of inputs and outputs. If j columns of the matrix $\mathbf{V}$ (or $\mathbf{W}$) are related to the k-th input (or output), then the k-th column (or row) of the moment matrix matches up to at least the $j-1$-st moment.

5. GUARANTIED STABILITY

In using Krylov subspace methods to reduce the order of a stable large scale model, there is no guaranty to find a stable reduced model. There exists a guaranty only for some types of systems which are related to passive systems (see the concept of positive realness as mentioned in (Freund 2000)). As mentioned in (Freund 2000) a one-sided method can preserve passivity. This result can be generalized to second order system using the state space model (2):

Theorem 6. In system (2), if $\mathbf{A} + \mathbf{A}^T \preceq 0$ and $\mathbf{E} = \mathbf{E}^T \succeq 0$, then the reduced model using one-sided state space Krylov subspace method with $\mathbf{W} = \mathbf{V}$, is stable and furthermore, the transfer function $H(s) = \mathbf{B}^T\mathbf{V}\left(s\mathbf{V}^T\mathbf{E}\mathbf{V} - \mathbf{V}^T\mathbf{A}\mathbf{V}\right)^{-1}\mathbf{V}^T\mathbf{B}$ is passive.

$\tilde{\mathbf{A}} \succeq 0$ for a symmetric matrix $\tilde{\mathbf{A}} \in \mathbb{R}^{N\times N}$ denotes that $\tilde{\mathbf{A}}$ is nonnegative definite; i.e. $\mathbf{x}^T\tilde{\mathbf{A}}\mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathbb{R}^N$. By considering the projection matrices (12) and (13) applied to the system (2) with $\mathbf{F} = \mathbf{K}$, the result of theorem 6 can easily be generalized to second order system to extract necessary conditions to preserve stability.

Theorem 7. In system (1), if $\mathbf{D} + \mathbf{D}^T \succeq 0$, $\mathbf{K} = \mathbf{K}^T \succ 0$ and $\mathbf{M} = \mathbf{M}^T \succeq 0$, a one-sided method with the choice $\mathbf{W} = \mathbf{V}$ results in a stable reduced model.

6. RATIONAL INTERPOLATION

Matching the moments of the second order model around a point $s_0 \neq 0$, can also be done by applying a projection to the original model (1). The transfer function of system (1) by direct Laplace transform is $\mathbf{H}(s) = \mathbf{L}(s^2\mathbf{M} + s\mathbf{D} + \mathbf{K})^{-1}\mathbf{G}$. The moment of $\mathbf{H}(s)$ around s_0 is equal to the moments of the following system around zero,

$$\begin{aligned}\mathbf{H}(s+s_0) &= \\ &\mathbf{L}\left((s+s_0)^2\mathbf{M} + (s+s_0)\mathbf{D} + \mathbf{K}\right)^{-1}\mathbf{G} \\ &= \mathbf{L}\left(s^2\mathbf{M} + s(\mathbf{D} + 2s_0\mathbf{M})\right. \\ &\quad \left. + (\mathbf{K} + s_0\mathbf{D} + s_0^2\mathbf{M})\right)^{-1}\mathbf{G}\end{aligned}$$

By using equation (3), the moments of $\mathbf{H}(s+s_0)$ are calculated by substituting the matrix $\mathbf{K}$ by $\mathbf{K} + s_0\mathbf{D} + s_0^2\mathbf{M}$ and the matrix $\mathbf{D}$ by $\mathbf{D} + 2s_0\mathbf{M}$ in the definition of the moments around zero, the moments around s_0 is found. Therefore, to match the moments around s_0, the same substitution as in the moments should be done in the definition of input and output Krylov subspaces; i.e. the second order Krylov subspaces $\mathcal{K}_{q_1}(-(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-1}(\mathbf{D}+2s_0\mathbf{M}), -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-1}\mathbf{M}, -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-1}\mathbf{G})$ and $\mathcal{K}_{q_2}(-(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-T}(\mathbf{D}+2s_0\mathbf{M})^T, -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-T}\mathbf{M}^T, -(\mathbf{K}+s_0\mathbf{D}+s_0^2\mathbf{M})^{-T}\mathbf{L}^T)$ should be considered and then by finding the corresponding bases as projection matrices the reduced order system can be found.

This result can also be generalized to match the moments around different points $s_1, \cdots, s_k$ by considering k different second order Krylov subspace and finding a projection matrix whose columns form a bases of the union of the given second order Krylov subspaces. Such a projection matrix can be calculated by extending the second order Arnoldi algorithm.

7. APPLICATION TO ELECTROSTATICALLY ACTUATED BEAM

We apply the proposed approach for reduced order modelling of an electrostatically actuated beam which is used in RF switches or filters [2]. Given a simple shape, they often can be modelled as one-dimensional beams embedded in two or three dimensional space. This model describes a slender beam which is actuated by a voltage between the beam and the ground electrode below; see figure 1. On the beam, at least three degrees of freedom per node have to be considered. On the ground electrode, all spatial degrees of freedom are fixed, so only charge has to be considered. The damping matrix is calculated by a linear combination of the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$.

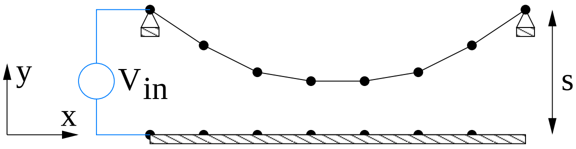

Fig. 1. A conducting beam supported at both ends with counter electrode below.

Based on the finite element discretization presented in (Weaver *et al.* 1990), an interactive matrix generator has been created. After modelling of the beam, a set of differential-algebraic equations of the following form (1) is found. Details of the implementation are available in (Lienemann *et al.* 2004). A typical input to this system is a step function; periodic on/off switching is also possible. The reduced model should thus both represent the step response as well as the possible influence of higher order harmonics.

Two types of model are considered: an undamped model ($\mathbf{D} = \mathbf{0}$) and a lightly damped model, both of order $N = 15992$ with $n = 7996$ second order differential equations. The original models are reduced to different order. It should be noted that the reduced system of the undamped model leads to an undamped system! In table 1, the maximum frequency that the reduced system is accurate is given. These result can be compared to the figures 2 and 3 where some of the reduced systems are plotted. By going to higher orders better accuracy at higher frequencies can be achieved. Because of preserving the second order structure, the slope of the bode plots at high frequencies is $-40dB/dec$..

[2] The model can be downloaded from Oberwolfach Model Reduction Benchmark Collection available online at http://www.imtek.uni-freiburg.de/simulation/benchmark/

Table 1. Maximum accurate frequency f_{max} compared to the reduced system of order 80

Model	Order q	s_0	f_{max}
Undamped model	8	0	1900Hz
	12	0.03	2550Hz
	16	30	5800Hz
	40	2×10^4	61kHz
Damped model	8	0	2280Hz
	12	0	5500Hz
	16	0.5	6600Hz
	20	2×10^3	15kHz
	40	2×10^4	60kHz

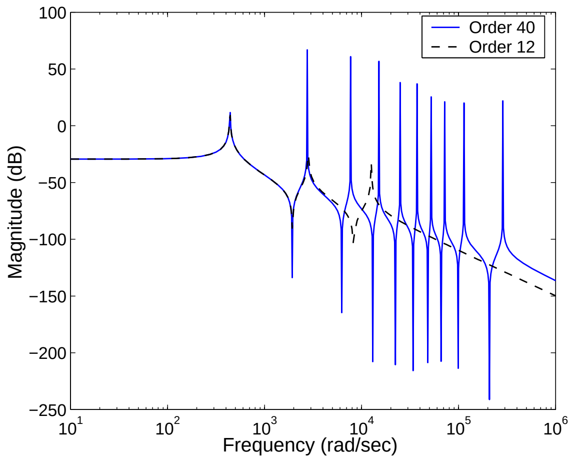

Fig. 2. Frequency response of the reduced systems of the undamped model.

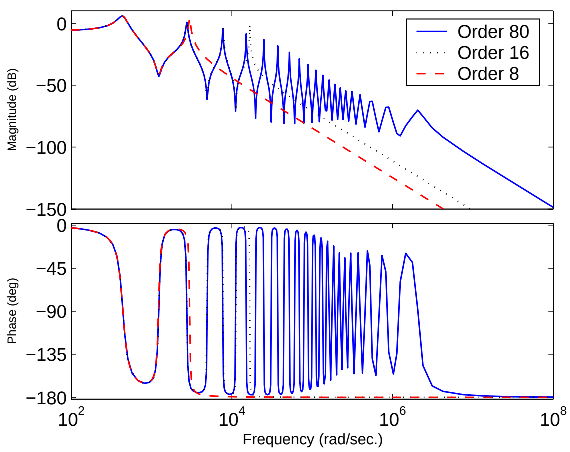

Fig. 3. Frequency response of the reduced systems of the damped model.

In figures 5 and 4, the step response of the reduced systems are compared to each other. For the undamped model, order 12 has an acceptable output however after order 16, the step response remains almost unchanged. For the damped model, the response of the order 8 reduced system is not far from the other systems but after order 16 the step response has very small changes by going to higher orders.

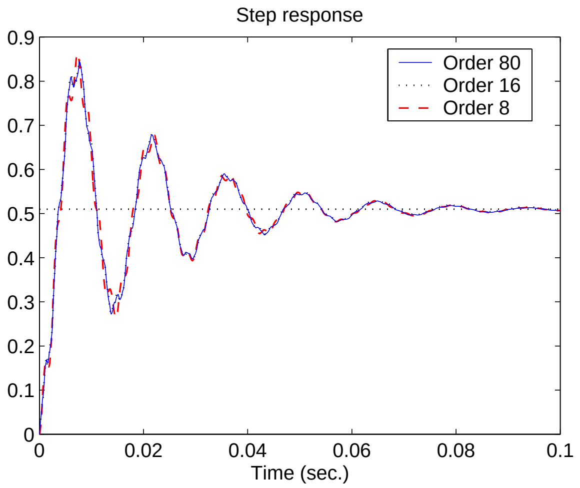

Fig. 4. Step Response of the reduced systems of the damped model.

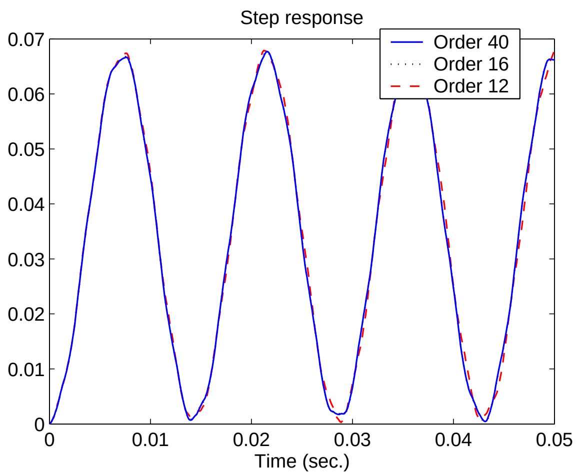

Fig. 5. Step Response of the reduced systems of the undamped model.

8. CONCLUSION

A generalization of the Krylov subspaces was applied to reduce large scale second order models, matching some of the first moments of the original and reduced order models. The advantages of the methods are:

- The second order structure is preserved.
- Compared to the Krylov subspace methods in state space, half number of iterations is necessary to reduce to the same order and the calculations are done in a half dimension.
- Some of the properties of the original systems are preserved; the undamped second order models are approximated with undamped reduced models, one sided methods preserve definiteness of the mass, damping and stiffness matrices.
- It has been proposed how to match the moments around $s_0 \neq 0$ or around different points .

The method was successfully applied for reduced order modelling of an electrostatically actuated beam model.

ACKNOWLEDGEMENTS

We would like to thank our partners at the university of Freiburg for producing the electrostatically actuated beam model.

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