TRACKING CONTROL FOR CONSTRAINED MONOTONE SYSTEMS

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Abstract: Tracking control of nonlinear systems subject to constraints on the input is a challenging issue in control design. Forcing saturation on a previously designed controller may in general lead to destabilization or at least result in performance losses. Hereby it is shown that for a certain class of nonlinear monotone systems it is possible to design a suitable static nonlinear output feedback which stabilizes the system and preserves stability under control saturation.

Keywords: Tracking, monotone systems, nonlinear systems, constraints, stabilization methods, output feedback.

1. INTRODUCTION

An important class of dynamical systems is that of monotone systems. Among the classical references in this area are the text-book (Smith, 1995) and the papers (Smale, 1976; Hirsch, 1985; Hirsch, 1988). Monotone systems are those for which trajectories preserve a partial ordering on the states. Recently the notion of monotonicity has been extended to systems with inputs and outputs (Angeli and Sontag, 2003; Angeli and Sontag, 2004*a*; Angeli and Sontag, 2004*b*) in order to understand system interconnections arising in mathematical biology. Monotone systems include certain classes of competitive and cooperative systems (De Leenheer and Aeyels, 2000; De Leenheer and Aeyels, 2001*a*) for which different state variables attenuate (negative feedback) or reinforce (positive feedback) each other respectively. More in general, for systems which preserve the partial order induced by an arbitrary given orthant, each pair of variables may affect each other in a mixed form (Angeli and Sontag, 2003). Examples of applications of this theory are in many different areas such as, for instance, chemistry (chemical reaction networks) (Angeli *et al.*, 2004; Volpert *et al.*, 1994), ecology, molecular biology and economy to name a few. When the dynamics are linear and the positivity cone is the positive orthant, monotone systems boil down to the so called positive linear systems. This is itself a very interesting class of systems which (in continuous as well as discrete time) has attracted a lot of attention in the control literature; see, for instance, (Luenberger, 1979; Muratori and Rinaldi, 1991; Valcher, 1996; Farina and Rinaldi, 2000; De Leenheer and Aeyels, 2001*b*; Piccardi and Rinaldi, 2002).

So far much attention has been devoted to the analysis of monotone systems and to the study of their interconnections; much less is known as far as specific control synthesis tools which could exploit monotonicity in some respect. One of the major problems in control theory is the design of an offset-free tracking control law for nonlinear systems subject to constraints on the input. The present paper shows that, for a certain class of nonlinear monotone systems, it is possible to design a static output controller in a straightforward way and then force saturation on the input without loss of stability and providing some optimality in the performance. The resulting control strategy is applicable in a simple way and its computational burden is very low. The paper is organized as follows. First a review of basic definitions and results on monotone systems is carried out in section 2. Then the main results on the convergence properties of the saturated control are presented and proved in section 3. The applicability of the method and its effectiveness is illustrated by means of a simulative example in section 4. Finally some conclusions are drawn in section 5.

2. PROBLEM FORMULATION

Consider the following continuous-time nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t)) \end{aligned} \tag{1}$$

where the map $f(x, u)$, defined for $x \in X \subseteq \mathbb{R}^n$ and $u \in U \subseteq \mathbb{R}$ is continuous in (x, u) and locally Lipschitz continuous in x locally uniformly on u and the map $h(x) \in Y \subseteq \mathbb{R}$ is continuous. The solution of (1) for the initial state $x_0 \in \mathbb{R}^n$ and the input signal $u \in \mathcal{U}$ will be denoted by $x(t, x_0, u)$. In the Euclidean space a partial order $\succeq$ induced by a positivity cone C is defined. That is, $C \subseteq \mathbb{R}^n$ is a nonempty, closed, convex, pointed ($C \bigcap -C = \{0\}$) cone with nonempty interior, and $x_1 \succeq x_2$ means that $x_1 - x_2 \in C$. Strict ordering is denoted by $x_1 \succ x_2$, meaning that $x_1 \succeq x_2$ and $x_1 \neq x_2$. The input signal is a locally essentially bounded Lebesgue measurable function $u(\cdot) : \mathbb{R}_{\geq 0} \longrightarrow U$ and the statement $u_1 \succeq u_2$ means that $u_1(t) \succeq u_2(t)$ for almost all $t \geq 0$. The system (1) is said monotone if the following property holds, with respect to the orders on the state and the inputs for all $x_1, x_2 \in X$ and input signals $u_1, u_2 \in \mathcal{U}$:

$$\begin{aligned} & x_1 \succeq x_2 \text{ and } u_1 \succeq u_2 \Rightarrow \\ & x(t, x_1, u_1) \succeq x(t, x_2, u_2) \quad \forall t \geq 0 \end{aligned} \tag{2}$$

and the function $h : \mathbb{R}^n \longmapsto \mathbb{R}$ is a continuous and monotone map with respect to the partial order on the state and output space, i.e. $h(x(t, x_1, u_1)) \geq h(x(t, x_2, u_2))$ for all $t \geq 0$. It is important to be able to check monotonicity without having to compute the trajectories of (1); some useful results in this respect are reported in (Angeli and Sontag, 2004*a*) and references therein. An important characterization of monotone systems from (Angeli and Sontag, 2003) is given in the subsequent theorem. This characterization uses the concept of convex tangent cone to the set $S \subset \mathbb{R}^n$ at the point $x \in \mathbb{R}^n$ denoted as $\mathcal{T}_x S$ and defined hereafter.

Definition 1. The tangent cone to S at x is the set $\mathcal{T}_x S$ of all limits of the type

$$\lim_{i \to \infty} \frac{x_i - x}{t_i}$$

such that $x_i \to x$ as $i \to \infty$ while $x_i \in S$ and $t_i \to 0$ while $t_i > 0$. ■

Theorem 1. The system $\dot{x} = f(x, u)$ is monotone, with respect to the positivity cone C on the states, if and only if

$$\begin{aligned} & x_1 \succeq x_2 \text{ and } u_1 \succeq u_2 \Rightarrow \\ & f(x_1, u_1) - f(x_2, u_2) \in \mathcal{T}_{x_1 - x_2} C \end{aligned} \tag{3}$$

In this paper, the control objective is that

(1) the output $y(t)$ track a piecewise constant reference $r(t)$, i.e. a signal switching among different constant set-points;
(2) the input $u(t)$ satisfy the constraints

$$\underline{u} \leq u(t) \leq \bar{u} \tag{4}$$

For the subsequent developments the following assumption is made.

Assumption 1. For each constant set-point r there is associated an unique (state,input) equilibrium pair $(x_e(r), u_e(r))$ such that

$$f(x_e(r), u_e(r)) = \mathbf{0}, \qquad r = h(x_e(r)) \tag{5}$$

Clearly the constraints (4) restrict the statically admissible set-points r to the ones that belong to the set

$$R = \{r : \underline{u} \leq u_e(r) \leq \bar{u}\}. \tag{6}$$

3. SYSTEM STABILIZATION AND MAIN RESULTS

In order to design a suitable tracking policy for (1) under the constraints (4) it is relevant to find how to stabilize such a system. A useful stability result for systems without external input is reported in (De Leenheer *et al.*, 2004)

Theorem 2. Suppose that:

(i) the dynamical system $\dot{x} = f(x)$ is monotone;
(ii) its trajectories are continuous and bounded in X;
(iii) X contains exactly one equilibrium point x_e;
(iv) for every compact subset S of X, both $inf(S)$ and $sup(S)$ belong to X (see (Davey and Priestley, 2002) for a rigorous definition of $inf(S)$ and $sup(S)$).

Then x_e is asymptotically stable globally in X, i.e. $\lim_{t \to \infty} x(t, x_0, 0) = x_e$ for all $x_0 \in X$. ■

However the open-loop system (1) needs not have unique and stable equilibria. Its steady state behaviour will be useful in order to design a controller.

Definition 2. The system (1) admits a (possibly multivalued) input to state (I/S) steady-state response curve defined as follow

$$k^X(u) \stackrel{\triangle}{=} \{x \in X \ : \ f(x,u) = 0\} \tag{7}$$

if assumption 1 is satisfied. If (1) admits an I/S characteristic, its input/output (I/O) characteristic is by definition the composition

$$k^Y(u) \stackrel{\triangle}{=} \{y \in Y \ : \ y = h(x) \text{ and } f(x,u) = 0\} \tag{8}$$

■

In the present paper SISO systems are considered, as the results are easier to state in this context. Without loss of generality (otherwise, it is always possible to consider $-u$ as an input or $-y$ as an output), the considered order on the input and output spaces is $C^U = C^Y = \mathbb{R}_{\geq 0}$. Our interest will be in the design of static nonlinear output feedback $u = \ell(y,r)$ solving the control objective stated in the previous section.

Theorem 3. Suppose that the system (1) is monotone with respect to C in X, with $C^U = C^Y = \mathbb{R}_{\geq 0}$ and it has an I/O characteristic $k^Y(u)$. Moreover assume that $\dot{x} = f(x,\underline{u})$ and $\dot{x} = f(x,\overline{u})$ admit a unique asymptotically stable equilibrium point in $x \in X$. Design an output feedback $u = \ell(y,r)$ with the following properties.

(1) It admits, for each fixed r, only one intersection point in the plane (y,u) with the I/O characteristic $k^Y(u)$.
(2) It is such that the closed-loop system

$$\dot{x} = f(x, \ell(h(x),r)) \tag{9}$$

is monotone with respect to the same partial order and has bounded trajectories.

Then the saturated control

$$\mathrm{sat}(\ell(h(x),r)) = \begin{cases} \underline{u} & \text{if } \ell(h(x),r) < \underline{u} \\ \ell(h(x),r) & \text{if } \underline{u} \leq \ell(h(x),r) \leq \overline{u} \\ \overline{u} & \text{if } \ell(h(x),r) > \overline{u} \end{cases} \tag{10}$$

is such that the output asymptotically tracks any constant reference $r \in R$ globally in X, i.e. for all initial states $x_0 \in X$. ■

In order to prove the above theorem, the following Lemma is fundamental.

Lemma 1. If both systems (1) and (9) are monotone with respect to the same partial order, then the closed-loop system under the saturated feedback (10) is monotone.

Proof - By theorem 1 one needs to show the following for all $r \in R$

$$\begin{aligned} x_1 \succeq x_2 \Rightarrow\ & f(x_1, \mathrm{sat}(\ell(h(x_1),r))) \\ & -f(x_2, \mathrm{sat}(\ell(h(x_2),r))) \in \mathcal{T}_{x_1 - x_2} C \end{aligned} \tag{11}$$

Only the following two cases need to be considered (the other are trivial)

$$(1) \qquad \begin{aligned} & \ell(h(x_1),r) \geq \mathrm{sat}(\ell(h(x_1),r)) \geq \\ & \geq \mathrm{sat}(\ell(h(x_2),r)) \geq \ell(h(x_2),r) \end{aligned} \tag{12}$$

$$(2) \qquad \begin{aligned} & \ell(h(x_1),r) \leq \mathrm{sat}(\ell(h(x_1),r)) \leq \\ & \leq \mathrm{sat}(\ell(h(x_2),r)) \leq \ell(h(x_2),r) \end{aligned} \tag{13}$$

In the first case, condition (11) is immediately verified by applying condition (3) to the open-loop system and letting $u_1 = \mathrm{sat}(\ell(h(x_1),r))$ and $u_2 = \mathrm{sat}(\ell(h(x_2),r))$. In the case (2) one has the following equality

$$\begin{aligned} & f(x_1, \mathrm{sat}(\ell(h(x_1),r))) - f(x_2, \mathrm{sat}(\ell(h(x_2),r))) = \\ & f(x_1, \mathrm{sat}(\ell(h(x_1),r)) - f(x_1, \ell(h(x_1),r)) + \\ & + f(x_1, \ell(h(x_1),r)) - f(x_2, \ell(h(x_2),r)) + \\ & + f(x_2, \ell(h(x_2),r)) - f(x_2, \mathrm{sat}(\ell(h(x_2),r))) \end{aligned} \tag{14}$$

Considering the relations in (13) and the monotonicity of the open-loop and closed-loop systems, it is straightforward to conclude that

$$\begin{aligned} & f(x_1, \mathrm{sat}(\ell(h(x_1),r)) - f(x_1, \ell(h(x_1),r)) \in \mathcal{T}_0 C \\ & f(x_1, \ell(h(x_1),r)) - f(x_2, \ell(h(x_1),r)) \in \mathcal{T}_{x_1 - x_2} C \\ & f(x_2, \ell(h(x_2),r)) - f(x_2, \mathrm{sat}(\ell(h(x_2),r))) \in \mathcal{T}_0 C \end{aligned} \tag{15}$$

Since $\mathcal{T}_0 C \subseteq \mathcal{T}_{x_1 - x_2} C$ for all $x_1 - x_2 \in C$, the condition (11) for monotonicity is verified by convexity of the cones, i.e $f(x_1, \mathrm{sat}(\ell(h(x_1),r)) - f(x_1, \ell(h(x_1),r)) \in \mathcal{T}_{x_1 - x_2} C$. ■

Proof of theorem 3 - Since under the feedback $u = \ell(y,r)$ the closed-loop system (9) is monotone and has bounded trajectories then, from theorem 2, the unique equilibrium point $x_e(r) \in X$ is globally asymptotically stable in X. Lemma 1 asserts that also the closed-loop system under the saturated feedback (10) is monotone. Under the assumption that $\dot{x} = f(x,\underline{u})$ and $\dot{x} = f(x,\overline{u})$ admit a unique asymptotically stable equilibrium point in X, the system $\dot{x} = f(x, \mathrm{sat}(\ell(h(x),r)))$ has only an equilibrium point for all $r \in R$ and its trajectories are bounded. Once again theorem 2 applies and $\dot{x} = f(x, \mathrm{sat}(\ell(h(x),r)))$ is globally asymptotically stable in $x_e(r)$ for all $r \in R$ and globally in X. ■

Hereafter a sufficient condition for monotonicity preservation under feedback control is given. Recall that a matrix is Metzler if its off-diagonal entries are non negative.

Proposition 1. Given the monotone system (1) with $y = h(x_i)$, x_i being the i-th component of x, a sufficient condition for the closed-loop system under a stabilizing controller $u = \ell(h(x_i), r)$ to be still monotone is the following:

(1) $\dfrac{\partial f_i}{\partial u} > 0$ and $\dfrac{\partial f_i}{\partial u} = 0 \quad$ for $j \neq i$, $\forall x \in X, \forall u \in U$.

(2) $\dfrac{\partial f}{\partial x}$ is Metzler $\forall x \in X,\ \forall u \in U$.

Proof - The Jacobian of the system (9) is

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial \ell}{\partial h}\frac{\partial h}{\partial x} \tag{16}$$

It is immediate to see, evaluating (16) in (x, r) for each r and exploiting (1) and (2), that one gets the following equation structure

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\gamma e_i^T = \frac{\partial f}{\partial x} + \beta e_i \gamma e_i^T \tag{17}$$

where e_i is a vector with all 0s except for a 1 in the $i-th$ entry, $\beta = \frac{\partial f_i}{\partial u} \in \mathbb{R}$ and $\gamma = \frac{\partial \ell}{\partial h}\frac{\partial h}{\partial x_i} \in \mathbb{R}$. This means that the term $\beta e_i \gamma e_i^T$ does not affect the off-diagonal elements of $\frac{\partial f}{\partial x}$ and, hence, the monotonicity of the system is preserved. ■

Remark 1. Under monotonicity of the map ℓ with respect to r the saturated control feedback corresponds to a one-step-ahead reference governor policy. In a more formal way it corresponds to the solution of the following optimizing control algorithm:
At time t, given the output $y(t)$ and the desired reference $r_d(t) = r_d$, the applied reference $r(t)$ is the solution of the following problem

$$\begin{aligned} r(t) = {} & \arg\min_{\overline{r} \in R} (r_d - \overline{r})^2 \\ & \text{subject to} \\ & \underline{u} \leq \ell(y, \overline{r}) \leq \overline{u} \end{aligned} \tag{18}$$

Indeed if the constraints are not active, the reference r_d is applied. Conversely, when the constraints are active, the saturated input, for each given output, may be seen as the input $\ell(y, r)$ for a different reference. It is straightforward to realize that under the condition of monotonicity of the scalar map ℓ with respect to r, the selected reference is the one that solves (18). Actually the cost in the optimization problem (18) is equivalent to $\min_r\ (\ell(y, r_d) - \ell(y, r))^2$. In a more formal way, the following relation holds:

$$\ell(y(t), r(t)) = \text{sat}(\ell(y(t), r_d)) \tag{19}$$

Remark 2. Under conditions of proposition (1) the design of a controller $u = \ell(y, r)$ turns out to be straightforward. Indeed any design that satisfies the requirement that the two sets of points $(u, k^Y(u))$, i.e. the plant I/O characteristic, and $(\ell(y, r), y)$, i.e. the controller I/O characteristic, admit just one intersection in the plane (u, y) for each r, does not destroy the monotonicity of the system. Therefore it is possible to choose graphically the shape of $u = \ell(y, r)$ so that the uniqueness of the equilibrium point is guaranteed for all r. In order to carry out easily a graphical choice of the feedback shape it is possible to reparametrize the feedback as $\ell(y, \alpha(r))$ for a suitable r-dependent parameter α. Given the desired structure of $\ell(y, \alpha(r))$ it is possible to determine $\alpha(r)$ by solving the following equation

$$k^Y(\ell(r, \alpha(r)) = r \tag{20}$$

The existence of a suitable $\ell(y, r)$ is ensured by assumption 1. Then, for each value of r, the computation of the matched value $\alpha(r)$ is performed on-line, so that offset-free tracking is ensured.

4. EXAMPLE

An interesting situation for the application of theorem 3 is when the monotone system presents a steady state characteristic with hysteresis as in the following example

$$\begin{aligned} \dot{x}_1 &= 1 - x_1 - 200\ x_1 \frac{u^4 x_2^4}{30 + u^4 x_2^4} \\ \dot{x}_2 &= 1 - x_2 - 10\ x_2 \frac{x_1^4}{1 + x_1^4} \end{aligned} \tag{21}$$

where $x \in \mathbb{R}_{\geq 0}$, $\mathbb{R}_{\geq 0}$ being an invariant region for the above system. It is possible to describe a parametrized family of equilibrium points for (21) through the choice of x_1 as scheduling variable.

$$\begin{aligned} x_{2e} &= \frac{1 + x_1^4}{1 + 11x_1^4} \\ u_e &= \frac{1 + 11x_1^4}{1 + x_1^4} \sqrt[4]{\frac{(1 - x_1)30}{201x_1 - 1}} \end{aligned} \tag{22}$$

This parametrization suggests the choice

$$y = x_1 \tag{23}$$

as output map. Notice that u_e is defined in the output range $(1/201, 1]$. It is straightforward to check that the system (21) is monotone with respect to the order induced by the positivivity cone $C = \mathbb{R}_{\leq 0} \times \mathbb{R}_{\geq 0}$ in $X = \mathbb{R}^2_{\geq 0}$. The I/O characteristic of the system (21), (23) is not well-defined. It is an hysteresis as shown in figure 1 which presents multiple equilibria for some values

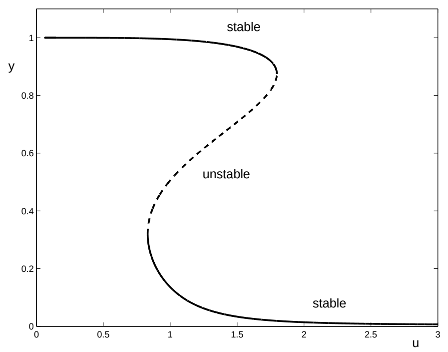

Fig. 1. The plant I/O characteristic $k^Y(u)$

of u. In order to stabilize the branch of unstable equilibrium points in $\mathbb{R}_{\geq 0}$ it is straightforward to design a controller satisfying the conditions of theorem 3. The simplest choice is

$$u = \alpha y^2 \tag{24}$$

Chosen the desired reference $r \in R$, the corresponding $\alpha(r)$ is easily computed as

$$\alpha(r) = \frac{(1+11r^4)\sqrt[4]{(1-r)30}}{r^2(1+r^4)\sqrt[4]{201r-1}} \tag{25}$$

The obtained behaviour of the proposed tracking strategy is shown by simulation experiments choosing the following input constraints

$$0.8 \leq u \leq 1.8 \tag{26}$$

The constraints (26) guarantee the existence of a unique equilibrium point for all $r \in R$ (see figure 2). The output response to a square wave set-point, applying the control law $u = \mathrm{sat}(\alpha(r)y^2)$, is shown in figure (3). The input (24) and the saturated one are reported in figures (4) and, respectively, (5) Notice that in this case the input has been saturated in the usual way since in $\mathbb{R}^2$ it is straightforward to show that the closed-loop system is still monotone. Finally in figure 6 the choice of the feasible $\alpha(r)$ is reported and it is evident how the saturated control is equivalent to a reference governor.

5. CONCLUSIONS

The paper has addressed tracking control of monotone nonlinear system in the presence of input constraints. It has been shown that for a certain class of nonlinear monotone systems, it is possible to design a static output controller in a straightforward way and then force saturation on the input without loss of stability and, under certain condition, some optimality in the performance is guaranteed. The implementation of the resulting control strategy is simple and its

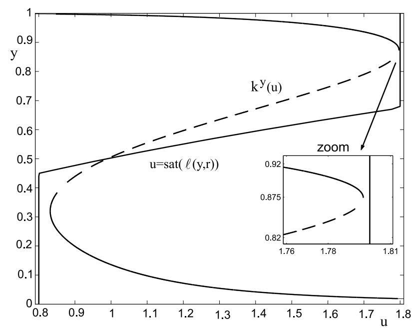

Fig. 2. The intersection between the plant I/O characteristic $k^Y(u)$ and $u = \mathrm{sat}(\ell(y, r)))$ for $r = 0.5$

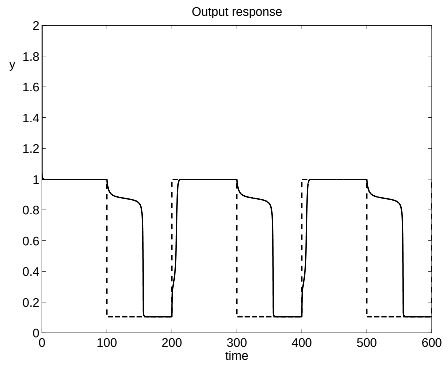

Fig. 3. Desired output (dashed) and output response (solid)

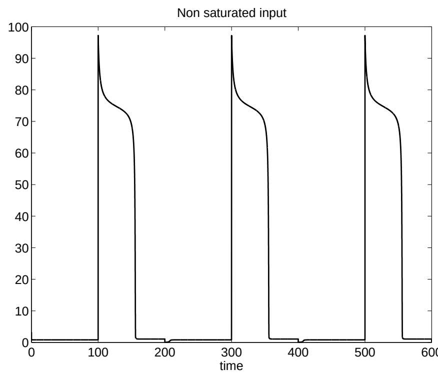

Fig. 4. Non-saturated input

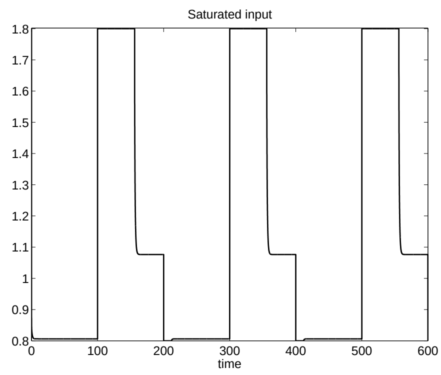

Fig. 5. Saturated input

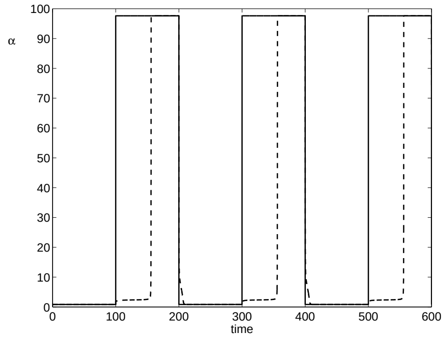

Fig. 6. Desired α (dashed) and values of $\alpha(r(t))$ that correspond to $\text{sat}(\alpha(r)y^2) \subseteq [0.8, 1.8]$ (solid)

computational burden is very low. Future work will be devoted to consider discrete time systems and the presence of state constraints.

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