MANUAL CONTROL AND STABILIZATION OF AN INVERTED PENDULUM

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Abstract: The problem of stabilizing an inverted pendulum on a cart while enabling manual control of the cart velocity is treated. Introduction of an input saturation nonlinearity makes the problem challenging in the sense that the system may be driven to a state where recovery is not possible. A controller based on the controllability set of the inverted pendulum, which ensures stability and tracking of constant reference velocities for the cart is developed. The controller also offers a trade-off between performance and robustness.

Keywords: Nonlinear control, Controllability set, Inverted pendulum

1. INTRODUCTION

In many control applications, a system is controlled by a combination of manual and automatic control. Typical examples are aircrafts, where stability augmentation systems are used to assist the pilot. The combination of manual and automatic control is particularly crucial for unstable systems with actuator constraints, because the system can be driven to such a state unintentionally by manual control. The problem is similar to the one encountered when controlling unstable aircrafts such as the Saab Gripen, where in some flight conditions the unstable mode is so fast that a pilot cannot stabilize the system. The aircraft dynamics is however more complex and the actuator rate is saturated, see (Rundqwist *et al.*, 1997; Patcher and Miller, 1998). The pendulum problem can however serve as a simple prototype for an interesting class of real problems.

The essence of the problem can be captured in the following formulation. Consider an unstable system with actuator saturation. Find a control strategy that stabilizes the system and provides facilities for manual control. The strategy should be such that the system can be controlled manually without driving it unstable.

There is an extensive literature on stabilizing a dynamical system subject to input or state constraints. For linear systems, the problem is well understood. For stable systems there are strong results stating that there always exist a controller that stabilizes the system globally. The result was proven for a chain of integrators in (Teel, 1992) and for the general case in (Sussmann *et al.*, 1994). For unstable systems, the situation is more involved. A key concept for control of unstable systems is the notion of *Controllability Sets*, which contain all points of the state space such that there exists a *feasible* control trajectory that brings the system to the origin. The problem is closely associated with that of minimum time optimal control. It can be shown that the controllability set of a linear exponentially unstable system is bounded in the directions of the unstable modes. Consequently, only semi-global stability may be achieved. An elegant result for calculation

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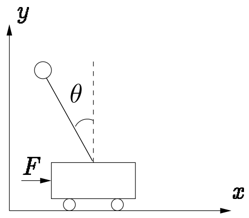

Fig. 1. A schematic picture of the inverted pendulum on cart.

of controllability sets for exponentially unstable systems as well as a method of semi-global stabilization are given in (Hu *et al.*, 2001).

For non-linear systems, the situation is different. Fewer results are available on stabilization with bounded controls, (Teel, 1996) being a notable exception. The problem of calculating controllability sets is significantly harder for non-linear systems.

Another branch of the theory deals with the problem of *anti-windup*. In this setting, a local *performance controller* is designed without taking the saturation nonlinearity into account. The problem is then to find an anti-windup modification of the controller that leaves the behavior of the local controller unaffected when there is no saturation, and limits the effects of saturation if it occurs, see for example (Rönnbäck, 1993). In (Teel and Kapoor, 1997), the problem was given a rigorous definition and solved for the case of stable linear systems. In (Teel, 1999) the anti-windup problem for exponentially unstable linear systems is addressed.

In this paper, the inverted pendulum, representing a non-linear unstable system is studied. The aim of the controller is to enable velocity tracking of the pivot point of the pendulum while ensuring stability. The controllability set of the system is explicitly characterized, and a controller based on this set is proposed. The paper is an extension of (Åkesson and Åström, 2001), where a linearized pendulum system was studied.

2. EQUATIONS OF MOTION

Consider the inverted pendulum on a cart in figure 1. Let the position of the cart be x, and the angle of the pendulum θ. Let l denote the distance from the pivot point to the center of mass of the pendulum, m_p the mass of the pendulum and J_p its moment of inertia w.r.t. the pivot point. Further, let m_c denote the mass of the cart, F the force acting on the cart and g the acceleration due to gravity. The equations of motion of the inverted pendulum may be written

$$\begin{aligned} J_p\ddot{\theta} - m_pl\ddot{x}\cos\theta - m_pgl\sin\theta = 0 \\ -m_pl\ddot{\theta}\cos\theta + (m_c+m_p)\ddot{x} - m_pl\dot{\theta}^2\sin\theta = F. \end{aligned} \tag{1}$$

By introducing the input transformation

$$\begin{aligned} F = \frac{1}{J_p}\Big[&v(m_cJ_p + m_pJ_p' + m_p^2l^2\sin^2\theta) \\ &-m_p^2gl^2\sin\theta\cos\theta + J_pm_pl\dot{\theta}^2\sin\theta\Big] \end{aligned}$$

where J_p' is the moment of inertia of the pendulum with respect to its center of mass, the control input to the system is transformed to the acceleration of the cart, v, rather than the acting force F. Notice that the transformation can be done globally in the state space since $J_p \leq m_pl^2$. Introducing the normalizations

$$\begin{aligned} &x_1 = \theta \quad x_2 = \sqrt{\frac{J_p}{m_pgl}}\,\dot{\theta} \quad x_3 = \sqrt{\frac{m_pl}{J_pg}}\,\dot{x} \\ &u = \frac{v}{g} \qquad \tau = \sqrt{\frac{m_pgl}{J_p}}\,t \end{aligned} \tag{2}$$

the dynamics of the system may be written

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin\theta + u\cos\theta \\ \dot{x}_3 &= u. \end{aligned} \tag{3}$$

Notice that the state x has been excluded, because the aim of the control system is to enable velocity control of the cart.

The equilibria of the pendulum are $x_1 = 0$ and $x_1 = \pi$ which represents a saddle (unstable) and a center (stable) respectively. Linearization of the model (3) with respect to the unstable equilibrium point $x = (0,0,0)$ is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u. \tag{4}$$

3. CONTROLLABILITY SET ANALYSIS

The controllability set plays an important role for design of controllers for unstable systems subject to input saturation, because stability is lost if the state leaves this set. A point in the state space belongs to the controllability set if there exists a feasible control signal such that the state of the system is brought to the origin. The set of all such points constitutes the controllability set.

In the following it will be assumed that the control input of the system (3) is subject to the following standard saturation

$$\mathrm{sat}_{u_0}(u) = \begin{cases} u_0 & u \geq u_0 \\ u & -u_0 \leq u \leq u_0 \\ -u_0 & u \leq -u_0. \end{cases} \tag{5}$$

The controllability set of the planar pendulum was studied in (Brufani, 1997), where the controllability set for $|x_1| \leq \frac{\pi}{2}$ was calculated. For completeness, this derivation is given below, as well as its extension to the case when $|x_1| \leq \pi$.

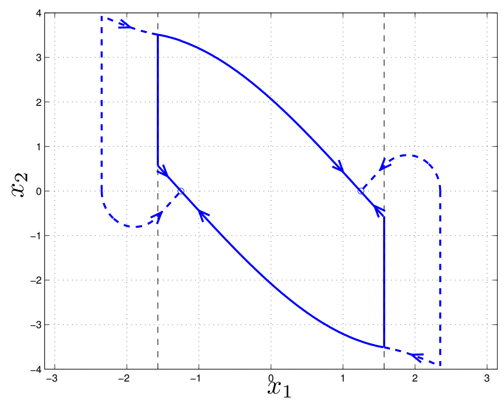

Fig. 2. Boundaries of the controllability regions for $|x| < \pi/2$ in full lines and for $|x_1| \le \alpha = 2.35$ in dashed lines. The circles show the unstable equilibria and the arrows show the direction of the vector field. In this case $u_0 = 3$.

We first notice that a for a constant acceleration u_0 there is an equilibrium at $x_1^0 = \arctan u_0$ and $x_2^0 = 0$. When the acceleration has the constant value u_0, the equation of motion of the pendulum can be integrated to give

$$\begin{aligned} \frac{1}{2}x_2^2 &= -\cos x + u_0 \sin x_1 + C \\ \frac{1}{2}x_2^2 &= \frac{-\cos x_1 \cos x_1^0 + \sin x_1 \sin x_1^0}{\cos x_0} + C \\ &= -\frac{\cos(x_1 + x_1^0)}{\cos x_1^0} + C. \end{aligned} \quad (6)$$

The controllability set is essentially given by (6). To explore the details we will consider two cases.

3.1 Case 1: $|x| < \pi/2$

This case correspond to the situation where the pendulum is never allowed to pass the horizontal plane through the pivot. In this case, the boundaries of the controllability set is given by the trajectories through the unstable equilibria $(x_1^0, 0)$ and $(-x_1^0, 0)$. Linearization around the equilibria shows that they are saddles. The trajectories are the stable solutions of (6) through the equilibria. This gives $C = \pm 1/\cos x_1^0$ and the expressions

$$f_{\pi/2}^+(x_1) = \begin{cases} \sqrt{2\frac{1-\cos(x_1 - x_1^0)}{\cos x_1^0}}, & -\frac{\pi}{2} \le x_1 \le x_1^0 \\ -\sqrt{2\frac{1-\cos(x_1 - x_1^0)}{\cos x_1^0}}, & x_1^0 \le x_1 \le \frac{\pi}{2} \end{cases}$$

for the upper boundary of the controllability region. Because of symmetry, the lower boundary is the mirror of the upper boundary, hence

$$f_{\pi/2}^-(x_1) = -f_{\pi/2}^+(-x_1). \quad (7)$$

Figure 2 shows the controllability region for this case with $u_0 = 3$ in solid curves.

3.2 Case 2 $|x| < \pi$

It follows from the analysis in (Åström and Furuta, 2000) that if the acceleration is larger than 4/3 it is possible to have a controllability set which allows the pendulum to go below the horizontal plane through the pivot. Assume that the angle is restricted to $-\alpha \le x_1 \le \alpha$. This requires that the acceleration of the pendulum is sufficiently large to swing up a pendulum at rest from the angle α. The energy analysis in (Åström and Furuta, 2000) gives the following relation between α and u_0.

$$\alpha = \pi - \arctan u_0 + \arccos\left(\frac{2u_0}{\sqrt{1+u_0^2} - 1}\right) \quad (8)$$

It is somewhat counterintuitive that the smallest acceleration $u_0 = 4/3$ is obtained for the largest α, i.e. $\alpha = \pi$. Smaller values of α requires larger acceleration.

To find the controllability set we first observe that the boundary of controllability region goes through the point $x_1 = \pm\alpha$, $x_2 = 0$. In the case of $\alpha > 0$, the acceleration is positive for $|x| > \pi/2$ and negative for $|x| < \pi/2$. Using the energy equation (6) and matching the parameter C to the boundary conditions we obtain the following expression for the upper boundary $x_2 = f^+(x_1)$ of the controllability region.

$$f_\alpha^+(x_1) = \begin{cases} \sqrt{-2\frac{\cos(x_1 + x_1^0)}{\cos x_1^0} + C_1} & \text{if } \frac{\pi}{2} \le x_1 \le \alpha \\ \sqrt{-2\frac{\cos(x_1 - x_1^0)}{\cos x_1^0} + C_2} & \text{if } -\frac{\pi}{2} < x_1 < \frac{\pi}{2} \\ \sqrt{-2\frac{\cos(x_1 + x_1^0)}{\cos x_1^0} + C_3} & \text{if } -\alpha \le x_1 \le -\frac{\pi}{2} \end{cases} \quad (9)$$

where

$$C_1 = 2\frac{\cos(\alpha + x_1^0)}{\cos x_1^0}$$

$$C_2 = 2\frac{\cos(\pi/2 - x_1^0)}{\cos x_1^0} - 2\frac{\cos(\pi/2 + x_1^0)}{\cos x_1^0} + C_1$$

$$C_3 = 2\frac{\cos(-\pi/2 + x_1^0)}{\cos x_1^0} - 2\frac{\cos(-\pi/2 - x_1^0)}{\cos x_1^0} + C_2$$

The lower boundary of the controllability region, $f_\alpha^-(x_1)$, is defined as in equation (7). In Figure 2, the controllability region in the case of $u_0 = 3$ and α given by (8) is shown in dashed curves. Figure 3 shows controllability regions for $|x_1| \le \pi$. Notice that the region grows for larger values of u_0. The size of the controllability set depends on the saturation limit, u_0, and on the permissible range of x_1. The entire state space is the controllability set if there are no restrictions on x_1.

4. A STABILIZING CONTROLLER

As a first step towards the design of a controller enabling tracking of reference commands for the

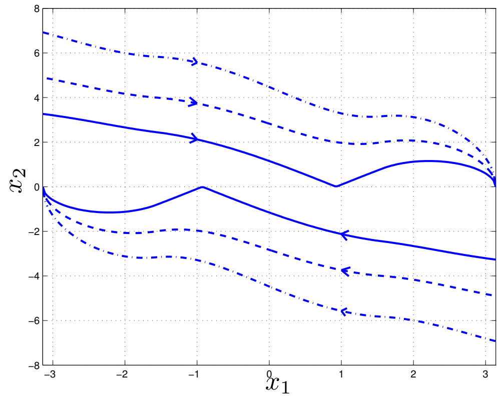

Fig. 3. Boundaries of the controllability regions for $|x_1| \leq \pi$ and $u_0 = 4/3$ (solid), $u_0 = 3$ (dashed) and $u_0 = 6$ (dash-dotted).

cart velocity, a stabilizing controller for the pendulum states x_1 and x_2 will be developed. It is clear from the previous analysis that such a controller may only stabilize the system in (a subset of) the controllable region, which is *a priori* known.

In the following, the case of $|x_1| \leq \pi/2$ will be considered. A simple but effective way to design such a controller is to use a linear design method based on the linearized model (4), resulting in a linear control law

$$u = \text{sat}_{u_0}(-l_1 x_1 - l_2 x_2), \qquad (10)$$

which locally stabilizes also the non-linear system (3). It is not clear that such a controller also achieves semi-global stabilization. Using an LQ design, however, it is possible to prove semi-global stability, given that the controller fulfills the following two sufficient conditions: Firstly, the region of the state space where the controller operates linearly must be entirely contained in the controllability set. Secondly, the solution of the algebraic Riccati equation, P, should produce a Lyapunov function candidate, $V(x) = x^T P x$, such that there is a sufficiently large region defined by $V(x) \leq c$ in which $\dot{V}(x) < 0$. From the Lyapunov stability theorem it follows that $\{x : x^T P x \leq c \,|\, x \to 0\}$. The first condition is to make sure that close to the boundaries of the controllability set, the controller is saturated. In this situation, trajectories will approach the center of the controllability set and the linear region. The second condition is to ensure that all trajectories starting outside of the ellipse defined by $x^T P x \leq c$ will actually enter it. It is not difficult to find a controller that fulfills the requirements. A typical situation is shown in Figure 4. As can be seen, an ellipse defined by $x^T P x \leq c$ (bold) can be fitted inside the region in which $\dot{V} \leq 0$ (dash-dotted bold). Further, the controller operates in linear mode in the region defined by the non-bold dash-dotted lines. It hence follows that all trajectories starting inside of the controllability region (dashed bold)

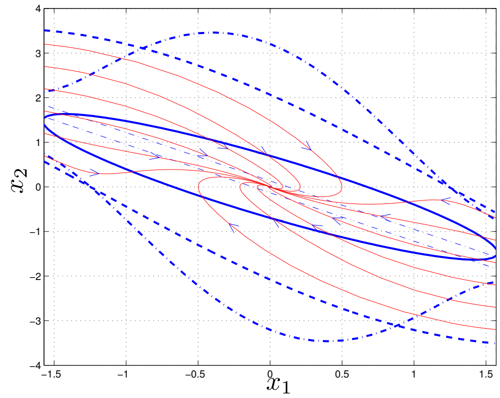

Fig. 4. The region of attraction of the linear saturated controller.

will inevitably enter the ellipse. Semi-global stability follows. By tuning the weights in the LQR-design, it is possible to shape the local behavior of the controller, and also obtain an ellipse that is better aligned with the controllability region. However, stability and the region of attraction of the controller will be unaffected as long as the two requirements stated above are fulfilled.

5. TRACKING

In this section we will design a controller that permits manual control of the cart velocity while stabilizing the pendulum. Consider the control law

$$u = \text{sat}_{u_0}(-l_1 x_1 - l_2 x_2 + m), \qquad (11)$$

where m represents the tracking term which will be defined below. First assume that m is constant. The equilibria of the perturbed system are then given by the equation

$$\tan x_1 = \text{sat}(-l_1 x_1 + m).$$

The curve representing the saturation is shifted horizontally when the manual control is changed. The number of equilibria then depends on m. In Figure 5, there are three equilibria marked by circles. The middle equilibrium is (controlled) stable and the others are unstable. For large positive or negative values of m there is only one equilibrium which is unstable. A necessary condition for semi-global stability is that the system has three equilibria for a constant m. To maintain stability it is necessary that the manual control actions are limited. From the point of view of performance it is desirable that the limits on the authority of manual control are as wide as possible. From the previous analysis, it is clear that a constant angle x_1^0, corresponds to a constant acceleration u^0. To enable fast tracking, i.e. large acceleration towards the reference velocity, it is thus desirable to allow for large values of m. By selecting the tracking term m as

$$m = \text{sat}_a(l_3(r - x_3)) \qquad (12)$$

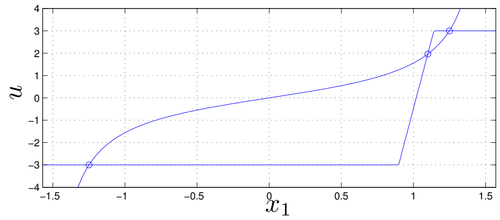

Fig. 5. Equilibria for the system subject to saturated control and constant reference tracking term m.

where r is the reference value of the cart velocity state x_3, it is possible to capture the trade-off between stability and performance. The feedback gain l_3 is conveniently calculated using LQR-design, that gives the desired local behavior. The choice of the saturation limit a is guided by

Lemma 1. Consider

$$a = \begin{cases} a^+(x_1) = l_2(f^-_{\pi/2}(x_1) + d) - u_0 + l_1 x_1 \\ a^-(x_1) = -l_2(f^+_{\pi/2}(x_1) - d) - u_0 - l_1 x_1 \end{cases} \quad (13)$$

where $a^+(x_1)$ and $a^-(x_1)$ are the positive and negative saturation limits of (12) and $0 \leq d \leq d_{max}$. Then the region bounded by $f^+_{\pi/2}(x_1) - d$ and $f^-_{\pi/2}(x_1) + d$ is positively invariant, i.e., trajectories starting in this region will remain in it regardless of the reference value r.

Proof: The proof is a straight forward application of Nagumos theorem, stating that for a closed set $\mathcal{S} \in R^n$, $\mathcal{S}$ is positively invariant for the system $\dot{x} = f(x)$, if and only if the field $f(x)$ points to the interior of $\mathcal{S}$ for all $x \in \partial\mathcal{S}$. See (Blanchini, 1999) for details.

The controller (11) with saturation limits defined by (13), operates in saturated mode whenever $x_2 \geq f^+_{pi/2}(x_1) - d$ or $x_2 \leq f^-_{pi/2}(x_1) + d$. It then follows from a phase plane argument that trajectories starting at the boundary curves $f^+_{pi/2}(x_1) - d$ and $f^-_{pi/2}(x_1) + d$ will approach the interior of the region. The same argument can be applied for the vertical line segments bounding the region at $x_1 = \pm\pi/2$.

Remark 1. To avoid an overly conservative design, the saturation limit a should depend on the angle x_1.

Remark 2. The value of d is used to control the size of the invariant region, yielding a safety margin for robustness. However, the region does not exist if d is too large.

Remark 3. The boundary functions $f^+_{\pi/2}(x_1)$ and $f^-_{\pi/2}(x_1)$ used in (13) can be approximated by simpler expressions, as long as the condition of Nagumos theorem are fulfilled.

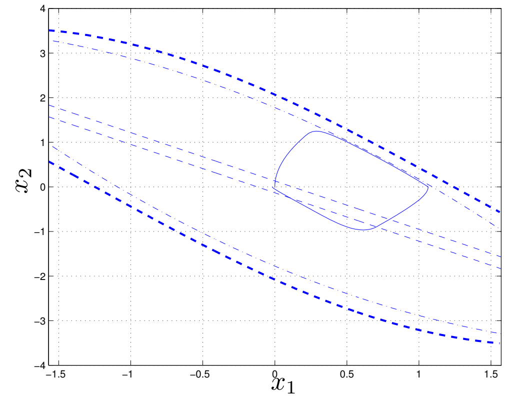

Fig. 6. Phase portrait of the system (3) subject to tracking control.

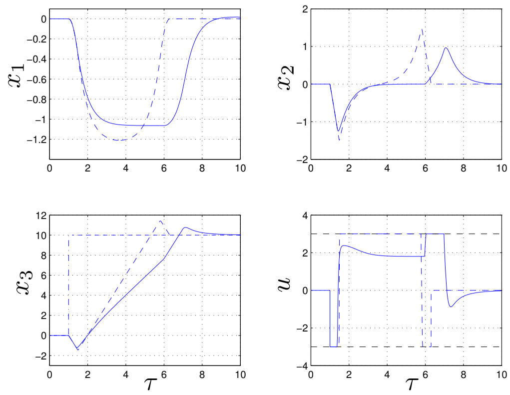

Fig. 7. Trajectories for a step reference change for the proposed controller (11) (solid) and the time-optimal solution (dashed).

Figures 6 and 7 show the phase portrait and the corresponding state trajectories when a step reference sequence is applied. Figure 6 shows that the state remains in the specified invariant set. Notice that the linear region marked by dashed lines is small compared to the invariant region. A strategy that avoids saturation is thus very conservative. The tracking behavior of x_3 in Figure 7 is reasonable as shown by a comparison with minimum time trajectories. The time optimal trajectories give a faster response for large set point changes, but lacks the robustness of the proposed feedback controller.

6. EXTENSIONS

The analysis above is valid for the system (3), where limited acceleration of the pivot was assumed. The true problem, however, is to device a controller for the system (1), assuming input saturation on F, i.e. limited force. This problem can be solved using insight gained from the analysis in the previous sections.

The controllability set of (1) subject to the input nonlinearity (5) can be found numerically through

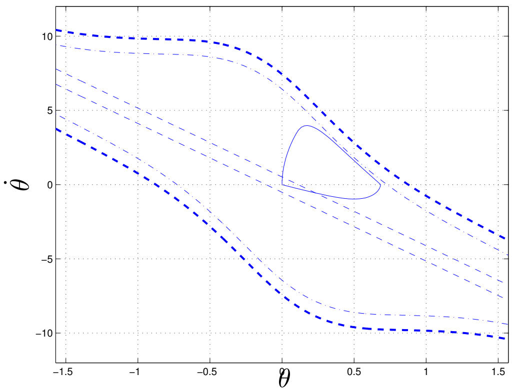

Fig. 8. Phase portrait of the system (1) subject to tracking control.

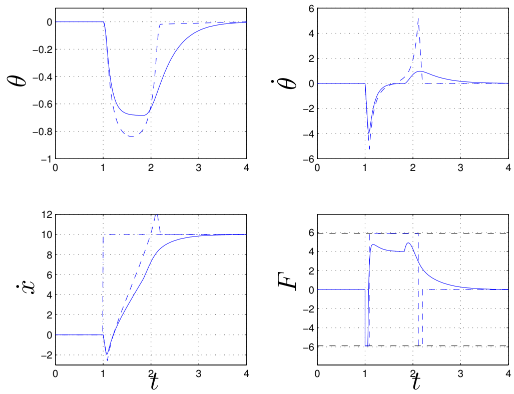

Fig. 9. Trajectories for a step reference change for the proposed controller (solid) and the time-optimal solution (dashed).

simulation. The set is indicated in Figure 8 in bold curves. Using this set, the controller (11) with saturation limits (13) can be employed. As previously, the controller renders the region defined by the boundary functions $f^+_{\pi/2}$ and $f^-_{\pi/2}$ and the parameter d invariant. Notice that the invariance argument holds also for approximations of the boundary functions, see Remark 3.

Figure 8 shows a typical phase portrait. The pendulum states do not leave the invariance region marked by dash dotted curves. Figure 9 shows the step response of the system. The minimum time solution is shown in dashed curves. Notice the different time scales in Figures 7 and 9, which is due to scaling. The following numerical values of the parameters of the system (1) were used in the simulations: $m_p = 0.3$ kg, $l = 0.5$ m, $m_c = 0.2$ kg, $g = 9.81\text{m/s}^2$ and $J_p = m_p l^2$.

7. CONCLUSIONS

An explicit characterization of the controllability set for an inverted pendulum on a cart subject to limited acceleration of the pivot has been given. A controller enabling tracking of constant pivot velocity references while stabilizing the pendulum has been proposed. A single parameter, d, is used to trade performance and robustness of the controller. The controller has also been generalized to the case of the actual pendulum system subject to limited force acting on the cart.

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