MOMENT ASYMPTOTIC STABILITY OF STOCHASTIC HYBRID DELAY SYSTEMS [1]

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Abstract: In this paper we investigate the moment asymptotic stability for the nonlinear stochastic hybrid delay systems. Sufficient criteria on the stabilization and robust stability are also established for linear stochastic hybrid delay systems.

Keywords: Hybrid delay system, moment asymptotic stability, stabilization, robust stability.

1. INTRODUCTION

Many physical systems are subject to frequent unpredictable structural changes, such as random failures, repairs of sudden environment disturbances, abrupt variation of the operating point on a nonlinear plant, etc. Such systems can best be modeled as hybrid systems or jump systems, with a state vector that has two components, $x(t)$ which is real set $S = \{1, 2, \ldots, N\}$. The first component is in general referred to as the state, and the second one is regarded as the mode.

The stability and control theory of hybrid systems has received a lot of attention over the past decade, see, for example, (Aubin, 1994; Ye *et al.*, 1998; Branicky, 1998; Michel and Hu, 1999; De Carlo *et al.*, 2000; Liberzon and Morse, 1999) and the references therein. This study has to a large extent been motivated by an interest in embedded and digital control. Embedded systems are naturally hybrid, since by definition they involve the interaction of a digital device with a predominantly analog environment. The recognition that in addition to discrete and continuous dynamics, embedded systems often involve considerable levels of uncertainty has recently motivated a research effort into stochastic extensions of hybrid systems see, for example, (Bensoussan, 2000; Glover and Lygeros, 2004; Hu *et al.*, 2004) to name a few. Much of the work in this area has been driven by an interest in control of communication networks (Hespanha, 2004), or control of distributed systems over communication networks. A number of stability results have been obtained for classes of stochastic hybrid systems, such as piecewise deterministic Markov processes (Davis,

[1] Work supported by the European Commission under COLUMBUS, IST-2001-38314 and project HYBRIDGE, IST-2001-32460

1993) and variants of switching diffusion processes (Yuan and Lygeros, 2004)

One feature of such networked embedded systems that has been overlooked in the stochastic stability literature is the presence of delays. Delays are unavoidable in control problems that involve communication networks; what is worse, the delay typically depends on the traffic of the network, and is therefore itself uncertain. It is well-known that time-delay often results in instability and poor performance.

The present paper attempts to address this problem by studying the stability and control theory of stochastic hybrid delay systems. In earlier work (Yuan and Lygeros, 2004) the authors have provided conditions to ensure the asymptotic stability and boundedness of a class of delay switching diffusion, where the evolution of the discrete state is governed by a Markov chain whose transition rates depend on the continuous state. In this paper we derive conditions for asymptotic moment stability for a simpler class of systems, ones where the switching sequence is fixed. We then use these conditions to derive stabilization and robust stability results for the special case where the diffusion process is governed by switched linear dynamics.

2. STOCHASTIC HYBRID DELAY SYSTEMS

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space with a filtration $\mathcal{F}_t$ which is right continuous and $\mathcal{F}_0$ contains all P-null sets. $B(t) = (B_t^1, \ldots, B_t^m)^T$ denotes an m-dimensional Brownian motion defined on this probability space. Let $\mathbb{Z}_+$ denote all nonnegative integer numbers. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions on $[-\tau, 0]$ with norm $\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$ (without any confusion with $\|\varphi\|$). If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \le \theta \le 0\}$. For any fix t, if $x(t+\theta)$ is a continuous $\mathbb{R}^n$-valued stochastic processes on $\theta \in [-\tau, 0]$, we let $x_t = \{x(t+\theta) : -\tau \le \theta \le 0\}$. For any real valued function $\gamma : [-\tau, +\infty) \to \mathbb{R}$, denote by $\gamma^+(t) = \max\{\gamma(t), 0\}$ and $\gamma^-(t) = -\min\{\gamma(t), 0\}$. Let $\sigma(t)$ be a right continuous piecewise constant function taking values in a finite state space $S = \{1, 2, ..., N\}$. Let $\{\rho_k\}_{k=0}^{\infty}$ be a sequence satisfying $\rho_0 = 0, \tau \le \rho_{k+1} - \rho_k$. Let $H_{ij} : C([-\tau, 0]; \mathbb{R}^n) \to C([-\tau, 0]; \mathbb{R}^n)$ be a function describing the discontinuous change in state at the transition times ρ_i.

Consider a stochastic hybrid delay systems of the form

$$\begin{cases} dx(t) = f(x(t), x(t-\tau), t, \sigma(t))dt \\ \qquad\qquad +g(x(t), x(t-\tau), t, \sigma(t))dB(t), \\ \sigma(t) = i_k, \quad \rho_k \le t < \rho_{k+1}, i_k \in S, k \in \mathbb{Z}_+, \\ x_{\rho_k} = H_{i_{k-1} i_k}(x_{\rho_k^-}), \end{cases} \tag{1}$$

on $t \ge 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $\sigma(0) = i_0$, where

$$f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$$

and

$$g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n \times m}.$$

Throughout this paper, we assume that $f(0, 0, t, i) \equiv 0$ and $g(0, 0, t, i) \equiv 0$ for $t \ge 0$ and $i \in S$. This implies that Eq. (1) has the solution $x(t) \equiv 0$ corresponding to the initial value $x_0 = 0$. This solution is called the trivial solution. The main aim of this paper is to investigate the stability of this trivial solution, in the following sense.

Definition 1. (i) The trivial solution of Eq.(1) is said to be stochastically stable in mean square if for any $\varepsilon > 0$, there exists a $\delta_1 = \delta_1(\varepsilon)$ such that

$$E|x(t; \xi, i_0)|^2 < \varepsilon \tag{2}$$

whenever $t \ge 0$ and $\|\xi\| < \delta_1$.

(ii) The trivial solution of Eq.(1) is said to be stochastically asymptotically stable in mean square if it is stochastically stable in mean square and there exists a δ_2 such that

$$\lim_{t \to \infty} E|x(t; \xi, i_0)|^2 = 0 \tag{3}$$

whenever $\|\xi\| < \delta_2$.

Subsequently we shall impose the following hypothesis:

Assumption 2.1. Given any initial data $x_0 = \{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, $\sigma(0) = i_0$, Eq. (1) has a unique solution denoted by $x(t; \xi, i_0)$ on $t \ge 0$. Moreover, there is an $M > 0$ such that

$$2x^T f(x, y, t, i) + 33|g(x, y, i, t)|^2 \le M(|x|^2 + |y|^2)$$

for $x, y \in \mathbb{R}^n, t \ge 0$ and $i \in S$.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denote the family of all non-negative functions $V(x, t, i) : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}_+$ which are continuously differentiable twice in x and once in t. For $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, define the operator $LV : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$ by

$$LV(x, y, t, i) = V_t(x, t, i) + V_x(x, t, i) f(x, y, t, i) \\ + \frac{1}{2}\mathrm{trace}[g^T(x, y, t, i) V_{xx}(x, t, i) g(x, y, t, i)], \tag{4}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t},$$

$$V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \dots, \frac{\partial V(x,t,i)}{\partial x_n}\right),$$

$$V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

3. MOMENT STABILITY

In this section we shall discuss stability in mean square for Eq. (1). Consider a general nonlinear stochastic differential delay equations

$$\begin{aligned} dx(t) = {} & F(x(t), x(t-\tau), t)dt \\ & + G(x(t), x(t-\tau), t)dB(t), \end{aligned} \quad (5)$$

on $t \geq 0$ with initial data $x_0 = \xi$. We assume that there exists a unique solution for Eq. (5) and denoted by $x(t;\xi)$. Moreover we assume

Assumption 3.1. There is an $h > 0$ such that

$$2x^T F(x,y,t) + 33|G(x,y,t)|^2 \leq h(|x|^2 + |y|^2)$$

for $x, y \in \mathbb{R}^n, t \geq 0$.

Lemma 3.1. Let c_1, c_2 be positive number and γ_1, γ_2 are real function such that $\gamma_1(s) : \mathbb{R}_+ \to \mathbb{R}$, $\gamma_2(s) : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that there exists a function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ such that

$$c_1|x|^2 \leq V(x,t) \leq c_2|x|^2 \quad (6)$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, and

$$\mathcal{L}V(x,y,t) \leq \gamma_1(t)|x|^2 + \gamma_2(t-\tau)|y|^2 \quad (7)$$

for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, where

$$\begin{aligned} \mathcal{L}V(x,y,t) = {} & V_t(x,t) + V_x(x,t)F(x,y,t) \\ & + \frac{1}{2}\mathrm{trace}[G^T(x,y,t)V_{xx}(x,t,i)G(x,y,t)], \end{aligned}$$

$\gamma_1(\theta) = \gamma_1(0), \gamma_2(\theta) = \gamma_2(0), -\tau \leq \theta \leq 0$. Then for any $t > t_1 \geq 0$, we have

$$\begin{aligned} E|x(t)|^2 \leq {} & \frac{\int_{t_1-\tau}^{t_1} \gamma_2(s)ds + c_2}{c_1} \\ & \times E\|x_{t_1}\|^2 \exp\left(\int_{t_1}^{t} (\gamma_1(s) + \gamma_2(s))ds\right). \end{aligned} \quad (8)$$

Moreover, if $t - t_1 \geq \tau$,

$$\begin{aligned} E\|x_t\|^2 \leq {} & 2(1+2h\tau)\frac{\int_{t_1-\tau}^{t_1} \gamma_2(s)ds + c_2}{c_1} \\ & \times E\|x_{t_1}\|^2 \exp\left(\int_{t_1}^{t} (\gamma_1(s) + \gamma_2(s))ds\right). \end{aligned} \quad (9)$$

Proof Fix any $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ and write $x(t;\xi) = x(t)$. Define $U(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+)$ by

$$U(x,t) = V(x,t) + \int_{t-\tau}^{t} \gamma_2(s)E|x(s)|^2 ds. \quad (10)$$

By Itô's formula (see Mao (Mao, 1997)), we can derive that

$$\begin{aligned} c_1 E|x(t)|^2 & \leq EU(x(t),t) = EU(x(t_1), t_1) \\ & + E\int_{t_1}^{t} \mathcal{L}U(x(s), x(s-\tau), s)ds \\ & \leq (\int_{t_1-\tau}^{t_1} \gamma_2(s)ds + c_2)E\|x_{t_1}\|^2 \\ & + \int_{t_1}^{t} (\gamma_1(s) + \gamma_2(s))E|x(s)|^2 ds. \end{aligned} \quad (11)$$

By Gronwall inequality, we have

$$\begin{aligned} E|x(t)|^2 \leq {} & \frac{\int_{t_1-\tau}^{t_1} \gamma_2(s)ds + c_2}{c_1} \\ & \times E\|x_{t_1}\|^2 \exp\left(\int_{t_1}^{t} (\gamma_1(s) + \gamma_2(s))ds\right). \end{aligned} \quad (12)$$

For any $\theta \in [-\tau, 0]$ and $t - t_1 \geq \tau$, by Itô's formula again, we obtain

$$\begin{aligned} & |x(t+\theta)|^2 = |x(t-\tau)|^2 \\ & + \int_{t-\tau}^{t+\theta} \Big(2x^T(s)F(x(s), x(s-\tau), s) \\ & + |G(x(s), x(s-\tau), s)|^2\Big)ds \\ & + 2\int_{t-\tau}^{t+\theta} x^T(s)G(x(s), x(s-\tau), s)dB(s). \end{aligned} \quad (13)$$

Using the Burkholder-Davis-Gundy's inequality (see Mao (Mao, 1997)) we can show that

$$\begin{aligned} & E \sup_{-\tau\leq\theta\leq 0} \left| \int_{t-\tau}^{t+\theta} x^T(s)G(x(s), x(s-\tau), s)dB(s) \right| \\ & \leq \sqrt{32}E\left(\int_{t-\tau}^{t} |x^T(s)G(x(s), x(s-\tau), s)|^2 ds\right)^{\frac{1}{2}} \\ & \leq \sqrt{32}E\Big(\sup_{-\tau\leq\theta\leq 0} |x(t+\theta)|^2 \\ & \times \int_{t-\tau}^{t} |G(x(s), x(s-\tau), s)|^2 ds\Big)^{\frac{1}{2}} \\ & \leq \frac{1}{4}E \sup_{-\tau\leq\theta\leq 0} |x(t+\theta)|^2 \\ & + 32E\int_{t-\tau}^{t} |G(x(s), x(s-\tau), s)|^2 ds. \end{aligned} \quad (14)$$

Substituting (14) into (13) and using Assumption 3.1, we obtain

$$
\begin{aligned}
&E \sup_{-\tau\le\theta\le 0} |x(t+\theta)|^2 \\
&\le 2E|x(t-\tau)|^2 \\
&+ 2E\int_{t-\tau}^{t} \Big(2x^T(s)F(x(s),x(s-\tau),s) \\
&+ 33|G(x(s),x(s-\tau),s)|^2\Big)ds \\
&\le 2E|x(t-\tau)|^2 \\
&+ 2h\int_{t-\tau}^{t} \Big(E|x(s)|^2 + E|x(s-\tau)|^2\Big)ds. \qquad (15)
\end{aligned}
$$

The required assertion (9) follows (12) and (15). The proof is therefore complete. □

We can now state our main result.

Theorem 3.1. Let Assumption 2.1 hold. Let α, c_3, c_4 be positive numbers and let $\gamma_3, \gamma_4, \gamma_5$ be real function such that $\gamma_3(s) : \mathbb{R}_+ \to \mathbb{R}$, $\gamma_4(s), \gamma_5(s) : \mathbb{R}_+ \to \mathbb{R}_+$ and $\int_{\rho_k}^{\rho_{k+1}} (\gamma_3+\gamma_4)^+(s)ds \le \alpha$ for all $k \ge 0$. Assume that there exists a function $V(x,t,i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ such that

(i) $c_3|x|^2 \le V(x,t,i) \le c_4|x|^2$ for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$;
(ii) $LV(x,y,t,i) \le \gamma_3(t)|x|^2 + \gamma_4(t-\tau)|y|^2$ for all $(x,y,t,i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S$;
(iii) $\|H_{i_{k-1}i_k}(x_{\rho_k^-})\|^2 \le \dfrac{c_3\gamma_5(\rho_k)}{\int_{\rho_k-\tau}^{\rho_k}\gamma_4(s)ds + c_4} \times \frac{1}{2(1+2M\tau)}\|x_{\rho_k^-}\|^2$ for all $k = 0,1,2,\ldots$;
(iv) $\sum_{k=0}^{+\infty} \gamma^+(\rho_k) < +\infty$, where $\gamma(\rho_k) = \ln(\gamma_5(\rho_k)) + \int_{\rho_k}^{\rho_{k+1}} (\gamma_3+\gamma_4)^+(s)ds$ for all $k = 0,1,2,\ldots$ and $\gamma_3(\theta) = \gamma_3(0), \gamma_4(\theta) = \gamma_4(0), \theta \in [-\tau, 0]$.

Then the trivial solution of Eq. (1) is stochastically stable in mean square.

Proof Fix any $\xi \in C^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n), i_0 \in S$, write $x(t;\xi,i_0) = x(t)$ for simplicity. For any $\varepsilon > 0$, let

$$
\begin{aligned}
\delta =& \frac{c_3}{2(1+2h\tau)\big(\int_{-\tau}^{\infty}\gamma_4(s)ds + c_4\big)} \\
&\times \varepsilon e^{-(\alpha+\sum_{k=0}^{\infty}\gamma^+(\rho_k))}. \qquad (16)
\end{aligned}
$$

If $E\|\xi\|^2 < \delta$. Let $t \in [\rho_0, \rho_1)$, by Lemma 3.1, we have

$$
\begin{aligned}
E|x(t)|^2 \le& \frac{\tau\gamma_4(0)+c_4}{c_3} E\|\xi\|^2 \\
&\times \exp\left(\int_0^t (\gamma_3(s)+\gamma_4(s))ds\right) \\
&\le \varepsilon \exp\left(-\sum_{k=1}^{\infty}\gamma^+(\rho_k)\right) \qquad (17)
\end{aligned}
$$

and

$$
\begin{aligned}
E\|x_{\rho_1^-}\|^2 \le& 2(1+2\tau)\frac{\tau\gamma_4(0)+c_4}{c_3} E\|\xi\|^2 \\
&\times \exp\left(\int_0^t (\gamma_3(s)+\gamma_4(s))ds\right) \\
&\le \varepsilon \exp\left(-\sum_{k=1}^{\infty}\gamma^+(\rho_k)\right). \qquad (18)
\end{aligned}
$$

In the following, we will prove

$$
E|x(t)|^2 \le \varepsilon \exp\left(-\sum_{k=l+1}^{\infty}\gamma^+(\rho_k)\right) \qquad (19)
$$

for all $t \in [\rho_l, \rho_{l+1})$ and

$$
E\|x(t_{\sigma_{l+1}^-})\|^2 \le \varepsilon \exp\left(-\sum_{k=l+1}^{\infty}\gamma^+(\rho_k)\right). \qquad (20)
$$

By (17) and (18) , we know that (19) and (20) holds when $l = 0$. We now assume (19) and (20) are true when $l = i$. Using Lemma 3.1 and condition (iii), for $t \in [\rho_{i+1}, \rho_{i+2})$

$$
\begin{aligned}
E|x(t)|^2 \le& \frac{\int_{\rho_{i+1}-\tau}^{\rho_{i+2}}\gamma_4(s)ds + c_4}{c_3} \\
&\times E\|x(\rho_{i+1})\|^2 \exp\left(\int_{\rho_{i+1}}^t (\gamma_3(s)+\gamma_4(s))ds\right) \\
&\le \varepsilon \exp\left(-\sum_{k=i+2}^{\infty}\gamma^+(\rho_k)\right) \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
E\|x_{\rho_{i+2}^-}\|^2 \le& 2(1+2\tau)\frac{\int_{\rho_{i+1}-\tau}^{\rho_{i+2}}\gamma_4(s)ds + c_4}{c_3} \\
&\le \varepsilon \exp\left(-\sum_{k=i+2}^{\infty}\gamma^+(\rho_k)\right). \qquad (22)
\end{aligned}
$$

By Mathematical induction, (19) and (20) are true and therefore we have

$$
E|x(t)|^2 < \varepsilon \quad \text{for all } t \ge 0.
$$

The proof is therefore complete. □

Theorem 3.2. Under the conditions of Theorem 3.1, if $\sum_{k=0}^{+\infty}\tilde{\gamma}^-(\rho_k) = +\infty$, where $\tilde{\gamma}^-(\rho_k) = \int_{\rho_k}^{\rho_{k+1}}(\gamma_3+\gamma_4)^-(s)ds$ for all $k = 0,1,2,\ldots$, then the trivial solution of Eq.(1) is stochastically asymptotically stable in mean square.

Proof We know from Theorem 3.1 that the trivial solution of Eq.(1) is stochastically stable in mean square. For any $\varepsilon > 0$, let δ is defined by (16). If $E\|\xi\|^2 < \delta$ and $t \in [\rho_0, \rho_1)$, in the same way as the proof of Theorem 3.1, we have

$$
\begin{aligned}
&E|x(t)|^2 \\
&\le \varepsilon \exp\left(-\sum_{k=1}^{\infty}\gamma^+(\rho_k) - \int_0^t (\gamma_3+\gamma_4)^-(s)ds\right) \qquad (23)
\end{aligned}
$$

and

$$E\|x_{\rho_1^-}\|^2 \le \varepsilon \exp\left(-\sum_{k=1}^{\infty}\gamma^+(\rho_k) - \int_0^t (\gamma_3+\gamma_4)^-(s)ds\right). \quad (24)$$

In the same way as the proof of Theorem 3.1, we can show

$$E|x(t)|^2 \le \varepsilon \exp\Big(-\sum_{k=i+2}^{\infty}\gamma^+(\rho_k) - \sum_{k=0}^{i}\tilde{\gamma}^-(\rho_k) - \int_{\rho_{i+1}}^t (\gamma_3+\gamma_4)^-(s)ds\Big) \quad (25)$$

whenever $t \in [\rho_{i+1}, \rho_{i+2})$.

since $\sum_{k=0}^{+\infty}\tilde{\gamma}^-(\rho_k) = +\infty$, we must have

$$\lim_{t\to\infty} E|x(t)|^2 = 0,$$

as required. □

4. STABILIZATION OF HYBRID LINEAR DELAY SYSTEMS

For systems where control inputs are available, a related question that is of great interest is stabilization (cf. (Gao and Ahmed, 1987; Moerder *et al.*, 1989; Willems and Willems, 1976)). The aim here is to select values for the control inputs (typically through a feedback mechanism) that ensure that the closed loop system possesses desirable stability properties. There are many results on stabilization for stochastic hybrid systems, which are concerned with the design of feedback controls under which the underlying equations become asymptotically stable in moment e.g. in mean square (cf. (Tomlin *et al.*, 2000; Wicks *et al.*, 1998)). From the mathematical point of view, the technique used in these papers is the method of the Lyapunov functionals and the conditions imposed are used to guarantee the diffusion operator acting on the Lyapunov functionals is negative-definite and hence follows the asymptotic stability in moment. To the best of our knowledge, there is only few results on stabilization for stochastic hybrid *delay* systems. In this section we will discuss the stabilization by feedback control in the sense that the underlying stochastic hybrid delay systems will become stable in moment.

Let us consider the following linear delay system

$$\begin{cases} dx(t) \\ \quad = [A(\sigma(t))x(t) + F(\sigma(t))x(t-\tau) + C(\sigma(t))u(t)]dt \\ \qquad + \sum_{l=1}^{m}[D_l(\sigma(t))x(t) + E_l(\sigma(t))x(t-\tau)]dB_l(t), \\ \sigma(t) = i_k, \quad \rho_k \le t < \rho_{k+1}, i_k \in S, k \in \mathbb{Z}_+, \\ x_{\rho_k} = H_{i_{k-1}i_k}(x_{\rho_k^-}). \end{cases} \quad (26)$$

Here u is an $\mathcal{F}_t$-measurable and $\mathbb{R}^p$-valued control law. For each mode $\sigma(t) = i \in S$, we write $A(i) = A_i$ etc. for simplicity, and A_i, F_i, D_{k_i}, E_{ki} are all $n \times n$ constant matrices while C_i is an $n \times p$ matrix.

The main aim of this section is to design a switched delay-independent memoryless state feedback controller of the form

$$u(t) = H(\sigma(t))x(t)$$

based on the state $x(t)$ and the mode $\sigma(t)$, such that the following closed-loop system of (26)

$$\begin{cases} dx(t) = [A(\sigma(t))x(t) + F(\sigma(t))x(t-\tau) \\ \quad + C(\sigma(t))H(\sigma(t))x(t)]dt \\ \quad + \sum_{l=1}^{m}[D_l(\sigma(t))x(t) + E_l(\sigma(t))x(t-\tau)]dB_l(t), \\ \sigma(t) = i_k, \quad \rho_k \le t < \rho_{k+1}, i_k \in S, k \in \mathbb{Z}_+, \\ x_{\rho_k} = H_{i_{k-1}i_k}(x_{\rho_k^-}). \end{cases} \quad (27)$$

becomes stochastically asymptotically stable in mean square. Here, for each mode $\sigma(t) = i \in S$, $H(i) = H_i$ is a $p \times n$ matrix. Let Q_i be positive definite matrix and denote by

$$\underline{\lambda}_i =: \lambda_{\min}\Big(-Q_iA_i - A_i^TQ_i - Q_iC_iH_i - (C_iH_i)^TQ_i - Q_iF_iF_i^TQ_i - 2\sum_{k=1}^{m}D_{ki}^TQ_iD_{ki}\Big)$$

and

$$\bar{\lambda}_i =: \lambda_{\max}\left(I + 2\sum_{k=1}^{m}E_{ki}^TQ_iE_{ki}\right).$$

Theorem 4.1. Let the condition (iii) of Theorem 3.1 hold. If there exist N positive definite matrices Q_i such that $\underline{\lambda}_i > \bar{\lambda}_i$ for all $i \in S$, then Eq. (26) is stochastically asymptotically stable with controller $u(t) = H(r(t))x(t)$.

The proof is based on Theorem 3.1.

5. ROBUST STABILITY OF HYBRID LINEAR DELAY SYSTEMS

In many practical situations the system parameters can only be estimated with a certain degree of uncertainty. The robustness of stability is therefore an important issue in the stability theory (cf. (Mao, 1997; Mao, 2000)).

Let us now consider the following equation

$$\begin{cases} dx(t) = [(A(\sigma(t)) + \Delta A(\sigma(t)))x(t) \\ \quad + (F(\sigma(t)) + \Delta F(\sigma(t)))x(t-\tau)]dt \\ \quad + \sum_{l=1}^{m}[D_l(\sigma(t))x(t) + E_l(\sigma(t))x(t-\tau)]dB_l(t), \\ \sigma(t) = i_k, \quad \rho_k \le t < \rho_{k+1}, i_k \in S, k \in \mathbb{Z}_+, \\ x_{\rho_k} = H_{i_{k-1}i_k}(x_{\rho_k^-}). \end{cases} \quad (28)$$

We shall simply write $A(i) = A_i$ etc. Assume that

$$\Delta A_i = M_i H_i N_i \quad \text{and} \quad \Delta F_i = G_i H_i R_i,$$

where $M_i, G_i \in \mathbb{R}^{n\times p}$ and $N_i, R_i \in \mathbb{R}^{q\times n}$ are known real constant matrices but H_i's are unknown $p \times q$-matrices such that

$$H_i^T H_i \leq I, \quad \forall i \in S. \tag{29}$$

Equation (28) can be regarded as the perturbed system of the following hybrid linear delay system

$$\begin{cases} \dot{x}(t) = [(A(\sigma(t)) + \Delta A(\sigma(t)))x(t) \\ \quad +(F(\sigma(t)) + \Delta F(\sigma(t)))x(t-\tau)]dt, \\ \sigma(t) = i_k, \quad \rho_k \leq t < \rho_{k+1}, i_k \in S, k \in \mathbb{Z}_+, \\ x_{\rho_k} = H_{i_{k-1}i_k}(x_{\rho_k^-}) \end{cases} \tag{30}$$

by taking into account the uncertainty of system parameter matrices as well as the stochastic perturbation. Assuming that system (30) is asymptotically stable, we are interested in finding the conditions under which the system can tolerate the parameter uncertainty and the stochastic perturbation without losing the stability property.

Theorem 5.1. Let the condition (iii) of Theorem 3.2 hold. If there exist constants $\mu_i > 0$ and N positive definite matrices Q_i such that $\underline{\lambda}_i > \bar{\lambda}_i$ for all $i \in S$, where

$$\begin{aligned}\underline{\lambda}_i =: \lambda_{\min}\big(&- Q_i A_i - A_i^T Q_i - \mu_i^2 Q_i M_i M_i^T Q_i \\ &- \mu_i^{-2} N_i^T N_i - Q_i F_i F_i^T Q_i \\ &- Q_i G_i G_i^T Q_i - 2\sum_{k=1}^m D_{ki}^T Q_i D_{ki}\big)\end{aligned}$$

and

$$\bar{\lambda}_i =: \lambda_{\max}\big(I + R_i^T R_i + 2\sum_{k=1}^m E_{ki}^T Q_i E_{ki}\big).$$

Then Eq. (28) is stochastically asymptotically stable.

The proof is based on Theorem 3.1.

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