A MODEL PREDICTIVE CONTROL BASED ON INPUT-OUTPUT CHARACTERIZATION OF FINITE-HORIZON LINEAR SYSTEMS

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Abstract: An LQ control problem for constrained discrete-time systems is discussed based on model predictive control with reduced order quadratic programming (QP) problem. By introducing a singular value decomposition for linear systems, where zero-order holder is embedded in the input signals, an approximation method of model predictive control is derived. The feature of resulting control system is illustrated with numerical examples.

Keywords: LQR control method, constraints, model predictive control, discrete-time system, singular value decomposition.

1. INTRODUCTION

Model Predictive Control (MPC) has become a standard control strategy for constrained multivariable systems and, especially for slow dynamical systems, the advantages are highlighted in numerous papers (Bemporad *et al.*, 1999). The key point of the MPC strategy is that the optimal control problem is solved, in the discrete-time setting, on-line over a finite-horizon and the first value of the resulting control signal is applied. At the next time step, the on-line computation is repeated starting from the new state and over a shifted horizon. Thus, the applicable control problems are limited by the complexity of the plant dynamics or the length of control horizon.

The LQ control problem has been mainly studied for discrete-time constrained systems (Chmielewski *et al.*, 1996; Scokaert *et al.*, 1998; Bemporad *et al.*, 2002) and, recently, a calculation method of sub-optimal LQ control is derived for continuous-time constrained systems by introducing a singular value decomposition (SVD) of linear systems (Kojima *et al.*, 2004). The SVD for the linear systems provides a sequence of dominant input signals, which effectively generate the responses, and enables us to design the input sequence based on the combination of the dominant input signals.

In this paper, we focus on an LQ control problem for discrete-time constrained systems and provide a calculation method of sub-optimal LQ control based on a fixed dimension of quadratic-programming (QP) problem. The proposed method is applicable to the LQ problems, which requires a huge size of QP-problem, and the resulting control has the following feature: 1) the dimension of QP-problem is a priori fixed by the number of singular vectors obtained by SVD of the system; 2) the unconstrained behavior of resulting system is indeed LQ-optimal and the approximation is made in the design of constrained control signals.

In the following, we first investigate the relation between the input signals, where 1 step zero-order holder is embedded, and the resulting responses. Then we derive a formula of singular value decomposition (SVD) for finite-horizon systems (Section 2). By employing a SVD representation for LQ control problem, we will show that the sub-optimal LQ control is obtained via fixed dimensional QP-problem (Section 3). The relation to the multi-parametric QP-problem is mentioned on the resulting control method and the result is illustrated with numerical examples (Section 4).

2. INPUT-OUTPUT CHARACTERIZATION OF LINEAR SYSTEMS

In the model predictive control, an optimization problem is solved on-line and the first value of control sequence is applied each time in the receding horizon policy. In this section, we focus on an abstract input sequence depicted by Fig.1 and clarify the input-output relation from the viewpoint of singular value decomposition (SVD) of the system. In Fig.1(b), the control signal is given by 1 step zero-order holder (ZOH), which value is applied for the control, and the arbitrary curve, which shape is employed in the optimization. This abstraction enables us to deal with the huge size of MPC problem and provides an approximation method of control law.

Focus on a continuous-time linear system

$$\Sigma: \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0 \qquad (1)$$
$$y(t) = Cx(t), \qquad 0 \le t \le h + T$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$ are the state, the control input, and the regulated output of the system respectively. Following assumptions are made for the system Σ.

(C1) (C, A) is observable; B is column full-rank.

Introducing function spaces $\mathcal{V} := \mathbb{R}^m \times L_2(h, h+T; \mathbb{R}^m)$, $\mathcal{Z} := \mathbb{R}^n \times L_2(0, h; \mathbb{R}^r) \times L_2(h, h+T; \mathbb{R}^r)$, we will describe the input and output signals depicted by Fig.1(b) in the following manner:

$$v = \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \in \mathcal{V}, \quad \begin{aligned} v^0 &= h^{\frac{1}{2}} R^{\frac{1}{2}} u \in \mathbb{R}^m \\ v^1 &= R^{\frac{1}{2}} u \in L_2(h, h+T; \mathbb{R}^m) \end{aligned} \qquad (2)$$

$$z = \begin{bmatrix} z^0 \\ z^1 \\ z^2 \end{bmatrix} \in \mathcal{Z}, \quad \begin{aligned} z^0 &= P^{\frac{1}{2}} x(h+T) \in \mathbb{R}^n \\ z^1 &= Q^{\frac{1}{2}} y \in L_2(0, h; \mathbb{R}^r) \\ z^2 &= Q^{\frac{1}{2}} y \in L_2(h, h+T; \mathbb{R}^r) \end{aligned} \qquad (3)$$

where $P > 0$, $Q > 0$, $R > 0$ are weighting matrices, which will be employed to reflect the cost-functional discussed in the model predictive control (Section3). Based on the representation (2),(3), the relaxed system Σ $(x(0) = 0)$ over the time horizon $[\,0, h+T\,]$ is given by the following integral operator.

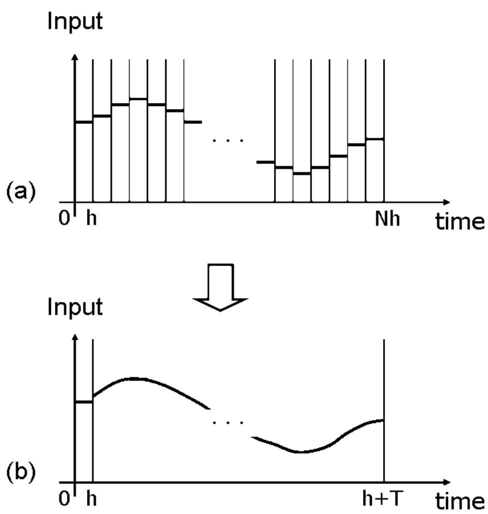


Fig. 1. Description of abstract input sequence

$$\mathcal{G} \in \mathcal{L}(\mathcal{V}, \mathcal{Z}): \begin{bmatrix} z^0 \\ z^1 \\ z^2 \end{bmatrix} = \begin{bmatrix} (\mathcal{G}v)^0 \\ (\mathcal{G}v)^1 \\ (\mathcal{G}v)^2 \end{bmatrix} \qquad (4)$$

$$\begin{aligned} (\mathcal{G}v)^0 \; &:= h^{-\frac{1}{2}} P^{\frac{1}{2}} \int_0^h e^{A(h+T-\xi)} B R^{-\frac{1}{2}} v^0 \, d\xi \\ &\quad + h^{-\frac{1}{2}} P^{\frac{1}{2}} \int_h^{h+T} e^{A(h+T-\eta)} B R^{-\frac{1}{2}} v^1(\eta) \, d\eta \\ (\mathcal{G}v)^1(\alpha) &:= h^{-\frac{1}{2}} Q^{\frac{1}{2}} C \int_0^\alpha e^{A(\alpha-\xi)} B R^{-\frac{1}{2}} v^0 \, d\xi \\ &\qquad 0 \le \alpha \le h \\ (\mathcal{G}v)^2(\beta) &:= h^{-\frac{1}{2}} Q^{\frac{1}{2}} C \int_0^h e^{A(\beta-\xi)} B R^{-\frac{1}{2}} v^0 \, d\xi \\ &\quad + Q^{\frac{1}{2}} C \int_h^\beta e^{A(\beta-\eta)} B R^{-\frac{1}{2}} v^1(\eta) \, d\eta \\ &\qquad h \le \beta \le h+T \end{aligned}$$

The singular values $\sigma > 0$ and vectors $(f, g) \neq 0$ for the operator (4) is defined by

$$\sigma > 0: \quad \sigma g = \mathcal{G} f, \quad \sigma f = \mathcal{G}^* g, \qquad (5)$$

and the calculation method is obtained as follows.

Theorem 1. The singular values $\{\sigma_i\}$ of $\mathcal{G} \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ are given by the root of the equation:

$$\sigma > 0 : \det \left\{ E(\sigma) \begin{bmatrix} e^{J(\sigma)T} & 0 \\ 0 & I \end{bmatrix} e^{K(\sigma)h} L \right\} = 0. \quad (6)$$

$$E(\sigma) := \begin{bmatrix} -\sigma^{-1}P & I_n & 0 & 0 \\ 0 & 0 & I_n & 0_{n\times n} \end{bmatrix}$$

$$L^T := \begin{bmatrix} 0_{n\times n} & I_n & 0 & 0 \\ 0 & 0 & I_n & I_n \end{bmatrix}$$

$$J(\sigma) := \begin{bmatrix} A & \sigma^{-1} B R^{-1} B^T \\ -\sigma^{-1} C^T Q C & -A^T \end{bmatrix}$$

$$K(\sigma) := \begin{bmatrix} A & 0 & 0 & \sigma^{-1} h^{-1} B R^{-1} B^T \\ -\sigma^{-1} C^T Q C & -A^T & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & 0_{n\times n} & 0 \end{bmatrix}$$

Furthermore the corresponding singular vectors $f_i = (f_i^0, f_i^1) \in \mathcal{V}$, $g_i = (g_i^0, g_i^1, g_i^2) \in \mathcal{Z}$ are given as follows.

$$\begin{aligned} f_i^0 &= h^{-\frac{1}{2}} R^{-\frac{1}{2}} B^T [\,0_{n\times n}\; I_n\,] w_i \\ f_i^1(\eta) &= R^{-\frac{1}{2}} B^T [\,0_{n\times n}\; I_n\,] r_i(\eta - h), \\ &\qquad h \le \eta \le h+T \\ g_i^0 &= P^{\frac{1}{2}} [\,I_n\; 0_{n\times n}\,] r_i(T) \\ g_i^1(\alpha) &= Q^{\frac{1}{2}} C [\,I_n\; 0_{n\times 3n}\,] e^{K(\sigma_i)\alpha} L w_i, \quad 0 \le \alpha \le h \\ g_i^2(\beta) &= Q^{\frac{1}{2}} C [\,I_n\; 0_{n\times n}\,] r_i(\beta - h), \; h \le \beta \le h+T \\ r_i(\xi) &:= e^{J(\sigma_i)\xi} [\,I_{2n}\; 0_{2n\times 2n}\,] e^{K(\sigma_i)h} L w_i \\ w_i &\neq 0 : E(\sigma_i) \begin{bmatrix} e^{J(\sigma_i)T} & 0 \\ 0 & I \end{bmatrix} e^{K(\sigma_i)h} L w_i = 0 \end{aligned} \qquad (7)$$

□

(Proof) Appendix A. □

The operator $\mathcal{G}$ has a countable number of singular values and they form a sequence which approaches 0 (Gohberg *et al.*, 1990). We will denote the singular values $\{\sigma_i\}$ in the decreasing order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_i \geq \cdots \geq 0$ and normalize the corresponding singular vectors (f_i, g_i) by $\|f_i\|_{\mathcal{V}} = 1$, $\|g_i\|_{\mathcal{Z}} = 1$ $(i = 1, 2, \cdots)$. Fundamental properties on the singular values and vectors are given by Remark 2, Lemma 3.

Remark 2. (Gohberg *et al.*, 1990) $\langle f_i, f_j \rangle_{\mathcal{V}} = 0$, $\langle g_i, g_j \rangle_{\mathcal{Z}} = 0$ $(i \neq j)$ holds. □

Lemma 3. Under the assumption (C1), the singular vectors $\{f_i\}$ form a complete basis in $\mathcal{V}$. □

(Proof) Appendix B. □

The singular values σ_i and vectors (f_i, g_i) provide an interpretation on the input-output relation of the system (1). If an input signal $v = f_i \in \mathcal{V}$ is applied to the relaxed system ($x_0 = 0$), the response is given by $\sigma_i g_i \in \mathcal{Z}$ since $\sigma_i g_i = \mathcal{G} f_i$ holds. As the singular value approaches 0 ($\sigma \simeq 0$), the corresponding singular vector f_i characterizes the direction of the input signal $v = f_i \in \mathcal{V}$, which does not affect the system response.

We will deal with the relaxed system (1) by the following relation:

$$v := \sum_{i=1}^{d} k_i \cdot f_i, \quad z := \sum_{i=1}^{d} k_i \cdot \sigma_i g_i \tag{8}$$

where the input signal v is represented in the ZOH embedded description (Fig.1(b)). Furthermore by Remark 2, the following cost-functional, which plays with the LQ control, is easily represented with the coefficients $\{k_i\}$.

$$\begin{aligned} J_c &= x^T(h+T)Px(h+T) \\ &\quad + \int_0^{h+T} \{y^T(t)Qy(t) + u^T(t)Ru(t)\}\, dt \\ &= \left\| \begin{bmatrix} v \\ z \end{bmatrix} \right\|^2_{\mathcal{V}\times\mathcal{Z}} = \sum_{i=1}^{d} (1+\sigma_i^2) \cdot k_i^2. \end{aligned} \tag{9}$$

In the case $h > 0$ is sufficiently small to the control horizon $[\,0,\, h+T\,]$, the cost-functional (9) provides an approximation to the discrete-time LQ control. In the sequel, we will discuss the LQ control for discrete-time constrained systems and, employing the SVD of linear systems, derive a calculation method of sub-optimal control via reduced order quadratic programming (QP) problem.

3. LQ CONTROL FOR CONSTRAINED DISCRETE-TIME SYSTEMS

Consider LQ control problem for the discrete-time system

$$\begin{aligned} \Sigma_d : \quad x_{k+1} &= A_d x_k + B_d u_k \\ y_k &= C x_k \end{aligned} \tag{10}$$

with the constraints

$$\begin{aligned} &\forall k \in \{0, 1, \cdots\} : \\ &\quad u_k \in \mathcal{U} := \{u \in \mathbb{R}^m : u_{\min} \leq u \leq u_{\max}\}, \\ &\quad y_k \in \mathcal{Y} := \{y \in \mathbb{R}^r : y_{\min} \leq y \leq y_{\max}\}. \end{aligned} \tag{11}$$

The control objective is to minimize the cost-functional

$$\begin{aligned} J_d &:= \sum_{k=0}^{\infty} \left\{ y_k^T Q_d y_k + u_k^T R_d u_k \right\}, \\ &\qquad Q_d > 0, R_d > 0. \end{aligned} \tag{12}$$

The system Σ_d is obtained via discretization of Σ with ZOH, which sample time is $h > 0$. We make following assumptions for the system Σ_d.

(C2) (A_d, B_d) is stabilizable.
(C3) $(u_k, y_k) = (0, 0)$ is an interior point of $\mathcal{U} \times \mathcal{Y}$.
(C4) For the initial state x_0, there exists a control u_k which drives the state to the origin under the constraints (11) with $J_d < \infty$.

For the bounded set of initial states $\{x_0\}$, the LQ control problem defined by (10)-(12) is reformulated with the cost-functional:

$$\begin{aligned} J_d &:= x_{N+1}^T P_d x_{N+1} + \sum_{k=0}^{N} \left\{ y_k^T Q_d y_k + u_k^T R_d u_k \right\}, \\ &\quad P_d > 0,\ Q_d > 0,\ R_d > 0 \end{aligned} \tag{13}$$

where $P_d > 0$ is a stabilizing solution to the following discrete-time Riccati equation(Chmielewski *et al.*, 1996; Scokaert *et al.*, 1998).

$$\begin{aligned} P_d &= A_d^T P_d A_d + C^T Q_d C \\ &\quad - A_d^T P_d B_d (R_d + B_d^T P_d B_d)^{-1} B_d^T P_d A_d \end{aligned} \tag{14}$$

In this section, we will derive a calculation method of sub-optimal LQ control via reduced order QP-problem. Define the control horizon of the system (1) by $T = Nh$ and the weighting matrices in (2),(3) by $P = h \cdot P_d$, $Q = Q_d$, $R = R_d$, then the following approximation

$$J_c \simeq h \cdot J_d \tag{15}$$

holds for $h \ll T$. Hence, by employing the SVD of the system Σ (Theorem 1), we derive a sub-optimal control for the problem (10)-(12).

Choose singular vectors (f_i, g_i) $(i = 1, 2, \cdots, d)$ and describe the control signals by

$$v = \tilde{v} + v_{\rm opt}, \qquad \tilde{v} = \sum_{i=1}^{d} k_i \cdot f_i \tag{16}$$

where $v_{\rm opt} = (v^0_{\rm opt}, v^1_{\rm opt}) \in \mathcal{V}$ denotes the unconstrained LQ optimal control as follows.

$$\begin{aligned}
v^0_{\rm opt} &:= V^0_{\rm opt} x_0 \qquad (17)\\
v^1_{\rm opt}(t) &:= V^1_{\rm opt}(t) x_0 \qquad (18)\\
V^0_{\rm opt} &:= h^{\frac{1}{2}} R^{\frac{1}{2}} K_{\rm LQ}\\
V^1_{\rm opt}(t) &:= \sum_{k=1}^{N} R^{\frac{1}{2}} K_{\rm LQ} A^k_{cl} \cdot \chi_{[kh,(k+1)h)}(t)\\
K_{\rm LQ} &:= -(R_d + B^T_d P_d B_d)^{-1} B^T_d P_d A_d\\
A_{cl} &:= A_d + B_d K_{\rm LQ}\\
\chi_{[a,b)}(t) &= \begin{cases} 1 & t \in [a,b) \\ 0 & t \notin [a,b) \end{cases}
\end{aligned}$$

Since the input signal $f_i \in \mathcal{V}$ generates the response $\sigma_i \cdot g_i \in \mathcal{Z}$, the system behavior is correspondingly described in the following form.

$$z = \tilde{z} + z_{\rm opt}, \qquad \tilde{z} = \sum_{i=1}^{d} k_i \cdot \sigma_i g_i \tag{19}$$

$$\begin{aligned}
z^0_{\rm opt} &:= Z^0_{\rm opt} x_0 \qquad (20)\\
z^1_{\rm opt}(t) &:= Z^1_{\rm opt}(t) x_0 \qquad (21)\\
z^2_{\rm opt}(t) &:= Z^2_{\rm opt}(t) x_0 \qquad (22)\\
Z^0_{\rm opt} &:= P^{\frac{1}{2}} A^{N+1}_{cl}\\
Z^1_{\rm opt}(t) &:= Q^{\frac{1}{2}} C \cdot \chi_{[0,h)}(t)\\
Z^2_{\rm opt}(t) &:= \sum_{k=1}^{N} Q^{\frac{1}{2}} C A^k_{cl} \cdot \chi_{[kh,(k+1)h)}(t)
\end{aligned}$$

Based on the description of system responses (16),(19), the calculation method of sub-optimal control law is obtained as follows.

Theorem 4. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_d\}$ be a subset of singular values of $\mathcal{G}$ and $(f_i, g_i)(i = 1, 2, \cdots, d)$ be the corresponding singular vectors. In the controls (16), the optimal control which minimizes

$$J_h = \left\| \begin{bmatrix} \tilde{v} \\ \tilde{z} \end{bmatrix} \right\|^2_{\mathcal{V}\times\mathcal{Z}} \tag{23}$$

is given by the solution $k = [\, k_1\, k_2 \cdots k_d \,]^T \in \mathbb{R}^d$ to the QP-problem:

$$k := \arg \min_{k\in\mathbb{R}^d} k^T \Lambda k$$

$$\text{subj.to} \begin{cases} h^{-\frac{1}{2}} R^{-\frac{1}{2}} \left(\tilde{V}^0 k + V^0_{\rm opt} x_0 \right) \in \mathcal{U} \\ Q^{-\frac{1}{2}} \left(\tilde{Z}^1 k + Z^1_{\rm opt}(0) x_0 \right) \in \mathcal{Y} \\ \forall\, t = ih \;\; (i = 1, 2, \cdots, N): \\ R^{-\frac{1}{2}} \left(\tilde{V}^1(t) k + V^1_{\rm opt}(t) x_0 \right) \in \mathcal{U} \\ Q^{-\frac{1}{2}} \left(\tilde{Z}^2(t) k + Z^2_{\rm opt}(t) x_0 \right) \in \mathcal{Y} \end{cases} \tag{24}$$

$$\begin{aligned}
\Lambda &:= {\rm diag}(1+\sigma_1^2, \cdots, 1+\sigma_d^2) \in \mathbb{R}^{d\times d} \qquad (25)\\
\tilde{V}^0 &:= [\, f^0_1 \;\; f^0_2 \;\; \cdots \;\; f^0_d \,] \qquad (26)\\
\tilde{V}^1(t) &:= [\, f^1_1(t) \;\; f^1_2(t) \;\; \cdots \;\; f^1_d(t) \,] \qquad (27)\\
\tilde{Z}^0 &:= [\, \sigma_1 g^0_1 \;\; \sigma_2 g^0_2 \;\; \cdots \;\; \sigma_d g^0_d \,] \qquad (28)\\
\tilde{Z}^1(t) &:= [\, \sigma_1 g^1_1(t) \;\; \sigma_2 g^1_2(t) \;\; \cdots \;\; \sigma_d g^1_d(t) \,] \qquad (29)\\
\tilde{Z}^2(t) &:= [\, \sigma_1 g^2_1(t) \;\; \sigma_2 g^2_2(t) \;\; \cdots \;\; \sigma_d g^2_d(t) \,] \qquad (30)
\end{aligned}$$

□

By Lemma 2 in (Kojima *et al.*, 2004), the equality

$$\lim_{h\to 0+} \left\langle \begin{bmatrix} v_{\rm opt} \\ z_{\rm opt} \end{bmatrix}, \begin{bmatrix} \tilde{v} \\ \tilde{z} \end{bmatrix} \right\rangle_{\mathcal{V}\times\mathcal{Z}} = 0 \tag{31}$$

holds for $\forall k \in \mathbb{R}^d$ and the cost-functional (13) is approximately represented as follows.

$$\begin{aligned}
& h \cdot J_d \simeq \\
& J_c = \left\| \begin{bmatrix} \tilde{v} + v_{\rm opt} \\ \tilde{z} + z_{\rm opt} \end{bmatrix} \right\|^2_{\mathcal{V}\times\mathcal{Z}} = \left\| \begin{bmatrix} \tilde{v} \\ \tilde{z} \end{bmatrix} \right\|^2_{\mathcal{V}\times\mathcal{Z}} + \left\| \begin{bmatrix} v_{\rm opt} \\ z_{\rm opt} \end{bmatrix} \right\|^2_{\mathcal{V}\times\mathcal{Z}}\\
& \quad = J_h + x^T_0 P x_0 \qquad (32)
\end{aligned}$$

Hence, by the relations (15),(32), Theorem 1,4 enable to approximate the optimization procedure in the model predictive control. The size of the QP-problem depends on the number of singular vectors which is characterized by Theorem 1.

It should be noted that the auxiliary input signal $\tilde{v}$ is introduced to make up the optimal control in the constrained region and does not affect in the linear control region since $\tilde{v} = 0$ is optimal there. Thus, the exact optimality in the unconstrained region is preserved in the design methods Theorem 1,4.

Finally we note that the sub-optimal control to the problem (24) is characterized by a piecewise affine function of the initial state. By employing a finite set of singular vectors (f_i, g_i) $(i = 1, 2, \cdots, d)$, it is shown by Theorem 4 that control problem (10)-(12) is solved via a QP-problem:

$$\min_{k\in\mathbb{R}^d} \frac{1}{2} \cdot k^T \hat{H} k \quad \text{subj. to} \;\; \hat{G} k \le \hat{W} + \hat{S} x_0, \tag{33}$$

with appropriate matrices $\hat{H} > 0$, $(\hat{G}, \hat{W}, \hat{S})$. We will introduce a fact that, if the QP-problem (33) is solved at one initial state $x_0 = x^*_0$, the optimal solution is extrapolated for a set of initial states $\{x_0\}$, which includes $x_0 = x^*_0$.

Lemma 5. [Bemporad et al., 2002] Let $k = k^*$ be the optimal solution to the QP-problem (33) for a given $x_0 = x^*_0$ and $(\tilde{G}, \tilde{W}, \tilde{S})$ be the rows of active constraints such that $\tilde{G} k^* = \tilde{W} + \tilde{S} x^*_0$ holds. Under the assumption such that the rows of $\tilde{G}$ are linearly independent, the optimal solution to the QP-problem (33) is expressed by

$$k = \hat{H}^{-1} \tilde{G}^T (\tilde{G} \hat{H}^{-1} \tilde{G}^T)^{-1} (\tilde{W} + \tilde{S} x_0) \tag{34}$$

in the polyhedral region

$$\hat{G}\hat{H}^{-1}\tilde{G}^T(\tilde{G}\hat{H}^{-1}\tilde{G}^T)^{-1}(\tilde{W}+\tilde{S}x_0) \le \hat{W}+\hat{S}x_0^*,$$
$$-(\tilde{G}\hat{H}^{-1}\tilde{G}^T)^{-1}(\tilde{W}+\tilde{S}x_0) \ge 0 \qquad (35)$$

which includes $x_0 = x_0^*$. □

By Lemma 5, the control (16) is given by a piecewise affine function of x_0. Hence the receding horizon control is equivalently given by a state feedback control law, which is constructed with the relation between the initial state x_0 and the resulting control u_0.

4. EXAMPLE

Consider LQ control problem for the double integrator.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad (36)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \qquad 0 \le t \le 5 \qquad (37)$$

$$-1 \le u(t) \le 1 \qquad (38)$$

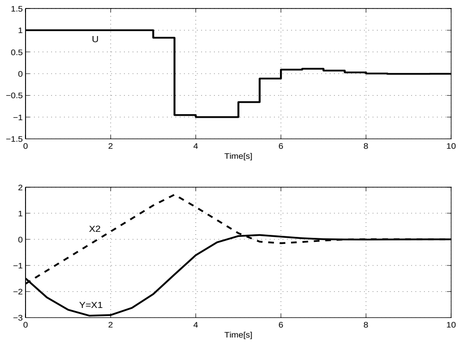
Based on the discretized system obtained with ZOH and following sample times:

(a) $h = 0.50$ ($N = 9$)
(b) $h = 1/3$ ($N = 14$)

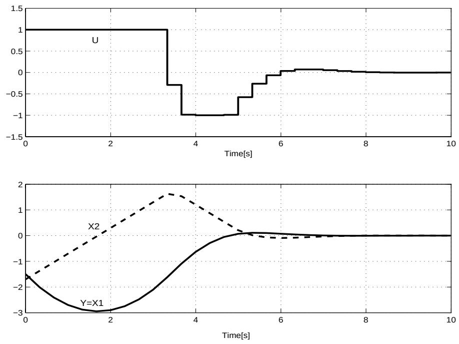
we will discuss the feature of resulting control system, which is obtained by Theorem 1,4. The cost-functional (13) is defined by the matrices $Q_d = 1$, $R_d = 0.1$ with (a),(b).

By employing 15 singular values in the decreasing order (Theorem 1), then applying the optimal control derived by Theorem 4 in the receding horizon policy, the system response is obtained by Fig.2. In this example, the similar shape of control signals is generated for given sample times. It should be noted that the control action in the unconstrained region ($t > 5$) is indeed optimal as the approximation is made in the calculation of auxiliary control signals $\tilde{v}$, which are introduced to make up the constrained LQ control to fulfill the system constraints.

Applying Lemma 5, the relations between the initial state x_0 and the resulting control signal u_0 are summarized by Fig.3. For example in Fig.3(a), the lower left areas are for the control $u_0 = +1$ and the upper right areas are for the control $u_0 = -1$. Both regions are continuously connected by 11 different polyhedral regions, whose representations are obtained by Lemma 5. As the sample time h is decreasing, it is observed that the number of regions grows rapidly however the the number of control law, which requires different representation, does not grow so fast. In the case (b), 21 different control laws are needed in the belt-shaped area, which connects the regions for $u_0 = -1$ and $u_0 = +1$.

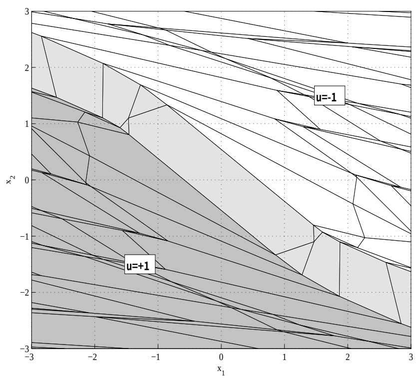


(a) $h = 0.5, h + T = 5$

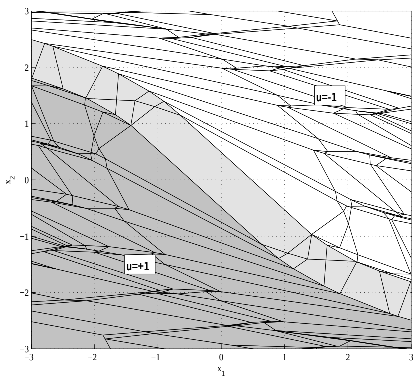


(b) $h = 1/3, h + T = 5$

Fig. 2. Response from $x_0 = (-1.5, -1.7)$



(a) $h = 0.5, h + T = 5$



(b) $h = 1/3, h + T = 5$

Fig. 3. Relation between x_0 and u_0

5. CONCLUSION

An LQ control problem for discrete-time constrained systems is discussed based on model predictive control with reduced order quadratic programming (QP).

By characterizing the dominant system responses, which are generated by the ZOH embedded input signals, an approximation method of LQ control law is derived. The proposed control law guarantees the exact optimality in the unconstrained behavior and enables us to approach the constrained LQ-control in a fixed dimension of QP-problem.

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APPENDICES

A. Proof of Theorem 1

By introducing auxiliary variables

$$p_1(\alpha) := \int_0^\alpha e^{A(\alpha-\xi)} BR^{-\frac{1}{2}} h^{-\frac{1}{2}} f^0 \, d\xi \qquad (39)$$

$$p_2(\beta) := \int_0^h e^{A(\beta-\zeta)} BR^{-\frac{1}{2}} h^{-\frac{1}{2}} f^0 \, d\zeta + \int_h^\beta e^{A(\beta-\zeta)} BR^{-\frac{1}{2}} f^1(\zeta) \, d\zeta \qquad (40)$$

$$q_1(\xi) := e^{A^T(h+T-\xi)} P^{\frac{1}{2}} g^0 + \int_\xi^h e^{A^T(\alpha-\xi)} C^T Q^{\frac{1}{2}} g^1(\alpha) \, d\alpha + \int_h^{h+T} e^{A^T(\beta-\xi)} C^T Q^{\frac{1}{2}} g^2(\beta) \, d\beta \qquad (41)$$

$$q_2(\zeta) := e^{A^T(h+T-\zeta)} P^{\frac{1}{2}} g^0 + \int_\zeta^{h+T} e^{A^T(\beta-\zeta)} C^T Q^{\frac{1}{2}} g^2(\beta) \, d\beta \qquad (42)$$

$$v(\xi) = \int_\xi^h q_1(\beta) \, d\beta \qquad (43)$$

$$w(\xi) = v(0), \quad 0 \le \xi \le h \qquad (44)$$

to the left and right equalities in (5), the relations are summarized by following differential equations with boundary conditions.

$$\begin{bmatrix} p_1'(\beta) \\ q_1'(\beta) \\ v'(\xi) \\ w'(\xi) \end{bmatrix} = K(\sigma) \begin{bmatrix} p_1(\xi) \\ q_1(\xi) \\ v(\xi) \\ w(\xi) \end{bmatrix} \qquad (45)$$

$$\begin{bmatrix} p_2'(\beta) \\ q_2'(\beta) \end{bmatrix} = J(\sigma) \begin{bmatrix} p_2(\beta) \\ q_2(\beta) \end{bmatrix} \qquad (46)$$

$$\begin{bmatrix} p_1(0) \\ q_1(0) \\ v(0) \\ w(0) \end{bmatrix} = L \begin{bmatrix} q_1(0) \\ v(0) \end{bmatrix}, \quad v(h) = 0 \qquad (47)$$

$$\begin{bmatrix} -\sigma^{-1}P & I \end{bmatrix} \begin{bmatrix} p_2(h+T) \\ q_2(h+T) \end{bmatrix} = 0 \qquad (48)$$

$$\begin{bmatrix} p_2(h) \\ q_2(h) \end{bmatrix} = \begin{bmatrix} p_1(h) \\ q_1(h) \end{bmatrix}. \qquad (49)$$

From (45)-(49), the following condition is obtained for the singular value $\sigma > 0$ and the vector $(q_1(0), v(0))$.

$$E(\sigma) \begin{bmatrix} e^{J(\sigma)T} & 0 \\ 0 & I \end{bmatrix} e^{K(\sigma)h} L \begin{bmatrix} q_1(0) \\ v(0) \end{bmatrix} = 0 \qquad (50)$$

Since $(q_1(0), v(0)) = 0$ implies $(p_1, q_1) \equiv 0$, $(p_2, q_2) \equiv 0$ and $(f, g) = 0$, $(q_1(0), v(0)) \neq 0$ is necessary for the existence of the singular value. This fact requires (6) for the singular value $\sigma > 0$. Sufficiency is verified by contradiction.

For each singular value $\sigma_i > 0$, replacing $(q_1(0), v(0)) \neq 0$ by w_i, the singular vectors (f_i, g_i) are given by (7). □

B. Proof of Lemma 3

Since $\sigma^2 f = \mathcal{G}^*\mathcal{G}f$ holds for the self-adjoint compact operators, the singular vectors $\{f_i\}$ form a complete basis iff $\mathcal{G}^*\mathcal{G}$ does not have an eigenvalue at 0 (Gohberg *et al.*, 1990). We will show that $\|\mathcal{G}f\| \neq 0$ holds for any given $f \neq 0$ ($f \in \mathcal{V}$). Introducing the variables (39),(40) for the equality $w = \mathcal{G}f$, we have the following relations.

$$\begin{aligned} p_1'(\xi) &= Ap_1(\xi) + BR^{-\frac{1}{2}}h^{-\frac{1}{2}}f^0, \; p_1(0) = 0 \\ p_2'(\xi) &= A(p_2(\xi) - p_1(h)) + BR^{-\frac{1}{2}}h^{-\frac{1}{2}}f^1(\xi), \\ p_2(h) &= p_1(h) \\ w^0 &= h^{-\frac{1}{2}}P^{\frac{1}{2}}p_2(h+T) \\ w^1(\xi) &= h^{-\frac{1}{2}}Q^{\frac{1}{2}}Cp_1(\xi), \quad 0 \le \xi \le h \\ w^2(\xi) &= h^{-\frac{1}{2}}Q^{\frac{1}{2}}Cp_2(\xi), \quad h \le \xi \le h+T \end{aligned} \qquad (51)$$

Since $BR^{-\frac{1}{2}}$ is column full-rank and $(Q^{\frac{1}{2}}C, A)$ is observable by (C1), $f^0 \neq 0$ yields $w^1 \neq 0$. Furthermore $f^0 = 0$, $f^1 \neq 0$ yields $w^2 \neq 0$ from the 2nd differential equation in (51). Thus the strict inequality: $\forall f \neq 0$: $\langle f, \mathcal{G}^*\mathcal{G}f \rangle_{\mathcal{V}} = \|w\|^2 > 0$ holds. □