A DESIGN METHOD OF COMPENSATION LAW FOR CONSTRAINED LINEAR SYSTEMS

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Abstract: This paper deals with a design method of compensation law for linear constrained systems. By employing singular value decomposition of linear systems, we provide a design method of compensation law which fulfills system constraints. The feature of resulting compensation law is illustrated with numerical examples.

Keywords: constrained systems, feedforward compensation, transient, singular value decomposition, deadbeat control

1. INTRODUCTION

In the synthesis of tracking control systems, a compensation input, which is applied to the controlled system over finite-horizon time, is effective to improve the transient response. From this point of view, the design methods of compensation law which attains favorable transient response are studied by various authors: (Ikeda and Suda, 1988; Izumi *et al.*, 2000; Kojima and Ishijima, 1999). However, in the presence of the system constraints such as input-output limitations, the calculation task of compensation signal is complicated and, especially for continuous-time systems, the design method of compensation law is not founded.

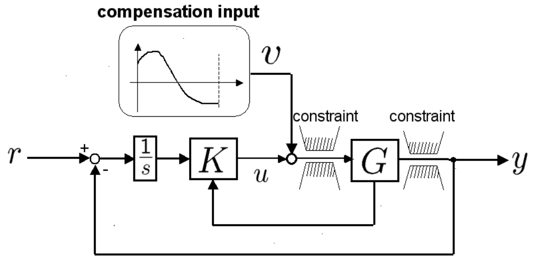
For constrained systems, several control methods are established in the paradigm of model predictive control (Bemporad and Morari, 1999; Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998) and, recently the dominant input-output relation for linear system is investigated in terms of the open-loop prediction (Kojima and Morari, 2004).

In this paper, we discuss a design method of compensation law such that the resulting system attains favorable transient response within the system constraints. In order to approach the problem, we first introduce a singular value decomposition (SVD) for finite-horizon linear systems. The SVD method provides a geometric interpretation of input and output relations for linear systems and, further, enables us to design a compensation law by the static linear combination of the singular vectors, which describe the system responses.

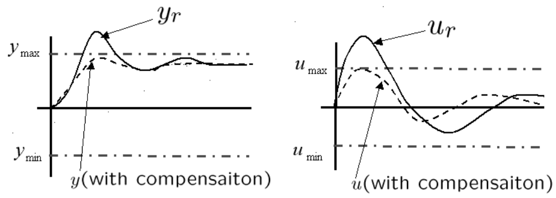
The paper is organized as follows. In Section 2, we formulate the problem in general form and illustrate the relation to typical control problems. In Section 3, a calculation method of singular value decomposition (SVD) is derived for finite-horizon linear systems. In Section 4, a design method of the compensation law is obtained based on the SVD approach. In Section 5, numerical examples are described based on the proposed design method.

2. PROBLEM FORMULATION

Let us begin with the typical servo system depicted by Fig.1. In Fig.1(a), G, K denote the plant and the controller respectively, and we prescribe input-output constraints for the plant G. The output response and corresponding input signal driven by the step reference are depicted by Fig.1(b), and let us assume



(a)Servo system with constraints



(b)Input and Output responses

Fig. 1. Constrained system and responses

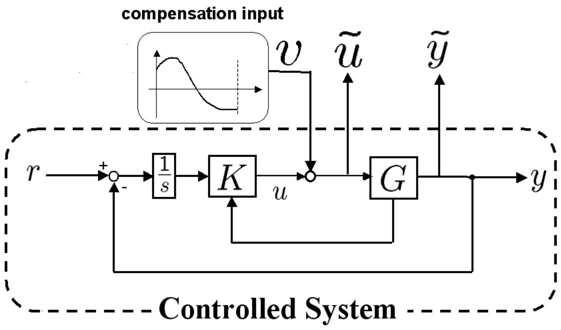


Fig. 2. Controlled system with compensation

both responses do not fulfill the constraints. If we can introduce an additional compensation input v to the controlled system over the finite time interval $[0, h]$, the performance of the control system will be improved from the following viewpoints:

(i) Modify the responses u and y so that the resulting responses fulfill the system constraints.
(ii) Drive the terminal state so that the response of the system over $[h, \infty]$ attains favorable performance.

Motivated by this observations, we investigate the relation between the auxiliary compensation input v and the difference of the responses $(\tilde{u}, \tilde{y})$, which are generated by the signal v(Fig.2). In the sequel, we formulate the problem in the general form (Fig.3) and clarify the design method of compensation law.

3. SINGULAR VALUES AND VECTORS OF LINEAR SYSTEMS

Consider a linear time invariant system driven by the input v, which is applied over the time interval $[0, h]$ (Fig.3).

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Dv(t),\ x(0) = 0 \\ z(t) = Ex(t) + F_0 v(t),\ 0 \le t \le h \end{cases} \tag{1}$$

In (1), $x(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^p$ are the state, the compensation input, and the regulated output respectively. For the system Σ, the following assumptions are made:

(A1) A is stable.
(A2) (E, A) is observable and (A, D) is controllable.
(A3) $F_0^{\mathrm{T}} F_0 = I$.

In the system Σ, it should be noted that the output $z(t)$ denotes the difference of input and output signals

$$z(t) = \begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix} \tag{2}$$

which are generated by the compensation input v. In the assumption (A1), we assume that the compensation input is applied to the stabilized control system. In (A3), the output $z(t)$ includes normalized penalty of compensation input $v(t)$. The design method of compensation input is easily extended to the case $F_0^{\mathrm{T}} F_0$ is nonsingular. In this section, we introduce a singular value decomposition (SVD) for the system Σ and clarify the relation between the compensation input v and the output z.

Let us describe the system input v and the output z on appropriate function spaces. Define spaces $\mathcal{V} := L_2(0, h; \mathbb{R}^m)$, $\mathcal{Z} := \mathbb{R}^n \times L_2(0, h; \mathbb{R}^p)$ with the inner products

$$\langle f_1, f_2 \rangle_{\mathcal{V}} := \int_0^h f_1^{\mathrm{T}}(\beta) f_2(\beta) d\beta \quad f_1, f_2 \in \mathcal{V} \tag{3}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} := g_1^{0^{\mathrm{T}}} g_2^0 + \int_0^h g_1^{1^{\mathrm{T}}}(\beta) g_2^1(\beta) d\beta$$

$$g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{4}$$

and denote input and output responses in the following manner:

$$\hat{v} := v \in \mathcal{V}, \quad \hat{z} := \begin{bmatrix} Fx(h) \\ z_{[0,h]} \end{bmatrix} \in \mathcal{Z}, \tag{5}$$

$$M =: F^{\mathrm{T}} F > 0 \tag{6}$$

where $M > 0$ is defined by the solution to the Lyapunov equation

$$A^{\mathrm{T}} M + MA + E^{\mathrm{T}} E = 0. \tag{7}$$

By including $Fx(h)$ in (5), the equality

$$\|\hat{z}\|_{\mathcal{Z}}^2 = \int_0^h z^{\mathrm{T}}(t) z(t) dt + x^{\mathrm{T}}(h) M x(h)$$
$$= \int_0^\infty z^{\mathrm{T}}(t) z(t) dt = \|z\|_{L_2(0,\infty;\mathbb{R}^p)}^2 \tag{8}$$

holds, the norm in $\mathcal{Z}$ is equivalent to the L_2 norm on $[0, \infty]$. The system behavior between v and z is described by the integral operator $\Gamma \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$:

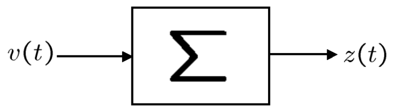


Fig. 3. Controlled system with compensation input

$$\hat{z} = \Gamma \hat{v} \tag{9}$$

$$\begin{bmatrix} (\Gamma\hat{v})^0 \\ (\Gamma\hat{v})^1(\xi) \end{bmatrix} := \begin{bmatrix} F \int_0^h e^{A(h-\beta)} D\hat{v}(\beta) d\beta \\ E \int_0^\xi e^{A(\xi-\beta)} D\hat{v}(\beta) d\beta + F_0 \hat{v}(\xi) \end{bmatrix} \tag{10}$$

$$(0 \le \xi \le h)$$

and the corresponding singular values and vectors are defined as follows:

$$\sigma \ge 0,\ (f, g) \ne 0 : \sigma g = \Gamma f,\ \sigma f = \Gamma^* g \quad f \in \mathcal{V}, g \in \mathcal{Z}. \tag{11}$$

Next theorem provides a calculation method of singular values and vectors for the operator Γ.

Theorem 1. The singular values of operator Γ are given by the roots of the following transcendental equation:

$$\det\left\{ \begin{bmatrix} -\sigma^{-1} M & I \end{bmatrix} e^{J(\sigma)h} \begin{bmatrix} 0 \\ I \end{bmatrix} \right\} = 0$$

$$J(\sigma) := \begin{bmatrix} A + \frac{1}{\sigma^2 - 1} D F_0^{\mathrm{T}} E & \frac{\sigma}{\sigma^2 - 1} D D^{\mathrm{T}} \\ -\frac{\sigma}{\sigma^2 - 1} E^{\mathrm{T}} E & -A^{\mathrm{T}} - \frac{1}{\sigma^2 - 1} E^{\mathrm{T}} F_0 D^{\mathrm{T}} \end{bmatrix} \quad (\sigma \ne 1). \tag{12}$$

Let σ_i be one of the singular values, define a non-zero vector $u_i \in \mathbb{R}^n$ which satisfies

$$\begin{bmatrix} -\sigma_i^{-1} M & I \end{bmatrix} e^{J(\sigma_i)h} \begin{bmatrix} 0 \\ I \end{bmatrix} u_i = 0, \tag{13}$$

then the singular vectors (f_i, g_i) correspond to σ_i are constructed as follows.

$$f_i(\xi) = \begin{bmatrix} \frac{1}{\sigma_i^2 - 1} F_0^{\mathrm{T}} E & \frac{\sigma_i}{\sigma_i^2 - 1} D^{\mathrm{T}} \end{bmatrix} e^{J(\sigma_i)\xi} \begin{bmatrix} 0 \\ I \end{bmatrix} u_i \tag{14}$$

$$g_i^0 = \sigma_i^{-1} \begin{bmatrix} F & 0 \end{bmatrix} e^{J(\sigma_i)h} \begin{bmatrix} 0 \\ I \end{bmatrix} u_i \tag{15}$$

$$g_i^1(\xi) = \begin{bmatrix} \frac{\sigma_i}{\sigma_i^2 - 1} E & \frac{1}{\sigma_i^2 - 1} F_0 D^{\mathrm{T}} \end{bmatrix} e^{J(\sigma_i)\xi} \begin{bmatrix} 0 \\ I \end{bmatrix} u_i \tag{16}$$

$$(0 \le \xi \le h).$$

PROOF. Appendix A. □

The singular values σ_i and vectors (f_i, g_i) characterize the input and output relations of the system Σ. If f_i is applied to the system, $\sigma_i g_i$ is generated in the output

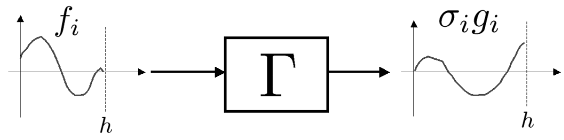


Fig. 4. Singular value and vector of linear systems

(Fig.4). Furthermore $\{f_i\}$ and $\{g_i\}$ form orthogonal basis in $\mathcal{V}$ and $\mathcal{Z}$ respectively (Gohberg *et al.*, 1990). In the following, we normalize the singular vectors f_i, g_i $(i = 1, 2, 3, \cdots)$ as $\|f_i\|_{\mathcal{V}} = 1$, $\|g_i\|_{\mathcal{Z}} = 1$.

4. DESIGN METHOD OF COMPENSATION LAW

By employing the singular value decomposition (SVD) obtained for the system Σ (Theorem1), we provide a design method of compensation law, which fulfills the system constraints.

We will describe the design method of compensation law based on the control system depicted in Fig.1. Let us describe the system constraints by

$$z_{\min}(t) \le z(t) \le z_{\max}(t)$$

$$z_{\min}(t) := \begin{bmatrix} u_{\min}(t) \\ y_{\min}(t) \end{bmatrix},$$

$$z_{\max}(t) := \begin{bmatrix} u_{\max}(t) \\ y_{\max}(t) \end{bmatrix},$$

$$z(t) := \begin{bmatrix} u(t) \\ y(t) \end{bmatrix},\ 0 \le t \le h \tag{17}$$

and u_r, y_r be input and output signals generated by the step reference. In case when the compensation signal $v(t)(0 \le t \le h)$ is applied to the controlled system (Fig.2), the system response is described by

$$z(t) = z_r(t) + \tilde{z}(t)$$

$$z_r(t) := \begin{bmatrix} u_r(t) \\ y_r(t) \end{bmatrix},$$

$$\tilde{z}(t) := \begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix}, 0 \le t \le h \tag{18}$$

where $\tilde{u}, \tilde{y}$ are the responses generated by v. Hence, in terms of the compensation signal, the system constraints are transformed to

$$\tilde{z}_{\min}(t) \le \tilde{z}(t) \le \tilde{z}_{\max}(t)$$

$$\tilde{z}_{\min}(t) := z_{\min}(t) - z_r(t)$$

$$\tilde{z}_{\max}(t) := z_{\max}(t) - z_r(t). \tag{19}$$

For the generalized system (1) (Fig.3) with the constraints (17), we will provide a design method of compensation law. Let σ_i, (f_i, g_i) $(i = 1, 2, \cdots, N)$ be the singular values and normalized vectors of Σ, and denote the candidate of compensation law by

$$\tilde{v} = \sum_{i=1}^{N} k_i f_i,\ k_i \in \mathbb{R}. \tag{20}$$

Then the output response is given by

$$\tilde{z} = (\Gamma\tilde{v})^1 = \sum_{i=1}^{N} k_i \sigma_i g_i^1 \qquad (21)$$

where the superscript denotes the second component of the singular vector in $\mathcal{Z}$. By discretizing the constraints (17) over the time interval $[0, h]$:

$$\begin{aligned} z_{\min}(t) \le z(t) \le z_{\max}(t) \\ t \in \{t_1, t_2, \cdots, t_s\} \subset [0, h], \end{aligned} \qquad (22)$$

an approximate compensation signal which minimizes

$$J = \int_0^\infty \left\{ \tilde{u}^{\mathrm{T}}(t)\tilde{u}(t) + \tilde{y}^{\mathrm{T}}(t)\tilde{y}(t) \right\} dt \qquad (23)$$

is obtained by the following theorem.

Theorem 2. Define a matrix Λ and a matrix-valued function $Z(t)$ by

$$\Lambda := \mathrm{diag}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_N^2) \qquad (24)$$

$$Z(t) := [\sigma_1 g_1^1(t), \sigma_2 g_2^1(t), \cdots, \sigma_N g_N^1(t)] \qquad (25)$$
$$t \in [0, h].$$

Then the coefficients $k = [k_1, k_2, \cdots, k_N]^{\mathrm{T}} \in \mathbb{R}^N$ of (20) which minimize cost index (23) under the discretized constraints (22) are given by the solution to the following quadratic optimization problem.

$$\begin{aligned} k_{\text{opt}} &:= \arg \min_{k \in \mathbb{R}^N} k^{\mathrm{T}} \Lambda k \\ \text{s.t. } & \tilde{z}_{\min}(t) \le Z(t)k \le \tilde{z}_{\max}(t) \\ & \forall t \in \{t_1, t_2, \cdots, t_s\} \subset [0, h] \end{aligned} \qquad (26)$$

PROOF. From the output response (21) and the definition (5), (23) is calculated as follows.

$$\begin{aligned} J &= \int_0^h \tilde{z}^{\mathrm{T}}(t)\tilde{z}(t)dt + \int_h^\infty \tilde{z}^{\mathrm{T}}(t)\tilde{z}(t)dt \\ &= \int_0^h \left\{ \sum_{i=1}^N k_i \sigma_i g_i^1(t) \right\}^{\mathrm{T}} \left\{ \sum_{i=1}^N k_i \sigma_i g_i^1(t) \right\} dt \\ &\quad + \left\{ \sum_{i=1}^N k_i \sigma_i g_i^0 \right\}^{\mathrm{T}} \left\{ \sum_{i=1}^N k_i \sigma_i g_i^0 \right\} \\ &= \sum_{i=1}^N k_i^2 \sigma_i^2 = k^{\mathrm{T}} \Lambda k. \end{aligned} \qquad (27)$$

Rewriting the constraints (22) with parameter k, the optimal coefficients k_{opt} is given by the quadratic programming (26). □

Remark 3. For any given $\epsilon > 0$, there exists a finite number of grid times $\{t_1, t_2, \cdots, t_s\}$ such that the uniform constraints:

$$\begin{aligned} \tilde{z}_{\min}(t) - \epsilon \cdot \mathbf{1} \le Z(t)k \le \tilde{z}_{\max}(t) + \epsilon \cdot \mathbf{1} \\ \mathbf{1} := [1, 1, \cdots, 1]^{\mathrm{T}} \in \mathbb{R}^p \end{aligned} \qquad (28)$$

are satisfied. This evaluation is obtained along [(Kojima and Morari, 2004), Theorem 7] with the fact the singular vectors (14), (15), (16) are Lipschitz functions. □

The inequalities stated in (26) are explicitly described in the following form.

$$Hk \le W \qquad (29)$$

$$H := \begin{bmatrix} Z(t_1) \\ -Z(t_1) \\ Z(t_2) \\ -Z(t_2) \\ \vdots \\ Z(t_s) \\ -Z(t_s) \end{bmatrix} \in \mathbb{R}^{2 \cdot p \cdot s \times N}, W := \begin{bmatrix} \tilde{z}_{\max}(t_1) \\ -\tilde{z}_{\min}(t_1) \\ \tilde{z}_{\max}(t_2) \\ -\tilde{z}_{\min}(t_2) \\ \vdots \\ \tilde{z}_{\max}(t_s) \\ -\tilde{z}_{\min}(t_s) \end{bmatrix} \in \mathbb{R}^{2 \cdot p \cdot s}$$

Thus the number of variables and constraints in the optimization linearly depends on N, s respectively.

Theorem 2 has a potential to be extended to other control problem since the response of the system defined by (5) includes the terminal state $Fx(h)$. A deadbeat control law is similarly obtained as follows.

Corollary 4. Let $x_r(t)$ be the state response of controlled system when step reference is applied and let $x_r(\infty)$ be the steady state. By including equality constraints

$$\bar{Z}k = F\{x_r(\infty) - x_r(h)\} \qquad (30)$$

$$\bar{Z} := [\sigma_1 g_1^0, \sigma_2 g_2^0, \cdots, \sigma_N g_N^0] \qquad (31)$$

to (26), we can design a deadbeat compensation law, which attains $x_r(h) = x_r(\infty)$ at $t = h$. □

5. NUMERICAL EXAMPLES

Let us illustrate the results (Theorem 2, Corollary 4) with numerical examples: 1) compensation law for the system constraints; 2) compensation law for the deadbeat control.

5.1 Improvement of transient response

Consider the servo system depicted by Fig.1(a) with

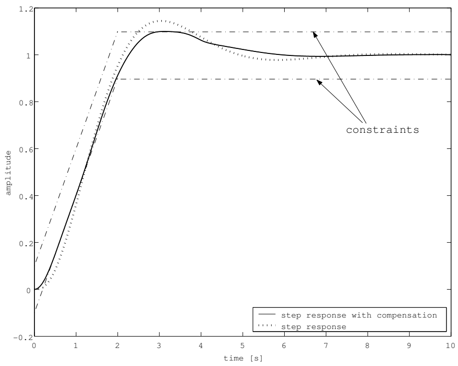
$$K = 8, \ G(s) = \frac{1}{(s+4)(s+2)}. \qquad (32)$$

The step response y is required to fulfill the following constraints (Fig.5(a)):

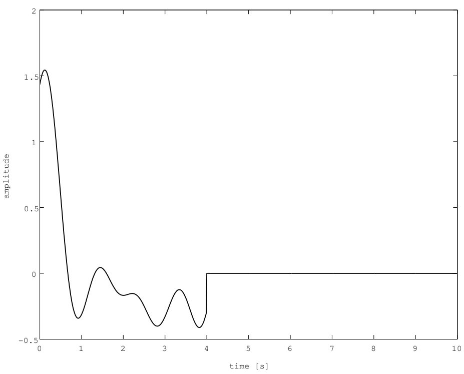
$$y_{\max}(t) = \begin{cases} \dfrac{t}{2} + 0.1 & (0 \le t \le 2) \\ 1.1 & (2 < t \le 4) \end{cases} \qquad (33)$$

$$y_{\min}(t) = \begin{cases} \dfrac{t}{2} - 0.1 & (0 \le t \le 2) \\ 0.9 & (2 < t \le 4) \end{cases} \qquad (34)$$

By employing 8 singular values in descending order (Table 1) and discretizing the uniform constraints by $t = nT$ $(T = 0.05 : n = 0, 1, 2, \cdots, 80)$, the compensation input is obtained based on Theorem 2 (Fig.5(b)). The improved response by the compensation law is depicted by Fig.5(a) (solid line) and it is observed that imposed constraints (33),(34) are satisfied.



(a) output response



(b) compensation input

Fig. 5. Example 5.1 Improvement of transient response

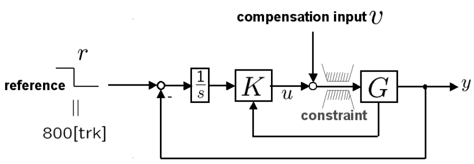


Fig. 6. Example 5.2 deadbeat control

5.2 Deadbeat control under input constraint

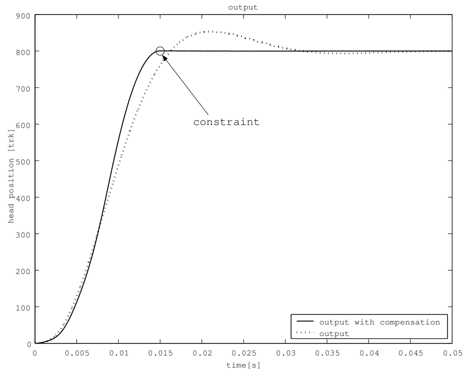
Consider a position control problem of the hard disk drive in Fig.6 (Hirata *et al.*, 1993):

$$G : \begin{cases} \dot{x}_p(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ K_p \end{bmatrix} u_p(t) \\ y_p(t) = \begin{bmatrix} 1/T_p & 0 \end{bmatrix} x_p(t) \end{cases}$$
$$K_p = 6.33 \times 10^{-2} [\mathrm{ms}^2/\mathrm{V}]$$
$$T_p = 12.7 \times 10^{-6} [\mathrm{m/trk}]. \quad (35)$$

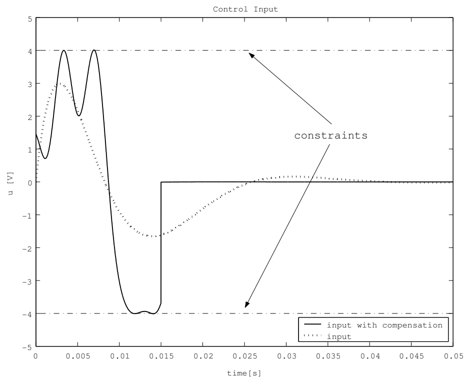
Controller K is designed by LQ control with the weighting matrices $Q = \mathrm{diag}(1, 1, 1)$, $R = 0.1$. Output and input responses when reference input 800[trk] is applied to the system are depicted in Fig.7(a) and (b) by dotted line respectively. We will design the deadbeat control such that the output y completely attains 800[trk] on time 0.015[s] and holding the value thereafter while satisfying control input constraint

$$-4 \leq u_p(t) \leq 4. \quad (36)$$

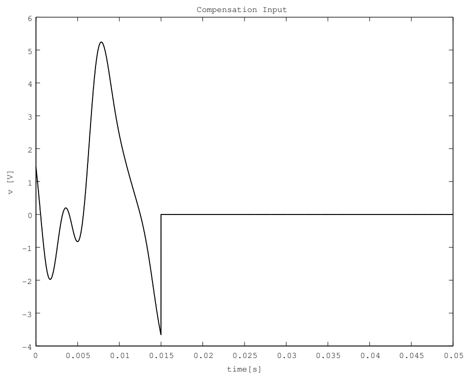
By employing 10 singular values in descending order (Table 2) and discretizing the uniform constraint by $t = nT \quad (T = 0.0001 : n = 0, 1, 2, \cdots, 150)$, the compensation input is obtained by Corollary 4 (Fig.7(c)). In Fig.7(a) solid line shows that the output response with compensation input, and it reached the desired reference within 0.015[s]. The control input is depicted in Fig.7(b) (solid line), and fulfills the constraints. The compensation input tends to large as the deadbeat control requires rapid change of the states in short period.



(a) output response



(b) control input



(c) compensation input

Fig. 7. Example 5.2 Deadbeat control under input constraint

6. CONCLUSION

A design method of compensation law for the linear constrained systems is derived. By introducing singu-

Table1 SingularValues σ_i of $\Gamma \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ on numerical example (a)

σ_i	Singular Values
σ_1	1.397157
σ_2	1.365168
σ_3	1.176565
σ_4	1.140163
σ_5	1.070774
σ_6	1.050191
σ_7	1.029234
σ_8	0.697683

Table2 SingularValues σ_i of $\Gamma \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ on numerical example (b)

σ_i	Singular Values
σ_1	36.865207
σ_2	35.857275
σ_3	19.570308
σ_4	12.147986
σ_5	7.228114
σ_6	4.931555
σ_7	3.490633
σ_8	2.683115
σ_9	2.138936
σ_{10}	1.802170

lar value decomposition for finite-horizon linear systems, a compensation law is obtained by the linear combination of singular vectors based on quadratic programming. Strength and limitation are discussed with numerical examples.

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Appendix A

Proof of theorem 1

The adjoint operator $\Gamma^* \in (\mathcal{Z}, \mathcal{V})$ is given as follows.

$$(\Gamma^* \hat{z})(\beta)$$
$$= D^{\mathrm{T}} e^{A^{\mathrm{T}}(h-\beta)} F^{\mathrm{T}} z^0 + \int_\beta^h D^{\mathrm{T}} e^{A^{\mathrm{T}}(\xi-\beta)} E^{\mathrm{T}} z^1(\xi) d\xi + F_0^{\mathrm{T}} z^1(\beta), \ (0 \le \beta \le h) \tag{A.1}$$

$$\hat{z} = \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in \mathcal{Z} \tag{A.2}$$

Introducing auxiliary variables

$$p(\xi) := \int_0^\xi e^{A(\xi-\beta)} D f(\beta) d\beta \tag{A.3}$$

$$q(\beta) := e^{A^{\mathrm{T}}(h-\beta)} F^{\mathrm{T}} g^0 + \int_\beta^h e^{A^{\mathrm{T}}(\xi-\beta)} E^{\mathrm{T}} g^1(\xi) d\xi \tag{A.4}$$

to (10), the following relations are obtained.

$$\sigma g^0 = F p(h) \tag{A.5}$$
$$\sigma g^1(\xi) = E p(\xi) + F_0 f(\xi), \ (0 \le \xi \le h) \tag{A.6}$$
$$p(0) = 0 \tag{A.7}$$
$$p'(\xi) = A p(\xi) + D f(\xi) \tag{A.8}$$
$$\sigma f(\beta) = D^{\mathrm{T}} q(\beta) + F_0^{\mathrm{T}} g^1(\beta), \ (0 \le \beta \le h) \tag{A.9}$$
$$q'(\beta) = -A^{\mathrm{T}} q(\beta) - E^{\mathrm{T}} g^1(\beta) \tag{A.10}$$
$$q(h) = F^{\mathrm{T}} g^0. \tag{A.11}$$

From (A.6), (A.8), (A.9), (A.10), the differential equation

$$\begin{bmatrix} p'(\xi) \\ q'(\xi) \end{bmatrix} = J(\sigma) \begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} \tag{A.12}$$

is derived and (A.5), (A.7), (A.11) yield boundary conditions:

$$p(0) = 0, \ q(h) = \sigma^{-1} M p(h). \tag{A.13}$$

Finally the next relation is obtained by (A.12) and (A.13)

$$\begin{bmatrix} -\sigma^{-1} M & I \end{bmatrix} e^{J(\sigma)h} \begin{bmatrix} 0 \\ I \end{bmatrix} q(0) = 0. \tag{A.14}$$

Since $q(0) = 0$ implies $(p, q) = 0$ and $(f, g) = 0$, the condition (12) is required if σ_i the singular value. Corresponding singular vectors f_i and g_i((14)-(16)) are obtained from the relations (A.5)-(A.11) with $u_i = q(0) \neq 0$ which satisfies (A.14).