MODELING AND DECENTRALIZED CONTROL OF INVENTORIES OF LINEAR DYNAMIC SUPPLY CHAINS

A. Rodriguez-Angeles [*,1] **A. Morales-Diaz** [*]
A. M. Blanco [*,2] **A. Sanchez** [**]

[*] *Programa de Matematicas Aplicadas y Computacion, Instituto Mexicano del Petroleo, Eje Central Lazaro Cardenas 152, Mexico D.F., 07730, Mexico*
[**] *Departamento de Ingenieria Electrica y Computacion, Centro de Investigacion y Estudios Avanzados (Cinvestav). Apdo. Postal 31-438, Guadalajara 45091, Mexico*

Abstract: Models for the production, storing and distribution elements of supply chains are proposed based on linear differential equations. The models consider the dynamics of inventories and production rates. A bounded control for the inventories of the mentioned elements is introduced for regulating the inventory level to a desired value manipulating production or incoming rates. Stability conditions are provided together with tuning rules. A simulation of a multi-product petrochemical supply chain illustrates the application of the dynamic models and the performance of the proposed controller.

Keywords: production systems, dynamics modelling, bounded PI control.

1. INTRODUCTION

Supply chains characterize the behavior, integration and relations between the players of a market network: suppliers, producers, distribution centers and customers. Figure 1 shows a typical arrangement. The network includes all activities related with the flow and transformation of goods from the raw material stage to the end user, and the associated information flows.

The study and control of supply chains have arisen interest on the research community because of implicit economic benefits. The markets, in which the supply chains are involved, demand a large variety of products, complying with high quality standards and competitive prices. These markets present quick changes on costumers demand, fast development of new products and high competitiveness of the players. Thus, new challenges for the participants on the global market environment have arisen, which run from minimizing inventories of raw material and finished products to quick

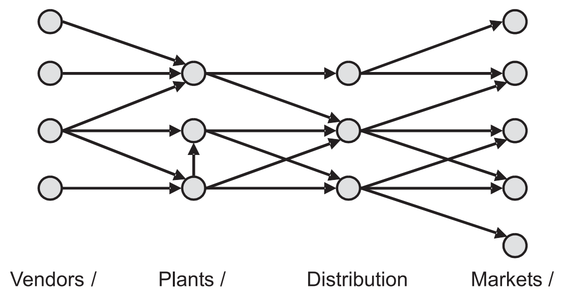

Fig. 1. Material flows in a supply chain.

[1] Corresponding author. e-mail address: arangele@imp.mx, Phone: +52 (55) 91757543, Fax: +52 (55) 91756277
[2] On leave from PLAPIQUI (UNS-CONICET), Argentina

distribution networks. A way to successfully address those challenges is to study and characterize the dynamic behavior of enterprise systems and all the elements on the market network, and to control the entire system to achieve a desired goal.

Most of the work on control of supply chains has been done on the planning and scheduling of the inventories and orders handling. For example, (Perea-Lopez *et al.*, 2001) introduced a decentralized control with a production policy. An extension to this scheme is presented by (Perea-Lopez *et al.*, 2003), where model predictive control is used to improve the forecast of the planning of inventories and handling of orders in a horizon of time. Few works have addressed the stability analysis of supply chains and the control of their elements as dynamic systems. (Nagatani and Helbing, 2004) propose a bounded controller for the production rate based on hyperbolic functions. It regulates the production rates up to a maximum of magnitude 1, while inventories are free to evolve in a bounded region, thus limiting the kind of systems for which the proposed controller applies.

This paper addresses the modelling of supply chains and inventory control to a desired value, by regulating the production or incoming rate. Different to the hyperbolic functions considered in (Nagatani and Helbing, 2004), exponential functions bounded between $[0, 1]$ are used. Thus a normalized production or incoming rate is controlled, then an scaling factor is introduced to recover the physical limitations of the system.

Section 2, following (Helbing, 2003), presents models for the dynamics of inventories and production rates of the elements of supply chains, based on linear differential equations. In section 3 a bounded control to regulate the inventory levels by manipulating the production or incoming rates is introduced. Stability conditions are provided in section 4. A simulation of a multi-product petrochemical supply chain illustrates the applicability and performance of the proposed controller.

2. DYNAMIC MODELS

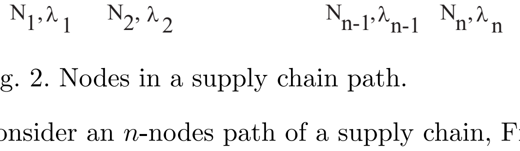

Fig. 2. Nodes in a supply chain path.

Consider an n-nodes path of a supply chain, Figure 2. Each node i (for $i = 1, \ldots, n$) is modeled by its inventory level N_i, and its production or incoming rate λ_i depending if it is a producer, warehouse, supplier or distribution center. Following (Helbing, 2003) and (Perea-Lopez *et al.*, 2003), the elements/nodes of the linear supply chain are modeled by linear differential equations as follows.

2.1 Model for producer nodes

The dynamics of a producer node $i \in \mathcal{C}_p$, (with $\mathcal{C}_p$ the set of indexes of producer nodes of the supply chain) is given by the change of its inventory level N_i and production rate λ_i

$$\frac{dN_i}{dt} = \lambda_i - \lambda_{d\_p,i} \quad (1)$$

$$\frac{d\lambda_i}{dt} = \frac{1}{T_i}\left(W_i - \lambda_i\right) \quad (2)$$

where T_i models the adaptation time of the production rate λ_i. W_i is the control action manipulating the production rate λ_i. $\lambda_{d\_p,i}$ represents the total demand of products for the producer i, and it is given by

$$\lambda_{d\_p,i} = \sum_{j \in \mathcal{C}_{d\_p,i}} F_{i,j}\lambda_j \quad (3)$$

where $\mathcal{C}_{d\_p,i}$ is the set of indexes of all nodes demanding products from producer i, λ_j is the production or incoming rate of demanding node j, and $F_{i,j}$ is the stoichiometric ratio of the product of node i that is required by node $j \in \mathcal{C}_{d\_p,i}$.

2.2 Model for non producer nodes

The dynamics of a supplier, warehouse or distribution center $i \in \mathcal{C}_{np}$, ($\mathcal{C}_{np}$ the set of indexes of non producer nodes on the supply chain) is given by the change of its inventory level N_i, because they do not produced anything, thus is given by

$$\frac{dN_i}{dt} = \lambda_i - \lambda_{d\_np,i} \quad (4)$$

where λ_i is the incoming rate on non producer node i, and it is the control action to regulate the inventory level N_i. $\lambda_{d\_np,i}$ represents the total demand of stock products in the supplier, warehouse or distribution center node i

$$\lambda_{d\_np,i} = \sum_{j \in \mathcal{C}_{d\_np,i}} F_{i,j}\lambda_j \quad (5)$$

whit $\mathcal{C}_{d\_np,i}$ the set of indexes of all nodes demanding products from the supplier, warehouse or distribution center i. λ_j is the production or incoming rate of the demanding node j. $F_{i,j}$ represents the stoichiometric ratio of the stock product in node i that is required by node $j \in \mathcal{C}_{d\_np,i}$.

3. INVENTORY LEVEL CONTROL

The controller is designed to keep inventory levels at a desired value by regulating the production or incoming rate. To capture the characteristics and limitations of the supply chain, there are some conditions to be considered on the control design.

Condition 1. Considering dynamic models (1), (2) and (4), the inventory levels N_i are to be controlled indirectly through the production or incoming rates λ_i.

Condition 2. The inventory levels N_i, production and incoming rates λ_i are bounded such that they hold the physical limitations on the producers, suppliers, warehouses and distribution centers. This means that

$$0 \leq N_i \leq N_{i,max} \tag{6}$$

$$0 \leq \lambda_i \leq \lambda_{i,max} \tag{7}$$

where $N_{i,max}, \lambda_{i,max}$ are the maximum inventory level on stock and maximum production or incoming rate respectively.

Condition 3. The control action to regulate the inventory level N_i is performed by the production or incoming rate λ_i (see (1), (2) and (4)). Since the production or incoming rate λ_i is bounded (7), then the control action must be bounded accordingly, such that for a producer node W_i must hold that $0 \leq W_i \leq \lambda_{i,max}$, where $\lambda_{i,max}$ is the maximum production rate. While for a supplier, warehouse and distribution center the incoming rate λ_i must hold that $0 \leq \lambda_i \leq \lambda_{i,max}$, where $\lambda_{i,max}$ is the maximum incoming rate.

3.1 Inventory level control for producer nodes

The control action W_i, (see 2), for a producer node i (for $i \in \mathcal{C}_p$) is given by

$$W_i = \lambda_{i,max} \times \left(2 - \frac{1}{1+e^{-\alpha_i(N_{e,i}-N_{c,i})}} - \frac{1}{1+e^{-\alpha_i N_{c,i}}}\right) \tag{8}$$

where $\lambda_{i,max}$ is the maximum production rate, α_i is a parameter that regulates the convergence rate of N_i. $N_{e,i}$ is a feedback term depending on the inventory levels accordingly to a policy strategy, as proposed by (Nagatani and Helbing, 2004) it is taken that $N_{e,i} = N_i$. The term $N_{c,i}$ is a parameter that sets the maximum bound for the inventory level N_i (Nagatani and Helbing, 2004). In this paper the term $N_{c,i}$ is used to induce regulation of the inventory level N_i to a desired value $N_{d,i}$, and is proposed as

$$N_{c,i} = N_{d,i} - K_{p,i}(N_i - N_{d,i}) - K_{I,i}\int(N_i - N_{d,i})dt \tag{9}$$

where $K_{p,i}$ and $K_{I,i}$ are the positive proportional and integral control gains respectively.

In order to produced a smooth control action exponential functions are considered in (8), where the parameter α_i provides a fast responses. The transient response is shaped by the proportional gain $K_{p,i}$, whilst the steady state error convergence is settled through the integral gain $K_{I,i}$.

3.2 Inventory level control for non producer nodes

The dynamics of a supplier, warehouse or distribution node i (for $i \in \mathcal{C}_{np}$) is controlled through its incoming rate λ_i (see 4) as

$$\lambda_i = \lambda_{i,max} \times \left(2 - \frac{1}{1+e^{-\alpha_i(N_{e,i}-N_{c,i})}} - \frac{1}{1+e^{-\alpha_i N_{c,i}}}\right) \tag{10}$$

where $\lambda_{i,max}$ is the maximum incoming rate in the node, α_i is a parameter that regulates the convergence rate of N_i, $N_{e,i}$ and $N_{c,i}$ are given as for the control of producer nodes (8).

Remark 1. Notice that the term in between parenthesis in (8) and (10) is bounded such that it can take values in $[0, 1]$. Therefore this term regulates the normalized production or incoming rate of the corresponding node. After multiplying this term by $\lambda_{i,max}$ the node recovers the physical limitations on its production or incoming rate, such that, different to (Nagatani and Helbing, 2004) systems with production rates larger than 1 can be considered.

4. STABILITY ANALYSIS

The goal of this section is to establish stability equilibrium conditions for the supply chain system in closed loop form.

4.1 Stability for producer nodes

Lets first determine the equilibrium point (λ_i^*, N_i^*) for the producer node modeled by (1) and (2) in closed loop form with the controller (8), (9).

From (1) it follows that the equilibrium condition is given by $0 = \lambda_i^* - \lambda_{d\_p,i}$. This means that the production rate in the equilibrium point is $\lambda_i^* = \lambda_{d\_p,i}$. The equilibrium point in the production rate is obtained from (2) as $0 = \frac{1}{T_i}(W_i - \lambda_i^*)$. This happens if and only if $W_i = W_i(N_i^*) = \lambda_i^*$ such that $W_i(N_i^*)$ is equal to a constant. Then by considering the definitions of W_i and $N_{c,i}$, equations (8) and (9), it follows that $W_i(N_i^*)$ is constant if and only if $N_i^* = N_{d,i}$.

In order to obtain stability conditions in system (1) and (2) in the equilibrium point (λ_i^*, N_i^*), the closed loop system is linearized as

$$\frac{d\delta N_i}{dt} = \delta\lambda_i \tag{11}$$

$$\frac{d\delta\lambda_i}{dt} = \frac{1}{T_i}\left(W_i'(N_i^*)\delta N_i - \delta\lambda_i\right) \tag{12}$$

where δN_i and $\delta\lambda_i$ denote small deviations from the equilibrium point (λ_i^*, N_i^*). System (11), (12) can be written in compact form as

$$\frac{dx}{dt} = Ax \tag{13}$$

where $x = (\delta N_i, \delta\lambda_i)^T$, with

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{T_i}W'(N_i^*) & -\frac{1}{T_i} \end{pmatrix} \tag{14}$$

The matrix A is stable, i.e. has both eigenvalues on the left half plane of the complex space, if the derivative of the control function $W_i'(N_i^*)$ fulfills the stability condition

$$W'_i(N_i^*) \leq -\frac{1}{4T_i} \tag{15}$$

Inequality (15) imposes conditions on the controller gains named $\alpha_i, K_{p,i}, K_{I,i}$. By replacing the regulation inventory control function (9) in the control action W_i, (8), it follows that

$$W'_i(N_i^*) = \alpha_i\lambda_{i,max}\left(-\frac{1}{4}\left(1 + K_{P,i} + tK_{I,i}\right) + \frac{(K_{P,i} + tK_{I,i})e^{-(\alpha_i N_{d,i})}}{(1 + e^{-(\alpha_i N_{d,i})})^2}\right) \tag{16}$$

where t represents the integration time. Considering that at the equilibrium point ($N_i^* = N_{d,i}$) there is no integral action, then it can be taken that $t = 0$. From (16) sufficient conditions for the positive proportional gain are provided as follows

$$K_{P,i} \leq \left|\frac{(1 + e^{-(\alpha_i N_{d,i})})^2(\alpha_i\lambda_{i,max}T_i - 1)}{\alpha_i\lambda_{i,max}T_i(e^{(-\alpha_i N_{d,i})} - 1)^2}\right| \tag{17}$$

To obtain conditions on the integral gain $K_{I,i}$ the regulation of the inventory level $N_{c,i}$ is evaluated at the equilibrium point. By defining the regulation error $e_{i,1} = (N_i - N_{d,i})$, replacing it in (9) and taking the first derivative with respect to time as

$$\frac{dN_{c,i}}{dt} = -K_{p,i}\frac{de_{i,1}}{dt} - K_{I,i}e_{i,1} \tag{18}$$

Because it is desired that at the equilibrium point the regulation inventory level $N_i \to N_i^*$, (18) is set equal to zero and Laplace transform is applied to obtain the pole induced by the controller, i.e.

$$s = -\frac{K_{I,i}}{K_{P,i}} \tag{19}$$

Considering that the pole in (19) must not be greater than the value settled in (15), then the sufficient condition on the integral gain is

$$K_{I,i} \leq \frac{1}{4}\frac{K_{P,i}}{T_i} \tag{20}$$

The last parameter left for tuning corresponds to the convergence regulation α_i, which can be chosen as $\alpha_i \geq 1$. With this it is ensured a minimum velocity in the convergence to the equilibrium point. This parameter can be done sufficiently large. However this may introduce overshoot problems in the convergence of the supply chain.

4.2 Stability for non producer nodes

The stability conditions for the inventory level control for suppliers, warehouses and distribution centers are obtained as in the previous section. Considering equilibrium conditions for the inventory level control of the supplier, warehouse or distribution center (10), first the system (4) is linearized in the equilibrium point $\lambda_i^* = \lambda_{d\_np,i}$ and $N_i^* = N_{d,i}$ to obtain

$$\frac{d\delta N_i}{dt} = \lambda'(N_i^*)\delta N_i \tag{21}$$

where δN_i denote small deviations from the equilibrium point. The stability condition in system (21) implies the following constraint on the inventory level control λ_i for the non producer nodes

$$\lambda'(N_i^*) < 0 \tag{22}$$

Which imposes conditions for the controller gains $\alpha_i, K_{p,i}, K_{I,i}$, as follows from λ_i',

$$\lambda_i'(N_i^*) = \alpha_i\lambda_{i,max}\left(-\frac{1}{4}\left(1 + K_{P,i} + tK_{I,i}\right) + + \frac{(K_{P,i} + tK_{I,i})e^{-(\alpha_i N_{d,i})}}{(1 + e^{-(\alpha_i N_{d,i})})^2}\right) \tag{23}$$

Now, by setting integration time $t = 0$ in (23) it can be proved that for the positive proportional gain $K_{P,i}$, the following inequality must be satisfied

$$K_{P,i} < \left|\frac{\left(1 + e^{-\alpha_i N_{d,i}}\right)^2}{\left(e^{-\alpha_i N_{d,i}} - 1\right)^2}\right| \tag{24}$$

The condition on the integral gain is obtained as in the regulation inventory level function $N_{c.i}$, (19), thus $s = -\frac{K_{I,i}}{K_{P,i}}$. Since $K_{P,i} > 0$, then $K_{I,i} > 0$. As in the producer nodes the convergence regulation parameter α_i must be chosen equal or greater than 1.

5. SIMULATION EXAMPLE

The dynamic models proposed in (1), (2), (4), and the inventory level controls (8), (10), are applied to the supply chain of a typical multi product petrochemical company proposed in (Lababidi *et al.*, 2004). Figure 3 shows the supply chain network and Table 1 identifies each of the participating nodes. Hexene and catalyst are ordered, whereas ethane is obtained from a local refinery. The production of ethylene and butene is carried out in separate production plants. Intermediate storages are available for the hexene, ethylene, and butene feedstocks. There are two reactors, R1 and R2, which can produce different products denoted by A1 and A2 for reactor R1, and B1 and B2 for reactor R2. Production volumes are shipped to retailers D1 to D5, and stock volumes are kept in the warehouses and storage facilities in each one of the production plants and reactors.

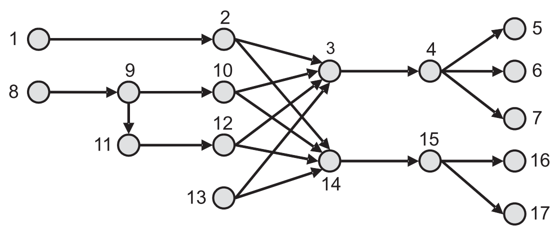

Fig. 3. Node diagram of the petrochemical plant.

Table 1. Node nomenclature.

node	element	node	element
1	hexene ship	2	hexene store
3	reactor R1	4	store A1, A2
5, 6, 7	client D1, D2, D3	8	refinery
9	ethylene plant	10	ethylene store
11	butene plant	12	butene store
13	catalyst store	14	reactor R2
15	store B1, B2	16, 17	client D4, D5

The set of producer nodes $\mathcal{C}_p$ is given by $\mathcal{C}_p = \{3, 8, 9, 11, 14\}$, while for non producer nodes $\mathcal{C}_{np} = \{1, 2, 4, 5, 6, 7, 10, 12, 13, 15, 16, 17\}$.

To clarify the modeling strategies presented in Section 2 lets consider the warehouse node 2 and the producer node 9. For the non producer node 2 ($i = 2$) accordingly to (4) and (5)

$$\frac{dN_2}{dt} = \lambda_2 - \lambda_{d\_np,2} \tag{25}$$

with $\mathcal{C}_{d\_np,2} = \{3, 14\}$, such that

$$\lambda_{d\_np,2} = F_{2,3}\lambda_3 + F_{2,14}\lambda_{14} \tag{26}$$

and from (1) - (3) for the node 9

$$\frac{dN_9}{dt} = \lambda_9 - \lambda_{d\_p,9}$$
$$\frac{d\lambda_9}{dt} = \frac{1}{T_9}(W_9 - \lambda_9)$$

with $\mathcal{C}_{d\_p,2} = \{10, 11\}$, such that

$$\lambda_{d\_p,9} = F_{9,10}\lambda_{10} + F_{9,11}\lambda_{11} \tag{27}$$

The dynamic models for the rest of the elements are obtained in a similar fashion.

The dynamic models of the supply chain elements for the petrochemical plant have been programed in SIMULINK (MATLAB). The simulated period corresponds to 24 hrs. Originally the plant produces A1 and B2, then at $t = 8$ hrs. Reactor R1 changes from A1 to A2, and at $t = 12$ hrs reactor R2 changes from B2 to B1. The stoichiometric ratios for the four considered products are listed in Table 2.

Table 2. Stoichiometric ratios for products A1, A2, B1 and B2.

Factor	A1	A2	Factor	B1	B2
$F_{2,3}$	0.25	0.4	$F_{2,14}$	0.6	0.4
$F_{10,3}$	0.15	0.2	$F_{10,14}$	0.15	0.1
$F_{12,3}$	0.5	0.3	$F_{12,14}$	0.1	0.3
$F_{13,3}$	0.2	0.1	$F_{13,14}$	0.15	0.2

For warehouses the stoichiometric ratios are 1, therefore $F_{1,2} = F_{3,4} = F_{9,10} = F_{11,12} = F_{14,15} = 1$. Since warehouses send only the amount of products required by the consumers, then the stoichiometric ratios for the costumer sources are 1, then $F_{4,5} = F_{4,6} = F_{4,7} = F_{15,16} = F_{15,17} = 1$. Finally, accordingly to the production plant requirements the stoichiometric ratios for the refinery and the ethylene plant are $F_{8,9} = 0.9$ and $F_{9,11} = 0.8$ respectively.

The plant capacity per reactor is 34.24 [MT/hr]. The demands of product A1, A2, B1 and B2 (in [MT/hr]) from the consumers during the simulated period of time are listed in Table 3. Note that the demanded product is supplied to the costumer only during the production time of the corresponding product.

Table 3. Consumer demands D1, D2, D3, D4, D5 in [MT/hr].

Product	D1	D2	D3	D4	D5
A1	5	3	12		
A2	8	6	6		
B1				9	13
B2				14	8

The storage capacity is 2000 [MT] for nodes 2, 10, and 12; 10000 [MT] for nodes 4, 13 and 15, and 500 [MT] for the storage in the producer elements (nodes 3, 8, 9, 11 and 14). According to a monthly schedule with daily resolution, it has been determined that the inventory levels in nodes 1 and 13 at the beginning of the day must be of 3000 [MT] and 2500 [MT] respectively. The inventory levels for the other nodes must be kept during the day in the values listed in Table 4.

Table 4. Desired inventory levels [MT].

Node	2	3	4	8	9
Inv. level	1000	400	7500	370	450
Node	10	11	12	14	15
Inv. level	1800	360	1600	420	5300

The initial values at $t = 0$, the maximum production and incoming rates and the adaptation parameter T_i for the production rates are listed in Table 5. The initial values for the inventories are close to the desired ones during operation (see Table 4), to generate curves with small oscillations and fast convergence. Nevertheless, the proposed controller can deal with large differences between the initial inventories and productions rates with respect to the desired ones.

Table 5. Initial values of N_i [MT], T_i [hr] and $\lambda_{i,max}$ [MT/hr].

Node	1	2	3	4	8	9
$N_i(0)$	3000	992	405	7495	377	443
$\lambda_i(0)$			24		36	39
T_i			0.008		0.001	0.001
$\lambda_{i,max}$		120	34.24	60	120	120
Node	10	11	12	13	14	15
$N_i(0)$	1808	356	1555	2500	415	5310
$\lambda_i(0)$		33			22.8	
T_i		0.003			0.001	
$\lambda_{i,max}$	120	120	120		34.24	60

For brevity of space only results for the producer nodes 3, 11 and 14 are presented.

Note that the production rates for the producer nodes, see Figure 4, satisfy the physical bounds of the system (condition (7) and Table 5), while the inventories converge to their desired values as shown in Figure 5

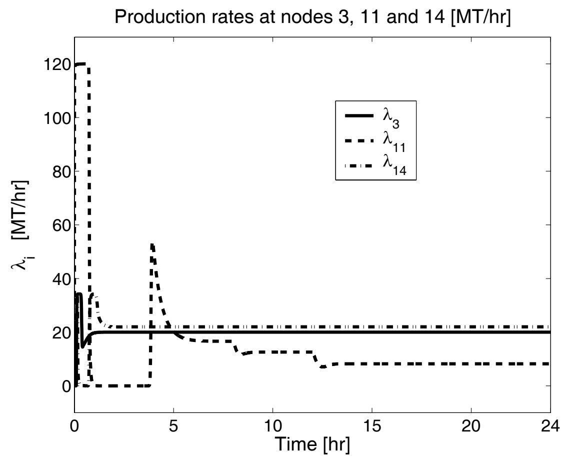

Fig. 4. Production rates.

As a result of changing the production from A1 to A2 and from B2 to B1, at t=8 and t=12 hrs. respectively, the production rate λ_{11} (4) changes its value. Therefore the proposed controller keeps the desired inventory levels by manipulating the production or incoming rates. Also notice how the

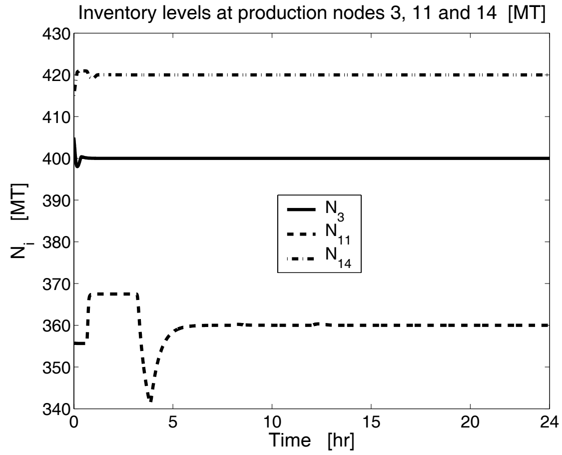

Fig. 5. Inventory levels of production nodes.

production rates evolve during the transient period (approximately for $t \in [0, 5)$) always holding condition (7), and in an smooth way due to the exponential functions in (8).

6. CONCLUSIONS AND FURTHER WORK

Dynamic linear models for the different elements of supply chains (suppliers, producers, warehouses, distribution centers and costumers) have been proposed together with a regulation control for inventory levels, by manipulating the production or incoming rates. The proposed modeling approach has been applied to a multi product petrochemical plant, showing the applicability of the proposed models and controller. The stability of the closed loop system has been validated based on a linearized system.

Supply chains considering recycle systems and nonlinearities will be address in future works.

REFERENCES

Helbing, D. (2003). Modeling supply networks and business cycles as unstable transport phenomena. *New J. of Physics* **5**(90), 1–28.

Lababidi, H.M.S., M.A. Ahmed, I.M. Alatiqi and A.F. Al-Enzi (2004). Optimizing the supply chain of a petrochemical company under uncertain operating and economic conditions. *Ind. Eng. Chem. Res* **43**, 63–73.

Nagatani, T. and D. Helbing (2004). Stability analysis and stabilization strategies for linear supply chains. *Physica A* **335**, 644–660.

Perea-Lopez, E., B.E. Ydstie and I.E. Grossmann (2003). A model predictive control strategy for supply chain optimization. *Computers and Chemical Engineering* **27**, 1201–1218.

Perea-Lopez, E., I.E. Grossmann and B.E. Ydstie (2001). Dynamic modeling and decentralized control of supply chains. *Ind. Eng. Chem. Res.* **40**, 3369–3383.