AN EXPLICIT SOLUTION TO THE DISCRETE-TIME SINGULAR LQ REGULATION PROBLEM FOR NON-SQUARE PLANT

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Abstract: In this paper, an explicit solution to the discrete-time singular LQ problem for a non-square plant will be discussed. It will be shown that the solution will be given by an inverted interactorizing gain for the minimum phase image of some squarizing systems. An interactor matrix plays an important role for squarizing of a given plant.

Keywords: LQ regulation problem, singular weightings, explicit solution, discrete-time systems, interactor matrix.

1. INTRODUCTION

It is well known that the solution to the discrete-time LQ regulation problem or the discrete-time optimal filtering problem can be obtained by solving a discrete-time algebraic Riccati equation. Although the solution cannot be obtained in terms of the system parameters explictly, it can be found in some limiting case where the input weighting matrix or the covariance matrix of measurement noise tends to zero, i.e., an explicit solution can be found in the singular weighting case (Peng and Kinnaert, 1992, Bittanti *et al*, 1995, Huang and S.L. Shah, 1997).

In the above papers, a relation between the singular problem and an interactor matrix (Wolovich and Falb, 1976) was pointed out. From the view point of exact model matching (Elliott and Wolovich, 1984), the explicit solution is given by a special feedback gain of inverted interactorizing (Mutoh and Nikifork 1992), where the nilpotent interactor (Rogozinski, *et al* 1987) is used. Unfortunately, the above literatures only consider the square transfer function matrix case, and there is no discussion about non-square case.

Recently, a simple method to calculate an interactor which has all-pass property in discrete-time was presented for a plant having more inputs than the outputs (i.e., FAT plant), using the result of Kase *et al* (1999). Then, a design of inverted interactorizing and model matching was reported (Kase, Watanabe and Mutoh 2004). Using the results, it was reported an explicit solution to the discrete-time optimal LQ regulation problem with singular weightings for a FAT plant (Kase, Miyoshi and Mutoh 2004). In this paper, it will be extended to the plant having non-square transfer function matrix.

To show the explicit solution, it is necessary to investigate an inductive property of some Toeplitz matrices, which was shown in Kase *et al* (1999) for square transfer function matices and was not important to derive the interactor. Then, an explicit solution will be given by using pseudoinverse of such Toeplitz matrix. It will be also pointed out that the solution will be given by an inverted interactorizing gain for the minimum phase image of some squarizing systems. An interactor matrix plays an important role for squarizing of a given plant.

The paper is organized as follows. In the next section, problem statement is given. In section III, the psuedoinverse of some Toeplitz matrices, which is used to derive an interactor with all-pass property, is discussed. In section IV, the explicit solution to the discrete-time singular LQ optimal problem is derived. However, the solution may not stabilize the system. In section V, a stabilizing solution is given from the view point of inverted interactorizing. Conclusions are given in secion VI.

2. PROBLEM STATEMENT

Consider the following p-inputs, m-outputs linear discrete-time system:

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

$$y(t) = Cx(t), \ x_0 := x(0) \tag{2}$$

where $u(t)$, $y(t)$ and $x(t)$ are the input, output and state vector of the system, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$, and $C \in \mathbf{R}^{m\times n}$. The feedback control law is given by

$$u(t) = -Fx(t). \tag{3}$$

The problem is to find a stabilizing feedback gain matrix F which minimizes the following cost function J:

$$J = \sum_{i=0}^{\infty} y^T(t)Qy(t), \quad Q = Q^T \geq 0. \tag{4}$$

In the followings, it is assumed without loss of generality that $Q = I_m$. If F which minimizes $y^T(t)y(t)$ is independent of time t, it also minimizes the cost J. Thus, consider the problem to find F which minimizes $y^T(t)y(t)$, and then show the time independence of F.

Multiplying z to both side of eqn.(2) successively, employing eqn.(1) for substitutiton, yields the following relation:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(i) \end{bmatrix} = \boldsymbol{O}_i(C,A)x_0 + \begin{bmatrix} 0_{m\times pi} \\ \boldsymbol{T}_{i-1} \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(i-1) \end{bmatrix}, \tag{5}$$

where

$$\boldsymbol{O}_i(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^i \end{bmatrix},$$
$$\boldsymbol{T}_{i-1} = \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{i-1}B & CA^{i-2}B & \cdots & CB \end{bmatrix}. \tag{6}$$

Multiplying z to both side of eqn.(3) successively, employing eqn.(1) for substitutiton, yields the following relation:

$$\begin{aligned} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(i-1) \end{bmatrix} &= - \begin{bmatrix} F \\ FA_F \\ \vdots \\ FA_F^{i-1} \end{bmatrix} x_0 \\ &= -\boldsymbol{O}_{i-1}(F, A_F)x_0, \end{aligned} \tag{7}$$

where $A_F := A = -BF$. Therefore, the cost J yields

$$\begin{aligned} J &= \|Cx_0\|^2 \\ &+ \lim_{i\to\infty} \| \{\boldsymbol{O}_{i-1}(C,A)A - \boldsymbol{T}_{i-1}\boldsymbol{O}_{i-1}(F,A_F)\} x_0\|^2 \end{aligned} \tag{8}$$

and thus, F which minimizes J is obtained by solving the following optimization problem for a large natural number i:

$$\begin{aligned} &\min_F \|\boldsymbol{O}_{i-1}(C,A)A - \boldsymbol{T}_{i-1}\boldsymbol{O}_{i-1}(F,A_F)\|^2 \\ &= \min_F \|\boldsymbol{O}_{i-1}(C,A_F)A_F\|^2. \end{aligned} \tag{9}$$

Using the pseudoinverse $\boldsymbol{T}_{i-1}^{\dagger}$ of $\boldsymbol{T}_{i-1}$, the optimal matrix $\boldsymbol{O}_{i-1}^{opt}$ of $\boldsymbol{O}_{i-1}(F, A_F)$ will be given by

$$\boldsymbol{O}_{i-1}^{opt} := \boldsymbol{T}_{i-1}^{\dagger}\boldsymbol{O}_{i-1}(C,A)A. \tag{10}$$

Since F can be calculated by

$$F = \begin{bmatrix} I_p & 0_{p\times p(i-1)} \end{bmatrix} \boldsymbol{O}_{i-1}(F, A_F),$$

define F by

$$\begin{aligned} F &= \begin{bmatrix} I_p & 0_{p\times p(i-1)} \end{bmatrix} \boldsymbol{O}_{i-1}^{opt} \\ &= \begin{bmatrix} I_p & 0_{p\times p(i-1)} \end{bmatrix} \boldsymbol{T}_{i-1}^{\dagger}\boldsymbol{O}_{i-1}(C,A)A. \end{aligned} \tag{11}$$

Example 1 Consider the following system:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -0.1 & -1.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.22 & -1.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.36 & -1.5 \end{bmatrix},$$

$$B = \begin{bmatrix} 0\ 0\ 0 \\ 1\ 0\ 0 \\ 0\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 0 \\ 0\ 0\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1.1 & 1 & 1.2 & 1 \\ 0.1 & 1 & 0.2 & 1 & 0.3 & 1 \end{bmatrix}.$$

For the above system, it is assumed that $i = 3$. Then, the feedback gain F is defined by

$$\begin{aligned} F &= \left[\, I_3 \ 0_{3\times 6} \,\right] \boldsymbol{T}_2^{\dagger} \boldsymbol{O}_2(C, A)A \\ &= \begin{bmatrix} -.0833 & .7223 & -.0733 & 1.5036 & .0600 & 2.4849 \\ -.0333 & -.2142 & -.0733 & -.2532 & -.1200 & -.2921 \\ .0167 & 1.1507 & -.0733 & -2.0099 & -.3000 & -3.0691 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} &FA_F \\ &= \begin{bmatrix} -.0033 & 1.7819 & .0147 & 2.2611 & -.0120 & 2.6036 \\ .0000 & -.0333 & .0000 & -.0733 & .0000 & -.1200 \\ .0033 & -1.8486 & -.0147 & -2.4077 & .0120 & -2.8436 \end{bmatrix}. \end{aligned}$$

On the other hand, for $i = 3$, the optimal approximate solution of eqn.(10) using pseudoinverse is given by

$$\begin{aligned} \begin{bmatrix} F \\ FA_F \\ FA_F^2 \end{bmatrix}^{opt} &= \begin{bmatrix} CB & 0 & 0 \\ CAB & CB & 0 \\ CA^2B & CAB & CB \end{bmatrix}^{\dagger} \begin{bmatrix} CA \\ CA^2 \\ CA^3 \end{bmatrix} \\ &= \begin{bmatrix} & & & \vdots & & \\ 0 & -.0306 & 0 & -.0692 & 0 & -.1146 \\ 0 & -.0333 & 0 & -.0733 & 0 & -.1200 \\ 0 & -.0361 & 0 & -.0774 & 0 & -.1254 \\ & & & \vdots & & \end{bmatrix}, \end{aligned}$$

which is differ from FA_F which was calculated through F. However, by calculating FA_F from the second block of

$$\begin{bmatrix} F \\ FA_F \\ FA_F^2 \\ FA_F^3 \end{bmatrix}^{opt} = \begin{bmatrix} CB & 0 & 0 & 0 \\ CAB & CB & 0 & 0 \\ CA^2B & CAB & CB & 0 \\ CA^3B & CA^2B & CAB & CB \end{bmatrix}^{\dagger} \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix},$$

it can be obtained FA_F by eqn.(11).

More explict form of F will be obtained by analysing $\boldsymbol{T}_{i-1}^{\dagger}$. In the following sections, It will be discussed that a property of $\boldsymbol{T}_{i-1}^{\dagger}$, and then it will be shown that the feedback gain F given in eqn.(11) satisfies eqn.(10), which means that F is the optimal solution.

3. A PROPERTY OF PSEUDO INVERSE FOR SOME TOEPLITZ MATRICES

For a temporary, consider the problem to find a polynomial matrix $L(z)$ satisfying

$$\lim_{z\to\infty} L(z)G(z) = K \quad (full\ rank) \tag{12}$$

for a given transfer function matrix $G(z) = C(zI - A)^{-1}B$. Such $L(z)$ is called an interactor matrix for $G(z)$ (Wolovich and Falb 1976). A derivation method of the interactor and full rank matrix K for FAT transfer function matrix was reported in Kase, Watanabe and Mutoh (2004), where it was shown that the space spanned by K does not depend on the derivation method of the interactor.

The coefficient matrix of $L(z)$ satisfies the following relation (Mutoh and Ortega, 1993):

$$\boldsymbol{L}\boldsymbol{T}_{w-1} = K\boldsymbol{J}_{w-1} \tag{13}$$

where

$$\begin{aligned} &L(z) = L_0 + zL_1 + z^2L_2 + \cdots + z^wL_w, \\ &\boldsymbol{L} = \left[\, L_1 \ \cdots \ L_w \,\right], \\ &\boldsymbol{J}_{w-1} = \left[\, I_p \ 0_{p\times p(w-1)} \,\right], \quad L_i \in \mathbf{R}^{m\times m}. \end{aligned}$$

and see Kase, Watanabe and Mutoh (2004) to find the integer w.

In order to solve eqn.(13), the pseudoinverse of $\boldsymbol{T}_{w-1}$ is important. For this, the following Lemma holds. The proof will be found in Kase *et al* (1999) for a square plant and in Kase, Miyoshi and Mutoh (2004) for a general case.

Lemma 1. For the integer $k \geq w-1$, the following equation holds:

$$\boldsymbol{T}_k^{\dagger} = \begin{bmatrix} \boldsymbol{M}_k \\ \boldsymbol{Z}_k - \boldsymbol{T}_{k-1}^{\dagger}\boldsymbol{O}_k\boldsymbol{M}_k \end{bmatrix} \tag{14}$$

where

$$\begin{aligned} \boldsymbol{M}_k &= K^{\dagger}\left[\, \boldsymbol{L} \ \boldsymbol{0}_k \,\right], \\ \boldsymbol{Z}_k &= \left[\, 0_{kp\times m} \ \boldsymbol{T}_{k-1}^{\dagger} \,\right], \\ \boldsymbol{O}_k &= \boldsymbol{O}_{k-1}(C, A)AB, \\ \boldsymbol{0}_k &= 0_{m\times m(k-w+1)}. \end{aligned} \tag{15}$$

Corollary 1. Let $\boldsymbol{T}_j^{\dagger}(i)$ denote the i-th row block of $\boldsymbol{T}_j^{\dagger}$, i.e.,

$$\boldsymbol{T}_j^{\dagger}(i) := \left[\, 0_{p\times p(i-1)} \ \boldsymbol{J}_{j-i+1} \,\right] \boldsymbol{T}_j^{\dagger}.$$

Then, the following relation holds for the positive integer i and j:

$$\boldsymbol{T}_{w+i+j-2}^{\dagger}(i) = \left[\, \boldsymbol{T}_{w+i-2}^{\dagger}(i) \ 0_{p\times mj} \,\right]. \tag{16}$$

(Proof). The result can be obtained by the following calculations.

$$
\begin{aligned}
&\boldsymbol{T}^{\dagger}_{w+i+j-2}(i)\\
&= \left[\, 0_{p\times p(i-1)} \;\; \boldsymbol{J}_{w+j-1} \,\right] \boldsymbol{T}^{\dagger}_{w+i+j-2}\\
&= \left[\, 0_{p\times p(i-2)} \;\; \boldsymbol{J}_{w+j-1} \,\right]\\
&\quad \times (\boldsymbol{Z}_{w+i+j-2} - \boldsymbol{T}^{\dagger}_{w+i+j-3}\boldsymbol{O}_{w+i+j-2}\boldsymbol{M}_{w+i+j-2})\\
&= \left[\, 0_{p\times p(i-2)} \;\; \boldsymbol{J}_{w-1} \,\right] (\left[\, \boldsymbol{Z}_{w+i-2} \;\; 0_{p(w+i-2)\times mj} \,\right]\\
&\quad - \left[\, \boldsymbol{T}^{\dagger}_{w+i-3} \;\; 0_{p(w+i-2)\times mj} \,\right] \boldsymbol{O}_{w+i+j-2}\boldsymbol{M}_{w+i+j-2})\\
&= \left[\, 0_{p\times p(i-2)} \;\; \boldsymbol{J}_{w-1} \,\right] (\left[\, \boldsymbol{Z}_{w+i-2} \;\; 0_{p(w+i-2)\times mj} \,\right]\\
&\quad -\boldsymbol{T}^{\dagger}_{w+i-3}\boldsymbol{O}_{w+i-2} \left[\, \boldsymbol{M}_{w+i-2} \;\; 0_{p(w+i-2)\times mj} \,\right])\\
&= \left[\, 0_{p\times p(i-2)} \;\; \boldsymbol{J}_{w-1} \,\right]\\
&\times \left[\boldsymbol{Z}_{w+i-2} - \boldsymbol{T}^{\dagger}_{w+i-3}\boldsymbol{O}_{w+i-2}\boldsymbol{M}_{w+i-2} \;\; 0_{p(w+i-2)\times mj} \right]\\
&= \left[\, \boldsymbol{T}^{\dagger}_{w+i-2}(i) \;\; 0_{m(w+i-2)\times mj} \,\right]
\end{aligned}
$$

Theorem 1. If the solution of eqn.(13) is given by

$$\boldsymbol{L} = K\boldsymbol{J}_{w-1}\boldsymbol{T}^{\dagger}_{w-1}, \tag{17}$$

and the interactor is given by

$$L(z) = K\boldsymbol{J}_{w-1}\boldsymbol{T}^{\dagger}_{w-1}\begin{bmatrix} zI_m \\ z^2 I_m \\ \vdots \\ z^w I_m \end{bmatrix},$$

then the following properties hold:

$$
\begin{aligned}
&\textbf{P1}\;\; L(z)L^{\sim}(z) = \boldsymbol{L}\boldsymbol{L}^T, && (18)\\
&\textbf{P2}\;\; \boldsymbol{O}_{w-1}(C, A_F)B = \boldsymbol{L}^{\dagger}, && (19)\\
&\textbf{P3}\;\; CA_F^w = 0 && (20)
\end{aligned}
$$

where $\boldsymbol{L}^{\dagger}$ is the pseudoinverse of $\boldsymbol{L}$, and

$$
\begin{aligned}
&L^{\sim}(z) = L^T(z^{-1}) = L_0^T + z^{-1}L_1^T + \cdots + z^{-w}L_w^T,\\
&F = \boldsymbol{L}\boldsymbol{O}_{w-1}(C, A)A.
\end{aligned}
$$

(Proof). See Kase *et al* (1999). Note that K can be determined before calculating $\boldsymbol{L}$ (Kase, Watanabe and Mutoh, 2004).

If $G(z)$ is TALL, then define $G_e(z)$ by

$$
\begin{aligned}
G_e(z) &= \left[\, G(z) \;\; G_{ap}(z) \,\right],\\
G_{ap}(z) &:= \begin{bmatrix} 0_{p\times(m-p)} \\ z^{-1}I_{m-p} \end{bmatrix}
\end{aligned}
$$

and calculate $L(z)$ for $G_e(z)$ (without loss of generality) assuming $\lim_{z\to\infty} L(z)G_e(z) = I_m$. It is clear that $L(z)$ satisfies eqn.(12) and $K = \begin{bmatrix} I_p \\ 0_{(m-p)\times p} \end{bmatrix}$.

For the square transfer function matrix case, an explicit solution to the singular LQ problem is given by the feedback gain of the inverted interactorizing (Mutoh and Nikifork, 1992), using the interactor with all-pass property (Peng and Kinnaert, 1992, Kase *et al*, 1999). The above Theorem shows that the interactor given here has all-pass property, and it implies that there exists a close relation between the above interactor and the singular LQ optimal problem.

4. DERIVATION OF FEEDBACK GAIN

In this section, it will be shown that the feedback gain F given in eqn.(11) minimizes the cost (9) for sufficient large $i > w$, using the result of Lemma 1.

Now, define F more explicitly by

$$
\begin{aligned}
F &:= \boldsymbol{J}_{w-1}\boldsymbol{O}^{opt}_{w-1}\\
&= \boldsymbol{J}_{w-1}\boldsymbol{T}^{\dagger}_{w-1}\boldsymbol{O}_{w-1}(C, A)A\\
&= \boldsymbol{M}_w\boldsymbol{O}_{w-1}(C, A)A.
\end{aligned} \tag{21}
$$

Since FA_F is given by the second block of $\boldsymbol{O}_w(F, A_F)$, it can be written by

$$FA_F = \left[\, 0_{p\times p} \;\; I_p \;\; 0_{p\times p(w-1)} \,\right] \boldsymbol{O}_w(F, A_F). \tag{22}$$

On the other hand, using eqn.(14),

$$
\begin{aligned}
&\left[\, 0_{p\times p} \;\; I_p \;\; 0_{p\times p(w-1)} \,\right] \boldsymbol{O}^{opt}_w\\
&= \left[\, 0_{p\times p} \;\; \boldsymbol{J}_{w-1} \,\right] \boldsymbol{T}^{\dagger}_w\boldsymbol{O}_w(C, A)A\\
&= \left[\, 0_{p\times p} \;\; \boldsymbol{J}_{w-1} \,\right] \begin{bmatrix} \boldsymbol{M}_w \\ \boldsymbol{Z}_w - \boldsymbol{T}^{\dagger}_{w-1}\boldsymbol{O}_w\boldsymbol{M}_w \end{bmatrix} \boldsymbol{O}_w(C, A)A\\
&= \boldsymbol{J}_{w-1}(\boldsymbol{Z}_w - \boldsymbol{T}^{\dagger}_{w-1}\boldsymbol{O}_w\boldsymbol{M}_w)\boldsymbol{O}_w(C, A)A\\
&= \left[\, 0_{p\times p} \;\; \boldsymbol{M}_{w-1} \,\right] \boldsymbol{O}_w(C, A)A - FBF\\
&= FA_F,
\end{aligned} \tag{23}
$$

i.e.,

$$FA_F = \left[\, 0_{p\times p} \;\; I_p \;\; 0_{p\times p(w-1)} \,\right] \boldsymbol{O}^{opt}_w.$$

Thus

$$
\begin{aligned}
FA_F &= \boldsymbol{T}^{\dagger}_w(2)\boldsymbol{O}_w(C, A)A = \cdots\\
&= \boldsymbol{T}^{\dagger}_{w+j}(2)\boldsymbol{O}_{w+j}(C, A)A
\end{aligned} \tag{24}
$$

by Corollary 1.

For $i = k > w$, assume that

$$
\begin{aligned}
FA_F^{k-w} &= \left[\, 0_{p\times p(k-w)} \;\; \boldsymbol{J}_{w-1} \,\right] \boldsymbol{O}^{opt}_{k-1}\\
&= \left[\, 0_{p\times p(k-w)} \;\; \boldsymbol{J}_{w-1} \,\right] \boldsymbol{T}^{\dagger}_{k-1}\boldsymbol{O}_{k-1}(C, A)A.
\end{aligned} \tag{25}
$$

Then, for $i = k + 1$,

$$
\begin{aligned}
&\left[\, 0_{p\times p(k-w+1)} \;\; \boldsymbol{J}_{w-1} \,\right] \boldsymbol{O}^{opt}_k\\
&= \left[\, 0_{p\times p(k-w+1)} \;\; \boldsymbol{J}_{w-1} \,\right] \boldsymbol{T}^{\dagger}_k\boldsymbol{O}_k(C, A)A\\
&= \left[\, 0_{p\times p(k-w+1)} \;\; \boldsymbol{J}_{w-1} \,\right] \begin{bmatrix} \boldsymbol{M}_k \\ \boldsymbol{Z}_k - \boldsymbol{T}^{\dagger}_{k-1}\boldsymbol{O}_k\boldsymbol{M}_k \end{bmatrix}\\
&\quad \times \boldsymbol{O}_k(C, A)A
\end{aligned}
$$

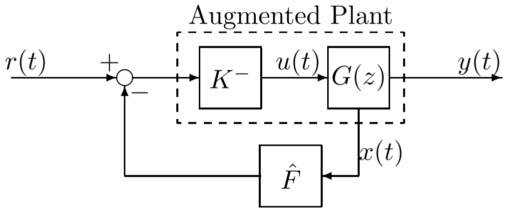

Fig. 1. Inverted Interactorizing Systems by State Feedback

$$
\begin{aligned}
&= \left[\, 0_{p\times p(k-w)} \;\; \boldsymbol{J}_{w-1} \,\right] (\boldsymbol{Z}_k - \boldsymbol{T}^{\dagger}_{k-1}\boldsymbol{O}_k\boldsymbol{M}_k) \\
&\quad \times \boldsymbol{O}_k(C,A)A \\
&= \left[\, 0_{p\times p(k-w)} \;\; \boldsymbol{J}_{w-1} \,\right] \boldsymbol{T}^{\dagger}_{k-1}\boldsymbol{O}_{k-1}(C,A)A \\
&\quad \times (A - B\boldsymbol{M}_k\boldsymbol{O}_k(C,A)A) \\
&= FA_F^{k-w}\cdot A_F
\end{aligned}
$$

Thus,

$$
\begin{aligned}
FA_F^{k-w} &= \boldsymbol{T}^{\dagger}_{k-1}(k-w+1)\boldsymbol{O}_{k-1}(C,A)A = \cdots \\
&= \boldsymbol{T}^{\dagger}_{w+j}(k-w+1)\boldsymbol{O}_{w+j}(C,A)A \qquad (26)
\end{aligned}
$$

for $i = k+1$. If A_F is stable, then $CA_F^i \to 0$ and thus the second term of the cost J converges to some fixed value from eqn.(9). Therefore, the feedback gain F does not depend on t and the minimum value of J only depends on the initial value of the state.

5. STABILITY COSIDERATIONS

Since the cost function defined by eqn.(4) does not contain the terms relating to input $u(t)$, it may permit the unbounded input. So it is important to discuss the stability condition and it will be considered in this section via the idea of inverted interactorizing (Mutoh and Nikifork, 1992). At first, the following result for inverted interactorizing holds.

Lemma 2. For a given $m \times p$ $(m \geq p)$ transfer function matrix $G(z)$, let (A, B, C) denote a realization of $G(z)$. Define the feedback gain by

$$
\begin{aligned}
\hat{F} &= \left[\, L_0 \;\; \boldsymbol{L} \,\right] \boldsymbol{O}_{w-1}(C,A) \\
&= \left[\, L_0 \; L_1 \; \cdots \; L_w \,\right] \begin{bmatrix} C \\ CA \\ \vdots \\ CA^w \end{bmatrix}. \qquad (27)
\end{aligned}
$$

Then, by the control law

$$
u(t) = -K^{-}(\hat{F}x(t) - r(t)), \qquad (28)
$$

the inverted interactorizing is achieved.

(Proof). See Kase, Watanabe and Mutoh (2004).

FAT plants

If $\boldsymbol{L}$ can be determined by eqn.(17) and $L_0 = 0$, then eqn.(28) yields to

$$
u(t) = -Fx(t) + K^{-}r(t). \qquad (29)
$$

Therefore, the feedback gain defined by eqn.(21) achieved the inverted interactorizing. In the control law (28), the generalized inverse K^{-} can be interpreted as the squarizing pre-compensator (see Fig.1). Then, $\hat{F}$ is the conventional inverted interactorizing feedback gain for the squarized plant $G(z)K^{-}$. Since the inverted interactorizing is a control strategy which eliminates the effect of zeros by poles-zeros cancellation, the closed-loop system is internally stable if and only if the given plant does not have unstable zeros. Therefore, the feedback gain, which makes the closed-loop be stable, can be calculated by the following procedures.

step 1 For a given plant, calculate an interactor $L(z)$ and its gain matrix K.

step 2 Let $G(z)$ denote the transfer function matrix of given plant. For the squarizing system $G(z)K^{\dagger}$, calculate its minimum phase image, i.e., find a square transfer function matrix $\hat{G}(z)$ such that

$$
(K^{\dagger})^T G^{\sim}(z)G(z)K^{\dagger} = \hat{G}^{\sim}(z)\hat{G}(z), \qquad (30)
$$

where $\hat{G}(z)$ is stably invertible.

step 3 Let $(A,\ B,\ \hat{C})$ denote a realization of $\hat{G}(z)$. The feedback gain F, which minimizes the cost J, is defined by

$$
F = K^{\dagger}\boldsymbol{L}\boldsymbol{O}_{w-1}(\hat{C},A)A. \qquad (31)
$$

By using the inverted interactorizing feedback gain F for a minimum phase image, the minimum value of the cost J is invariant. Since the interactor is common between a given plant $G(z)K^{\dagger}$ and its minimum phase image $\hat{G}(z)$ (Mutoh, 1995),

$$
\lim_{z\to\infty} L(z)G(z)K^{\dagger} = \lim_{z\to\infty} L(z)\hat{G}(z) = I_m.
$$

TALL plants

Although the inverted interactorizing will not be acheived for this case by applying control law (29), A-matrix of the closed-loop system is given by $A - BK^{\dagger}\boldsymbol{L}\boldsymbol{O}_{w-1}(C,A)A$. Now, consider the following transfer function matrix:

$$
K^{\dagger}L(z)G(z) = \left[\begin{array}{c|c} A & B \\ \hline K^{\dagger}\boldsymbol{L}\boldsymbol{O}_{w-1}(C,A)A & I_p \end{array}\right]. \qquad (32)
$$

For the transfer function matrix, the interactor matrix is I_p and the inverted interactorizing gain is given by $I_p \cdot K^{\dagger}\boldsymbol{L}\boldsymbol{O}_{w-1}(C,A)A$. Therefore, $A - BK^{\dagger}\boldsymbol{L}\boldsymbol{O}_{w-1}(C,A)A$ is the A-matrix of the inverted interactorizing system for the plant given by eqn.(32). Therefore, the feedback gain, which makes the closed-loop be stable, can be calculated by the following procedures.

step 1 For a given plant, calculate an interactor $L(z)$ and its gain matrix K.

step 2 Let $G(z)$ denote the transfer function matrix of given plant. For the squarizing system $K^{\dagger}L(z)G(z)$, calculate its minimum phase image, i.e., find a square transfer function matrix $\hat{G}(z)$ such that

$$G^{\sim}(z)L^{\sim}(z)(K^{\dagger})^{T}K^{\dagger}L(z)G(z) = \hat{G}^{\sim}(z)\hat{G}(z), \tag{33}$$

where $\hat{G}(z)$ is stably invertible.

step 3 Let $(A,\ B,\ \hat{C},\ I_p)$ denote a realization of $\hat{G}(z)$. The feedback gain F, which minimizes the cost J, is defined by

$$F = I_p \cdot \hat{C} = \hat{C}. \tag{34}$$

In the above procedures, the hardest part is to calculate a minimum phase image. A method to obtain a minimum phase image is given in Kase, Miyoshi and Mutoh (2004) using the derivation of generalized interactor (Mutoh and Nikifork, 1994).

6. CONCLUSIONS

In this paper, an explicit solution to the discrete-time optimal LQ regulation problem with singular weightings for a plant having non-square transfer function matrix was discussed. The optimal solution was given by an inverted interactorizing gain for the minimum phase image of some squarizing systems. An interactor matrix plays an important role for squarizing of a given plant. Although it is theoretically valuable to derive a state feedback control law for non-square system, it would be more interesting and useful to derive a dynamic output feedback control law. The problem will be presented in the forthcoming paper.

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