ROBUST $\mathcal{D}$-STABILITY ANALYSIS OF UNCERTAIN POLYNOMIAL MATRICES VIA POLYNOMIAL-TYPE MULTIPLIERS

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Abstract: This paper addresses robust $\mathcal{D}$-stability analysis problems of uncertain polynomial matrices. The underlying idea we follow is that a given polynomial matrix is $\mathcal{D}$-stable if and only if there exist polynomial-type multipliers that render the resulting polynomial matrices to be strictly positive over a specific region on the complex plane. By applying the generalized $\mathcal{S}$-procedure technique, we show that those positivity analysis problems can be reduced into feasibility tests of linear matrix inequalities (LMIs). Thus we can obtain varieties of LMI conditions for (robust) $\mathcal{D}$-stability analysis of polynomial matrices according to the degree/structure of the multipliers to be employed. In particular, we show that existing LMI conditions for robust $\mathcal{D}$-stability analysis can be viewed as particular cases of the proposed conditions, where the degree of the multipliers chosen to be the same as those of the polynomial matrices to be examined. It turns out that, by increasing the degree of the multipliers, we can readily obtain less conservative LMI conditions than the one found in the literature.

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Keywords: uncertain polynomial matrices, $\mathcal{D}$-stability, positive polynomials.

1. INTRODUCTION

Robust stability/performance analysis problems of linear time-invariant systems have been a central topic in control theory. Since the linear matrix inequalities (LMIs) have proved to be effective for dealing with such analysis problems, intensive research effort has been made in this direction. Among them, in the late 1990's, novel LMI conditions have been proposed for robust $\mathcal{D}$-stability analysis of matrix polytopes in (Oliveira *et al.*, 1999; Peaucelle *et al.*, 2000), where a matrix is said to be $\mathcal{D}$-stable if all of its eigenvalues lie in $\mathcal{D} \subset \mathbf{C}$. In these new LMI conditions, auxiliary matrix variables are introduced so that the system matrices have no multiplication relation with the Lyapunov matrices. This allows us to employ parameter-dependent Lyapunov functions to assess the robust $\mathcal{D}$-stability, which is very promising to alleviate the conservatism arising in the conventional quadratic-stability-based LMI conditions (Boyd *et al.*, 1994).

In spite of these successful achievements, in our opinion, the LMI conditions in (Oliveira *et al.*, 1999; Peaucelle *et al.*, 2000) are not thoroughly examined and leave a good deal for further investigation. In particular, since the auxiliary variables in (Oliveira *et al.*, 1999; Peaucelle *et al.*, 2000) are introduced via algebraic manipulations related to the Elimination Lemma (Boyd *et al.*, 1994), the interpretation on the roll of those auxil-

[1] This work is supported in part by the Ministry of Education, Culture, Sports, Science and Technology of Japan under Grant-in-Aid for Young Scientists (B), 15760314.

iary variables is not necessarily clear. Clarifying such interpretations would be very important to grasp essential logical structures that lead us to such new LMI conditions and to derive yet less conservative LMI conditions. This paper pursues this direction under the general setting of robust $\mathcal{D}$-stability analysis of uncertain polynomial matrices (Henrion *et al.*, 2003*a*; Henrion *et al.*, 2003*b*; Oliveira *et al.*, 2002), and indeed provides less conservative LMI conditions based on an intriguing interpretation on the role of the auxiliary variables. We note here that in (Dettori and Scherer, 2000), the role of those auxiliary variables has also been examined from the viewpoints of parameter-dependent multipliers.

The underlying idea we follow in this paper is that a given polynomial matrix is $\mathcal{D}$-stable iff there exist *polynomial-type multipliers* that render the resulting polynomial matrices to be strictly positive over $\mathcal{D}^c \subset \mathbf{C}$, where $\mathcal{D}^c$ stands for the complement of $\mathcal{D}$ in $\mathbf{C}$. By means of the recently developed generalized $\mathcal{S}$-procedure technique (Iwasaki *et al.*, 2000; Iwasaki and Hara, 2005; Ebihara *et al.*, 2004), we show that those positivity analysis problems can be reduced into feasibility tests of linear matrix inequalities (LMIs). Thus we can obtain wide varieties of LMI conditions for (robust) $\mathcal{D}$-stability analysis of polynomial matrices according to the degree/structure of the multipliers to be employed. It follows that those LMI conditions in (Henrion *et al.*, 2001; Henrion *et al.*, 2003*a*; Oliveira *et al.*, 1999; Peaucelle *et al.*, 2000) can be regarded as particular cases of the proposed conditions where the degree of the multipliers are fixed to the same as those of the polynomial matrices to be examined. By increasing the degree of the multipliers, we readily obtain less conservative LMI conditions. Interpretations of those new LMI conditions from the viewpoint of Lyapunov's theory are also provided.

We use the following notations in this paper. For a complex matrix A, its complex conjugate transpose is denoted by A^*. For $A \in \mathbf{C}^{n\times n}$, $\mathrm{He}\{A\}$ is a shorthand notation for $A + A^*$. The symbols $\mathbf{H}_n$ and $\mathbf{P}_n$ denote the sets of $n \times n$ Hermitian matrices and positive-definite Hermitian matrices, respectively. For matrices Ψ and P, we denote by $\Psi \otimes P$ their Kronecker product. Given a positive integer N, let $\mathbf{Z}_N$ denote the set of positive integers up to N, i.e., $\mathbf{Z}_N := \{1, \cdots, N\}$.

2. PROBLEM DESCRIPTION AND PRELIMINARY RESULTS

Given $A_k \in \mathbf{C}^{n\times m}$ $(k = 0, \cdots, N)$ with $n \geq m$, let us consider the $n \times m$ complex polynomial matrix

$$\mathcal{A}(s) = \sum_{k=0}^{N} s^k A_k \tag{1}$$

We assume that the normal rank of $\mathcal{A}(s)$ is m. The (finite) *zeros* of $\mathcal{A}(s)$ are defined as complex numbers $z \in \mathbf{C}$ such that $\mathrm{rank}(A(z)) < m$. Furthermore, for a given region $\mathcal{D} \subset \mathbf{C}$, we say that $\mathcal{A}(s)$ is $\mathcal{D}$-stable if the zeros of $\mathcal{A}(s)$ belong to $\mathcal{D}$ (Henrion *et al.*, 2001). Our goal is to derive necessary and sufficient conditions for the $\mathcal{D}$-stability of $\mathcal{A}(s)$ in terms of LMIs.

Note that $\mathcal{A}(s)$ is $\mathcal{D}$-stable iff $\mathcal{A}(s)$ is of full-column rank for all $s \in \mathcal{D}^c$. In the subsequent discussions, we restrict our attention to the regions defined below.

Definition 1. Given $\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{12}^* & \psi_{22} \end{bmatrix} \in \mathbf{H}_2$ with $\det(\Psi) < 0$, we define $\mathcal{D}_\Psi$ by

$$\mathcal{D}_\Psi := \{\lambda \in \mathbf{C} : \ \sigma(\lambda, \Psi) < 0\}, \ \sigma(\lambda, \Psi) := \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Psi \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

A useful tool for the $\mathcal{D}$-stability analysis of polynomial matrices is the generalized $\mathcal{S}$-procedure (Iwasaki *et al.*, 2000; Iwasaki and Hara, 2005; Ebihara *et al.*, 2004). Basically, the generalized $\mathcal{S}$-procedure concerns inequality conditions on a Hermitian matrix Θ and a subset of Hermitian matrices $\mathcal{S}$ given by

$$\begin{aligned} &\zeta^*\Theta\zeta > 0 \quad \forall \zeta \in \Omega, \\ &\Omega := \{\zeta \in \mathbf{C}^m : \ \zeta \neq 0, \quad \zeta^* S \zeta \geq 0 \quad \forall S \in \mathcal{S}\} \end{aligned} \tag{2}$$

It can be seen that a sufficient condition for (2) is

$$\exists S \in \mathcal{S} \text{ such that } \Theta > S \tag{3}$$

The procedure to replace the condition (2) by (3) is called the generalized $\mathcal{S}$-procedure. Generally, this replacement introduces conservatism since the condition (3) is only sufficient for (2) and may not be necessary. By exploring explicit conditions that render the generalized $\mathcal{S}$-procedure to be nonconservative, in (Ebihara *et al.*, 2004), the authors introduced the notion of *one-vector-lossless sets*. The definition is now reviewed.

Definition 2. (Ebihara *et al.*, 2004) A subset $\mathcal{S} \subset \mathbf{H}_m$ is said to be *one-vector-lossless* if it has the following properties:
(a) $\mathcal{S}$ is convex.
(b) $S \in \mathcal{S} \Rightarrow \tau S \in \mathcal{S} \quad \forall \tau > 0$.
(c) For each nonzero matrix $H \in \mathbf{C}^{m\times m}$ with rank r that satisfies

$$H = H^* \geq 0, \quad \mathrm{tr}(SH) \geq 0 \quad \forall S \in \mathcal{S},$$

there exist vectors $\zeta_i \in \mathbf{C}^m$ $(i = 1, \cdots, r)$ such that $H = \sum_{i=1}^{r} \zeta_i \zeta_i^*$ and the condition $\zeta_j^* S \zeta_j \geq 0$ $(\forall S \in \mathcal{S})$ holds for at least one index j.

It should be noted that the above definition has been introduced by relaxing the requirements for the *lossless sets* given in (Iwasaki *et al.*, 2000). See (Ebihara *et al.*, 2004) for mutual connections of these two sets and related discussions.

In (Ebihara *et al.*, 2004), the generalized $\mathcal{S}$-procedure was proved to be nonconservative if $\Theta \geq 0$ and the set $\mathcal{S}$ is one-vector-lossless. The next theorem can be regarded as a slight extension of this preceding result.

Theorem 3. For given $\hat{\mathcal{A}} \in \mathbf{C}^{n\times m}$ and a one-vector-lossless set $\mathcal{S} \subset \mathbf{H}_m$, the following statements are equivalent.

(i) There exists $\hat{\mathcal{F}} \in \mathbf{C}^{n\times m}$ such that

$$\begin{array}{l} \zeta^* \mathrm{He}\{\hat{\mathcal{A}}^* \hat{\mathcal{F}}\}\zeta > 0 \quad \forall \zeta \in \Omega, \\ \Omega := \{\zeta \in \mathbf{C}^m :\ \zeta \neq 0, \quad \zeta^* S \zeta \geq 0 \quad \forall S \in \mathcal{S}\} \end{array} \tag{4}$$

(ii) There exists $\hat{\mathcal{G}} \in \mathbf{C}^{n\times m}$ and $S \in \mathcal{S}$ such that

$$\mathrm{He}\{\hat{\mathcal{A}}^* \hat{\mathcal{G}}\} > S \tag{5}$$

Moreover, for every $\hat{\mathcal{G}}$ that satisfies (5) for some $S \in \mathcal{S}$, $\hat{\mathcal{F}} = \hat{\mathcal{G}}$ satisfies (4).

Proof 1. See the appendix section for the proof.

In Theorem 3, we should be careful on the relations among $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$ satisfying (4) and (5), respectively. In particular, we emphasize that for $\hat{\mathcal{F}}$ that satisfies (4), $\hat{\mathcal{G}} = \hat{\mathcal{F}}$ *does not* satisfy (5) in general. If the matrix $\hat{\mathcal{F}}$ satisfies $\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{F}}\} \geq 0$ in addition to (4), however, then we see from (Ebihara *et al.*, 2004) that $\hat{\mathcal{G}} = \hat{\mathcal{F}}$ indeed satisfies (5). Besides these delicate points, the results in Theorem 3 should be examined more carefully in comparison with (Iwasaki *et al.*, 2000; Iwasaki and Hara, 2005) and (Ebihara *et al.*, 2004). However, this is out of the scope of the paper and we do not pursue this direction in the sequel.

In the next section, we address robust $\mathcal{D}$-stability analysis problems of uncertain polynomial matrices by means of Theorem 3. The next lemma also plays an important role for this purpose.

Lemma 4. (Ebihara *et al.*, 2004) Let $\Psi \in \mathbf{H}_2$ with $\det(\Psi) < 0$ and $\Gamma \in \mathbf{C}^{2m\times l}$ be given. Define a subset of Hermitian matrices by

$$\mathcal{S} := \{\Gamma^*(\Psi \otimes P)\Gamma :\ P \in \mathbf{P}_m\} \tag{6}$$

Then the set $\mathcal{S}$ is one-vector-lossless.

Note that the matrix Ψ will be used to characterize the region $\mathcal{D}$ in the subsequent discussions.

3. $\mathcal{D}$-STABILITY ANALYSIS OF POLYNOMIAL MATRICES

3.1 $\mathcal{D}$-stability Analysis of Polynomial Matrices via Polynomial-Type Multipliers

The goal of this paper is to analyze the $\mathcal{D}$-stability of the polynomial matrix $\mathcal{A}(s)$ given in (1) by using polynomial-type multipliers. To explicate our basic ideas concisely, however, we restrict our attention to the simplest case where $\mathcal{A}(s)$ is given by a matrix pencil of the form $\mathcal{A}_1(s) = sI - A$. It is straightforward to extend the following results to handle general polynomial matrix cases. Note that the $\mathcal{D}$-stability of a matrix $A \in \mathbf{C}^{n\times n}$ is equivalent to the $\mathcal{D}$-stability of the matrix pencil $\mathcal{A}_1(s)$.

It is obvious that $\mathcal{A}_1(s)$ is $\mathcal{D}$-stable iff there exists a *polynomial-type multiplier* $\mathcal{F}(s)$ that satisfies

$$\mathrm{He}\{\mathcal{A}_1(s)^*\mathcal{F}(s)\} > 0 \quad \forall s \in \mathcal{D}^{\mathrm{c}} \tag{7}$$

Standard choice of the multiplier would be $\mathcal{F}(s) = F_1 s + F_0$ where $F_1, F_0 \in \mathbf{C}^{n\times n}$ are matrix variables to be determined. Indeed, it is enough to seek for a multiplier of this form since $\mathcal{A}_1(s)$ is $\mathcal{D}$-stable iff (7) holds with $\mathcal{F}(s) = \mathcal{A}_1(s)$, whose degree is one. Note however that we can seek for multipliers of higher-degree. For example, if we employ a multiplier $\mathcal{F}(s)$ of the form

$$\mathcal{F}(s) = \begin{bmatrix} sI \\ I \end{bmatrix}^* \begin{bmatrix} F_{21} & F_{11} & F_{01} \\ F_{20} & F_{10} & F_{00} \end{bmatrix} \begin{bmatrix} s^2 I \\ sI \\ I \end{bmatrix}, \tag{8}$$

we see from (7) that $\mathcal{A}_1(s)$ is $\mathcal{D}$-stable iff

$$\begin{bmatrix} s^2 I \\ sI \\ I \end{bmatrix}^* \mathrm{He}\left\{ \begin{bmatrix} I & 0 \\ -A^* & I \\ 0 & -A^* \end{bmatrix} \begin{bmatrix} F_{21} & F_{11} & F_{01} \\ F_{20} & F_{10} & F_{00} \end{bmatrix} \right\} \begin{bmatrix} s^2 I \\ sI \\ I \end{bmatrix} > 0 \quad \forall s \in \mathcal{D}^{\mathrm{c}} \tag{9}$$

The usefulness of those higher-degree multipliers will be clarified later on when studying robust $\mathcal{D}$-stability analysis problems.

Our primary concern is to give explicit formulas for the existence of such polynomial-type multipliers. As we have seen, this amounts to verifying the positivity of polynomial matrices of the form as in (9) over $\mathcal{D}^{\mathrm{c}}$. In the next theorem, we show how those positivity analysis problems can be reduced into feasibility tests of LMIs by means of Theorem 3 and Lemma 4.

Theorem 5. Let complex matrices $\hat{\mathcal{A}} \in \mathbf{C}^{Nn\times(M+1)m}$ of the form

$$\hat{\mathcal{A}} = \begin{bmatrix} \hat{\mathcal{A}}_{11} & \cdots & \hat{\mathcal{A}}_{1,M+1} \\ \vdots & \ddots & \vdots \\ \hat{\mathcal{A}}_{N,1} & \cdots & \hat{\mathcal{A}}_{N,M+1} \end{bmatrix}, \ \hat{\mathcal{A}}_{ij} \in \mathbf{C}^{n\times m} \ (n \geq m)$$

and $\Psi \in \mathbf{H}_2$ with $\det(\Psi) < 0$ be given. Suppose one of the following conditions holds:

1. $\hat{\mathcal{A}}_1 := [\ \hat{\mathcal{A}}_{11}^* \ \cdots \ \hat{\mathcal{A}}_{N,1}^*\ ]^*$ is of full-column rank.
2. $\psi_{11} < 0$.

Then the following conditions are equivalent.

(i) There exists $\hat{\mathcal{F}} \in \mathbf{C}^{Nn\times(M+1)m}$ such that

$$\begin{bmatrix} s^M I_m \\ \vdots \\ I_m \end{bmatrix}^* \mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{F}}\} \begin{bmatrix} s^M I_m \\ \vdots \\ I_m \end{bmatrix} > 0 \quad \forall s \in \mathcal{D}_\Psi^{\mathrm{c}} \tag{10}$$

(ii) There exists $\hat{\mathcal{F}} \in \mathbf{C}^{Nn\times(M+1)m}$ such that

$$\zeta^* \mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{F}}\}\zeta > 0 \quad \forall \zeta \in \Omega', \tag{11}$$

$$\begin{array}{l} \Omega' := \{\zeta = [\ \zeta_M^* \ \cdots \ \zeta_0^*\ ]^* \in \mathbf{C}^{(M+1)m} :\ \zeta_0 \neq 0, \\ \exists s \in \mathcal{D}_\Psi^{\mathrm{c}} \text{ such that } \zeta_{k+1} = s\zeta_k \ (k = 0, \cdots, M-1)\} \end{array}$$

(iii) There exists $\hat{\mathcal{F}} \in \mathbf{C}^{Nn\times(M+1)m}$ such that

$$\zeta^* \mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{F}}\}\zeta > 0 \quad \forall \zeta \in \Omega, \tag{12}$$

$$\Omega := \left\{\zeta \in \mathbf{C}^{(M+1)m} :\ \zeta \neq 0,\ \zeta^* S \zeta \geq 0\ \forall S \in \mathcal{S}_W\right\},$$

$$\mathcal{S}_W := \{W^*(\Psi \otimes P)W : \ P \in \mathbf{P}_{Mm}\},$$
$$W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad W_1 := \begin{bmatrix} I_{Mm} \\ 0_{m,Mm} \end{bmatrix}^*, \quad W_2 := \begin{bmatrix} 0_{m,Mm} \\ I_{Mm} \end{bmatrix}^*$$

(iv) There exists $\hat{\mathcal{G}} \in \mathbf{C}^{Nn\times(M+1)m}$ and $P \in \mathbf{P}_{Mm}$ such that
$$\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\} - W^*(\Psi \otimes P)W > 0 \qquad (13)$$

Moreover, for every $\hat{\mathcal{G}}$ that satisfies (13) for some $P \in \mathbf{P}_{Mm}$, $\hat{\mathcal{F}} = \hat{\mathcal{G}}$ satisfies (10), (11) and (12). If the matrices $\hat{\mathcal{A}}$ and Ψ are all real, the equivalence still holds when we restrict $\hat{\mathcal{F}}$ in (i), (ii), (iii) and $\hat{\mathcal{G}}$ and P in (iv) to be real.

Proof 2. The equivalence of (i) and (ii) is apparent and thus we prove only the equivalence of (ii), (iii) and (iv).

Equivalence of (ii) and (iii) We first prove that (iii) $\Rightarrow$ (ii) holds, which can be established by showing $\Omega' \subset \Omega$. To this end, suppose $\zeta = [\ \zeta_M^* \ \cdots \zeta_0^*\ ]^* \in \Omega'$ and define $\zeta_u, \zeta_l \in \mathbf{C}^{Mm}$ by

$$\zeta_u := [\ \zeta_M^* \ \cdots \zeta_1^*\ ]^*, \quad \zeta_l := [\ \zeta_{M-1}^* \ \cdots \zeta_0^*\ ]^* \quad (14)$$

Then we see from the definition of Ω' that the following inequality holds for all $P \in \mathbf{P}_{Mm}$.

$$\zeta^* W^*(\Psi \otimes P)W\zeta = \sigma(s,\Psi)\zeta_l^* P \zeta_l \geq 0$$

This shows that $\zeta \in \Omega$ and hence $\Omega' \subset \Omega$ holds.

On the other hand, to prove (ii) $\Rightarrow$ (iii), suppose $\zeta = [\ \zeta_M^* \ \cdots \zeta_0^*\ ]^* \in \Omega$ and define ζ_u and ζ_l by (14). Then, from the definition of Ω, we have

$$\begin{aligned}\mathrm{tr}\,((&\psi_{11}\zeta_u\zeta_u^* + \psi_{12}^*\zeta_u\zeta_l^* \\ &+\psi_{12}\zeta_l\zeta_u^* + \psi_{22}\zeta_l\zeta_l^*)P) \geq 0 \quad \forall P \in \mathbf{P}_{Mm}\end{aligned}$$

The above condition implies

$$\psi_{11}\zeta_u\zeta_u^* + \psi_{12}^*\zeta_u\zeta_l^* + \psi_{12}\zeta_l\zeta_u^* + \psi_{22}\zeta_l\zeta_l^* \geq 0 \quad (15)$$

If $\zeta_l \neq 0$, we see from (Rantzer, 1996) that (15) holds iff $\zeta_u = s\zeta_l$ for some $s \in \mathcal{D}_\Psi^{\mathrm{c}}$ and this implies $\zeta \in \Omega'$. With this fact and $\Omega' \subset \Omega$, we have

$$\begin{aligned}&\Omega = \Omega' \cup \Upsilon, \\ &\Upsilon := \{\zeta = [\ \zeta_M^* \ \cdots \ \zeta_0^*\ ]^* \in \mathbf{C}^{(M+1)m} : \ \zeta \in \Omega, \\ &\qquad \zeta_M \neq 0, \ \zeta_i = 0 \ (i = 0,\cdots,M-1)\}\end{aligned}$$

Under the assumption $\psi_{11} < 0$, however, the set Υ is empty and hence $\Omega = \Omega'$, which implies (ii) $\Rightarrow$ (iii). On the other hand, if $\psi_{11} \geq 0$, i.e., if $\mathcal{D}_\Psi^{\mathrm{c}}$ is unbounded, note that the upper left $m \times m$ block of $\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{F}}\}$ in (11) should be positive semidefinite. In addition, $\hat{\mathcal{A}}_1^*\hat{\mathcal{A}}_1 > 0$ holds from the assumption. Hence, for every $\hat{\mathcal{F}}$ that satisfies the condition in (ii), we see that the following inequality holds for $\alpha > 0$.

$$\zeta^*\mathrm{He}\{\hat{\mathcal{A}}^*(\hat{\mathcal{F}} + \alpha\hat{\mathcal{A}})\}\zeta > 0 \quad \forall \zeta \in \Omega' \cup \Upsilon \qquad (16)$$

This clearly shows that (ii) $\Rightarrow$ (iii).

Equivalence of (iii) and (iv) The equivalence follows immediately from Theorem 3 and Lemma 4. Indeed, since the set $\mathcal{S}_W$ is one-vector-lossless by Lemma 4, we can conclude that (iii) is equivalent to (iv) by Theorem 3.

The real case results can be shown by following similar lines to (Iwasaki *et al.*, 2000). This completes the proof. □

From the discussions around (7) and Theorem 5, we can obtain the following results.

Corollary 6. For given matrices $A \in \mathbf{C}^{n\times n}$ and $\Psi \in \mathbf{H}_2$ with $\det(\Psi) \neq 0$, the following statements are equivalent.

(i) The matrix A is $\mathcal{D}_\Psi$-stable.

(ii) There exists a multiplier $\mathcal{F}_1(s)$ of the form $\mathcal{F}_1(s) = F_1 s + F_0$ that satisfies
$$\mathrm{He}\{(sI - A)^*\mathcal{F}_1(s)\} > 0 \quad \forall s \in \mathcal{D}_\Psi^{\mathrm{c}} \qquad (17)$$

(iii) There exists G_0, G_1 and $P \in \mathbf{P}_n$ that satisfy
$$\mathrm{He}\left\{\begin{bmatrix} I \\ -A^* \end{bmatrix}\begin{bmatrix} G_1 & G_0 \end{bmatrix}\right\} - \Psi \otimes P > 0 \quad (18)$$

Moreover, for every $\{G_0, G_1\}$ that satisfies (18) for some $P \in \mathbf{P}_n$, $\mathcal{F}_1(s) = G_1 s + G_0$ satisfies (17).

The LMI condition (18) coincides with those obtained in (Peaucelle *et al.*, 2000). In these previous studies, the condition (18) was derived by applying the Elimination Lemma (Boyd *et al.*, 1994) to the standard Lyapunov inequalities and through such algebraic manipulations, the variables G_0 and G_1 are introduced as auxiliary variables. In stark contrast, Corollary 6 provides an intriguing interpretation on the role of those variables and shows a new insight to the condition (18). Namely, the LMI condition (18) implies the existence of the first-degree multiplier $\mathcal{F}_1(s) = G_1 s + G_0$ that ensures $\mathcal{D}_\Psi$-stability of the matrix A via (17).

Remark 7. It should be noted that, since Corollary 6 is based on Theorem 5, we cannot conclude that for every $\mathcal{F}_1(s) = G_1 s + G_0$ that satisfies (17), the coefficients G_0 and G_1 also satisfy (18). Namely, there might be a gap between the existence condition of the multipliers (17) and numerically tractable LMI condition (18). This delicate problem essentially stems from the fact that, for $\hat{\mathcal{F}}$ that satisfies (4) in Theorem 3, $\hat{\mathcal{G}} = \hat{\mathcal{F}}$ does not satisfy (5) in general. Around these points, we need to study in deeper detail in the future.

In addition to the intriguing interpretation of the auxiliary variables, it is obvious from the above discussions that we can readily obtain new LMI conditions for the $\mathcal{D}_\Psi$-stability of $\mathcal{A}_1(s)$ by employing different types of multipliers. Although varieties of LMI conditions follow according to the degree/structure of the multipliers, we give here only one example, which corresponds to (8).

Corollary 8. Let matrices $A \in \mathbf{C}^{n\times n}$ and $\Psi \in \mathbf{H}_2$ with $\det(\Psi) \neq 0$ be given. Then, A is $\mathcal{D}_\Psi$-stable iff there exists $G_{ij} \in \mathbf{C}^{n\times n}$ $(i = 0,1,2, \ j = 0,1)$ and $\Pi \in \mathbf{P}_{2n}$ that satisfy the LMI condition (19) given at the top of the next page. Moreover, for every $\{G_{ij} \ (i = 0,1,2, \ j = 0,1)\}$ that satisfies (19) for some $\Pi \in \mathbf{P}_{2n}$, $F_{ij} = G_{ij}$ $(i = 0,1,2, \ j = 0,1)$ satisfy (9).

$$\mathrm{He}\left\{\begin{bmatrix} I & 0 \\ -A^* & I \\ 0 & -A^* \end{bmatrix}\begin{bmatrix} G_{21} & G_{11} & G_{01} \\ G_{20} & G_{10} & G_{00} \end{bmatrix}\right\} - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T (\Psi \otimes \Pi) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0 \tag{19}$$

$$\mathrm{He}\left\{\begin{bmatrix} I & 0 \\ -\tilde{A}_i^* & I \\ 0 & -\tilde{A}_i^* \end{bmatrix}\begin{bmatrix} G_{21} & G_{11} & G_{01} \\ G_{20} & G_{10} & G_{00} \end{bmatrix}\right\} - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T (\Psi \otimes \Pi_i) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} > 0 \quad \forall i \in \mathbf{Z}_N \tag{22}$$

$$\tilde{A}_1 = \begin{bmatrix} -4.4850 & -0.7750 & -8.1000 \\ -0.3400 & -3.0000 & 0.5000 \\ 4.3250 & 8.1000 & 5.0200 \end{bmatrix}, \ \tilde{A}_2 = \begin{bmatrix} 0.4850 & 2.7750 & 6.1000 \\ 5.3400 & -3.0000 & 0.5000 \\ -6.3250 & -6.1000 & -12.0200 \end{bmatrix} \tag{25}$$

3.2 Robust $\mathcal{D}$-stability Analysis of Polynomial Matrices via Polynomial-Type Multipliers

Let us consider the robust $\mathcal{D}$-stability analysis problems of a matrix polytope $\boldsymbol{A}$ defined by

$$\begin{aligned} &\boldsymbol{A} = \{A_\delta : \ \delta \in \boldsymbol{\delta}\}, \quad A_\delta = \sum_{i=1}^N \delta_i \tilde{A}_i, \\ &\delta = [\ \delta_1, \ \cdots, \ \delta_N\ ]^T \in \boldsymbol{\delta}, \\ &\boldsymbol{\delta} := \left\{\delta : \ \delta_i \geq 0 \quad \forall i \in \mathbf{Z}_N, \quad \sum_{i=1}^N \delta_i = 1\right\} \end{aligned} \tag{20}$$

Here, $\tilde{A}_i \in \mathbf{C}^{n\times n}$ $(i \in \mathbf{Z}_N)$ are given matrices.

To deal with such robust $\mathcal{D}$-stability analysis problems in a less conservative fashion, Peaucelle et al. showed that the LMI condition (18) is very useful. Indeed, the following sufficient condition for the robust $\mathcal{D}$-stability of matrix polytope $\boldsymbol{A}$ readily follows from (18).

Proposition 9. (Peaucelle *et al.*, 2000) The matrix polytope $\boldsymbol{A}$ in (20) is robustly $\mathcal{D}_\Psi$-stable if there exist $G_0, \ G_1 \in \mathbf{C}^{n\times n}$ and $P_i \in \mathbf{P}_n$ that satisfy

$$\mathrm{He}\left\{\begin{bmatrix} I \\ -\tilde{A}_i^* \end{bmatrix}\begin{bmatrix} G_1 & G_0 \end{bmatrix}\right\} - \Psi \otimes P_i > 0 \ \forall i \in \mathbf{Z}_N \tag{21}$$

From the viewpoints of the polynomial-type multipliers, the implication of the above LMI condition is now obvious. Namely, if (21) holds, then there exists a first degree multiplier $sG_1 + G_0$ that ensures the robust $\mathcal{D}_\Psi$-stability of $\boldsymbol{A}$ via

$$\mathrm{He}\{(sI - A_\delta)^*(sG_1 + G_0)\} > 0 \quad \forall s \in \mathcal{D}_\Psi^c \quad \forall \delta \in \boldsymbol{\delta}$$

On the other hand, if we consider the higher-degree multiplier (8), we can obtain the following proposition that provides a new LMI condition for the robust $\mathcal{D}$-stability analysis of matrix polytopes.

Proposition 10. The matrix polytope $\boldsymbol{A}$ in (20) is robustly $\mathcal{D}_\Psi$-stable if there exist $G_{ij} \in \mathbf{C}^{n\times n}$ $(i = 0, 1, 2, \ j = 0, 1)$ and $\Pi_i \in \mathbf{P}_{2n}$ $(i \in \mathbf{Z}_N)$ that satisfy the LMIs (22) given at the upper part of this page.

It is worth mentioning that, if (21) is feasible, then (22) is always feasible. This is because whenever $G_0, \ G_1 \in \mathbf{C}^{n\times n}$ and $P_i \in \mathbf{P}_n$ satisfy (21), the conditions in (22) are satisfied by

$$\begin{bmatrix} G_{21} & G_{11} & G_{01} \\ G_{20} & G_{10} & G_{00} \end{bmatrix} = \begin{bmatrix} G_1 & G_0 & 0 \\ 0 & G_1 & G_0 \end{bmatrix}, \quad \Pi_i = \begin{bmatrix} P_i & 0 \\ 0 & P_i \end{bmatrix}$$

Namely, the new LMI condition (22) encompasses the existing condition (21) as a special case.

3.3 Interpretations of the New LMI Conditions from the Viewpoint of Lyapunov's Theorem

Although we have derived the LMI conditions (21) and (22) by using polynomial-type multipliers, it is meaningful to examine those LMI conditions from the viewpoint of Lyapunov's theorem. As shown in (Peaucelle *et al.*, 2000), if the LMI condition (21) holds, then the following Lyapunov inequality holds for all $\delta \in \boldsymbol{\delta}$.

$$\begin{bmatrix} A_\delta \\ I \end{bmatrix}^* (\Psi \otimes P_\delta) \begin{bmatrix} A_\delta \\ I \end{bmatrix} < 0 \tag{23}$$

Here, $P_\delta = \sum_{i=1}^N \delta_i P_i$. On the other hand, if (22) holds, then it is not hard to see that the Lyapunov inequality (23) holds with

$$P_\delta = \begin{bmatrix} A_\delta \\ I \end{bmatrix}^* \left(\sum_{i=1}^N \delta_i \Pi_i\right) \begin{bmatrix} A_\delta \\ I \end{bmatrix} \tag{24}$$

Namely, the new LMI condition (22) can also be interpreted in the way that it ensures the robust stability of $\boldsymbol{A}$ in (20) via the Lyapunov inequality (23), where the Lyapunov matrix P_δ depends cubically on the parameter δ as in (24).

4. NUMERICAL EXAMPLES

Given two vertex matrices (25) shown at the upper part of this page, let us define a matrix polytope $\boldsymbol{A}$ by (20). The problem here is to assess the robust Hurwitz stability of $\boldsymbol{A}$. This problem is borrowed from Example 2 of (Chesi *et al.*, 2003), where it was shown that $\boldsymbol{A}$ is indeed robustly Hurwitz stable whereas the conditions (21) derived in (Peaucelle *et al.*, 2000) and the one in (Leite and Peres, 2003) fail to conclude the robust stability. On the other hand, with the help of MATLAB LMI Control Toolbox, we can confirm that the new LMI condition (22) is indeed feasible. Thus, in this particular example, the condition (22) yields less conservative analysis results than (Leite and Peres, 2003; Peaucelle *et al.*, 2000). Note however that the condition (22) could be more conservative than (Leite and Peres, 2003) in other cases since there is no explicit inclusion relationship among these two conditions.

5. CONCLUSION

In this paper, we addressed robust $\mathcal{D}$-stability analysis problems of uncertain polynomial matrices. By means of the polynomial-type multipliers and the generalized $\mathcal{S}$-procedure technique, we showed a constructive way to derive LMI conditions for robust $\mathcal{D}$-stability analysis. It turned out that we can readily obtain less conservative LMI conditions by employing appropriate higher-degree multipliers, at the expense of the computational complexity of the resulting LMIs.

APPENDIX

Proof of Theorem 3. (ii) $\Rightarrow$ (i) Suppose (ii) holds. Then, there exists $\hat{\mathcal{G}}_0 \in \mathbf{C}^{n\times m}$ and $S_0 \in \mathcal{S}$ such that

$$\zeta^*(\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}_0\} - S_0)\zeta > 0 \quad \forall \zeta \neq 0$$

The above inequality implies

$$\zeta^*\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}_0\}\zeta > 0 \quad \forall \zeta \in \Omega_0,$$
$$\Omega_0 := \{\zeta \in \mathbf{C}^m : \ \zeta \neq 0, \quad \zeta^* S_0 \zeta \geq 0\}$$

Since $\Omega \subset \Omega_0$, we can conclude that the condition (ii) implies (i). The last statement of Theorem 3 also follows from these observations.

(i) $\Rightarrow$ (ii) Suppose (ii) does not hold, i.e., there is no $\hat{\mathcal{G}} \in \mathbf{C}^{n\times m}$ and $S \in \mathcal{S}$ such that $\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\} > S$. To examine this inequality condition carefully, let us first define the following set.

$$\mathcal{L} := \left\{\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\} - S : \ \hat{\mathcal{G}} \in \mathbf{C}^{n\times m}, \quad S \in \mathcal{S}\right\}$$

Since the set $\mathcal{S}$ is convex, the set $\mathcal{L}$ is also convex. Hence, if there is no $\hat{\mathcal{G}} \in \mathbf{C}^{n\times m}$ and $S \in \mathcal{S}$ such that $\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\} - S > 0$, i.e., if $\mathcal{L} \cap \mathbf{P}_m$ is empty, we see from the separating hyper-plane theorem (Iwasaki *et al.*, 2000) that there exists a nonzero matrix H that satisfies

$$\begin{aligned} &H = H^* \geq 0, \\ &\mathrm{tr}((\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\} - S)H) \leq 0 \quad \forall \hat{\mathcal{G}} \in \mathbf{C}^{n\times m} \quad \forall S \in \mathcal{S} \end{aligned} \tag{25}$$

From the property (b) of the one-vector-lossless set $\mathcal{S}$ in Definition 2, we see that the following conditions are necessary for the second condition in (25) to hold.

$$\mathrm{tr}((\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\})H) \leq 0 \quad \forall \hat{\mathcal{G}} \in \mathbf{C}^{n\times m},$$
$$\mathrm{tr}(SH) \geq 0 \quad \forall S \in \mathcal{S}$$

The second condition above implies the existence of the vectors ζ_i $(i = 1, \cdots, r)$ in property (c), where $r = \mathrm{rank}(H)$. On the other hand, the first condition implies $\hat{\mathcal{A}}\zeta_i = 0$ $(i = 1, \cdots, r)$. These facts in particular implies that, for any $\hat{\mathcal{G}} \in \mathbf{C}^{n\times m}$, the conditions $\zeta_j^*\mathrm{He}\{\hat{\mathcal{A}}^*\hat{\mathcal{G}}\}\zeta_j = 0$ and $\zeta_j \in \Omega$ hold for at least one index term j. This clearly contradicts the condition (i). □

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