INFINITE EIGENVALUE ASSIGNMENT BY OUTPUT-FEEDBACKS FOR SINGULAR SYSTEMS

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Abstract. The problem of infinite eigenvalue assignment by output-feedbacks is considered. Necessary and sufficient conditions for the existence of a solution to the problem are established. A procedure for computation of the output-feedback gain matrix is given and illustrated by a numerical example.

Keywords: infinite eigenvalue assignment, output-feedback, singular system, solvability condition, procedure

1. INTRODUCTION

It is well-known (Dai, 1989; Kailath, 1980; Wonham, 1979; Kučera, 1981; Kaczorek, 1993) that if a pair (A,B) of standard linear system $\dot{x} = Ax + Bu$ is controllable then there exist a state-feedback gain matrix K such that $\det[I_n s - A + BK] = p(s)$, where $p(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$ is a given arbitrary n degree polynomial. By changing K we may modify arbitrarily only the coefficients $a_0, a_1, ..., a_{n-1}$ but we are not able to change the degree n of the polynomial which is determined by the matrix $I_n s$. In singular linear systems we are also able to change the degree of the closed-loop characteristic polynomials by suitable choice of the state-feedback matrix K. The problem of finding of a state-feedback matrix K such that $\det[Es - A + BK] = \alpha \neq 0$ (α is independent of s) has been considered in (Delin and Ho, 1999; Kaczorek 2003). The infinite eigenvalue assignment problem by feedbacks is very important problem in design of the perfect observers (Kaczorek, 2000; Kaczorek, 2002b; Kaczorek, 2003). In this paper the problem of infinite eigenvalue assignment by output-feedbacks is formulated and solved.

This is an extension of the method given in (Kaczorek, 2003) for output feedback case. Necessary and sufficient conditions for the existence of a solution to the problem will be established and a procedure for computation of the output-feedback gain matrix will be presented.

2. PROBLEM FORMULATION

Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n = R^{n\times 1}$.

Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu\,,\; y = Cx \tag{1}$$

where $\dot{x} = \frac{dx}{dt}$, $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the semistate, input and output vectors and $E, A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$. The system (1) is called singular if $\det E = 0$ and it is called standard when $\det E \neq 0$.

It is assumed that $rank\,E = r < n, rank\,B = m$, $rank\,C = p$ and the pair (E, A) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C} \tag{2}$$

(the field of complex numbers)

Let us consider the system (1) with the output-feedback

$$u = v - Fy \tag{3}$$

where $v \in R^m$ is a new input and $F \in R^{m\times p}$ is a gain matrix.

From (1) and (3) we have

$$E\dot{x} = (A - BFC)x + Bv \tag{4}$$

Problem 1. Given matrices E, A, B, C of (1) and nonzero scalar α (independent of *s*). Find a $F \in R^{m\times p}$ such that

$$\det[Es - A + BFC] = \alpha \tag{5}$$

In this paper necessary and sufficient conditions for the existence of a solution to the problem will be established and a procedure for computation of F will be proposed.

3. PROBLEM SOLUTION

From the equality

$$Es - A + BFC = [Es - A, B]\begin{bmatrix} I_n \\ FC \end{bmatrix} = \\ = [I_n, BF]\begin{bmatrix} Es - A \\ C \end{bmatrix} \tag{6}$$

and (5) it follows that the problem has a solution only if

$$rank[Es - A, B] = n \text{ for all finite } s \in \mathbf{C} \tag{7}$$

and

$$rank\begin{bmatrix} Es - A \\ C \end{bmatrix} = n \text{ for all finite } s \in \mathbf{C} \tag{8}$$

The problem will be solved by the use of the following two steps procedure

Step 1. (subproblem 1). Given *E,A,B* of (1) and a scalar α. Find a matrix $K = FC$ such that

$$\det[Es - A + BK] = \alpha \tag{9}$$

Step 2. (subproblem 2). Given C and K depending of some free parameters $k_1, k_2, ..., k_l$ (found in Step 1). Find desired F satisfying the equation

$$K = FC \tag{10}$$

The solution of the subproblem 1 is based on the following lemma [2,7].

Lemma 1. If the condition (2) is satisfied then there exist orthogonal matrices U, V such that

$$U[Es - A]V = \begin{bmatrix} E_1 s - A_1 & * \\ 0 & E_0 s - A_0 \end{bmatrix}, \\ UB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \begin{matrix} E_1, A_1 \in R^{n_1\times n_1} \\ E_0, A_0 \in R^{n_0\times n_0} \end{matrix}, B_1 \in R^{n_1\times m} \tag{11a}$$

where the subsystem (E_1, A_1, B_1) is completely controllable, the pair (E_0, A_0) is regular, E_1 is upper triangular and * denotes an unimportant matrix.

Moreover the matrices E_1, A_1 and B_1 are of the forms

$$E_1 s - A_1 = \begin{bmatrix} E_{11}s - A_{11} & E_{12}s - A_{12} & \cdots & E_{1,k-1}s - A_{1,k-1} & E_{1k}s - A_{1k} \\ -A_{21} & E_{22}s - A_{22} & \cdots & E_{2,k-1}s - A_{2,k-1} & E_{2k}s - A_{2k} \\ 0 & -A_{32} & \cdots & E_{3,k-1}s - A_{3,k-1} & E_{3k}s - A_{3k} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots 0 & -A_{k,k-1} & E_{kk}s - A_{kk} \end{bmatrix} \tag{11b}$$

$$B_1 = \begin{bmatrix} B_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} E_{ij}, A_{ij} \in R^{\overline{n}_i\times \overline{n}_j}, \; i, j = 1, ..., k \\ B_{11} \in R^{\overline{n}_1\times m}, \sum_{i=1}^{n} \overline{n}_i = n_1 \end{matrix}$$

with $B_{11}, A_{21}, ..., A_{k,k-1}$ of full row rank and $E_{22}, ..., E_{kk}$ nonsingular.

Remark 1. The matrix $\overline{C} = CV$ has no special form.

Theorem 1. Let the condition (2) and (7) be satisfied and let the matrices E, A, B of (1) be transformed to the forms (11). There exists a matrix K satisfying the condition (9) if and only if

i) the subsystem (E_1, A_1, B_1) is singular, i.e.

$$\det E_1 = 0 \tag{12a}$$

ii) if $n_0 > 0$ then the degree of the polynomial $\det[E_0 s - A_0]$ is zero, i.e.

$$\deg \det[E_0 s - A_0] = 0 \text{ for } n_0 > 0 \tag{12b}$$

Proof. ***Necessity.*** From (9) and (11a) we have

$$\begin{aligned} &\det[Es - A + BK] = \det U^{-1} \det V^{-1} \\ &\det[E_1 s - A_1 + B_1 \overline{K}] \det[E_0 s - A_0] = \alpha \end{aligned} \tag{13}$$

where $\overline{K} = KV \in R^{m \times n}$ and $\det\left[E_0 s - A_0\right] = 1$ if $n_0 = 0$.

From (13) it follows that the condition (9) holds only if the conditions (12) are satisfied.

Sufficiency. First let us consider the single-input $(m = 1)$ case. In this case we have

$$E_1 = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n_1} \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \hdashline 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n_1-1} & a_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \hdashline 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix}, \tag{14}$$

$$B_1 = b_1 = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $e_{ii} \neq 0, a_{i,i-1} \neq 0$ for $i = 2, ..., n_1$ and $b_{11} \neq 0$.

The condition (12a) implies that $e_{11} = 0$. Premultiplying the matrix $[E_1 s - A_1, b_1]$ by orthogonal row operations matrix P_1 it is possible to make zero the entries $e_{12}, e_{13}, ..., e_{1n_1}$ of E_1 since $e_{ii} \neq 0$, $i = 2, ..., n_1$. By this reduction only the entries of the first row of A_1 will be modified.

$$\overline{E}_1 = P_1 E_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e_{22} & \cdots & e_{2n_1} \\ \hdashline 0 & 0 & \cdots & e_{n_1 n_1} \end{bmatrix},$$

$$\overline{A}_1 = P_1 A_1 = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1,n_1-1} & \overline{a}_{1n_1} \\ a_{21} & a_{22} & \cdots & a_{2,n_1-1} & a_{2n_1} \\ 0 & a_{31} & \cdots & a_{3,n_1-1} & a_{3n_{1a}} \\ \hdashline 0 & 0 & \cdots & a_{n_1,n_1-1} & a_{n_1 n_1} \end{bmatrix}, \tag{15}$$

$$\overline{b}_1 = P_1 b_1 = b_1$$

Let

$$\overline{k}_1 = \frac{1}{b_{11}} \left[-\overline{a}_{11}, -\overline{a}_{12}, ..., -\overline{a}_{1,n_1-1}, 1 - \overline{a}_{1n_1}\right] \tag{16}$$

Using (13), (15) and (16) we obtain

$$\det\left[\overline{E}_1 s - \overline{A}_1 + \overline{b}_1 \overline{k}_1\right] = \\ = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ -a_{21} & e_{22}s - a_{22} & \cdots & e_{2,n_1-1}s - a_{2,n_1-1} & e_{2n_1}s - a_{2n_1} \\ 0 & -a_{31} & \cdots & e_{3,n_1-1}s - a_{3,n_1-1} & e_{3n_{1a}}s - a_{3n_{1a}} \\ \hdashline 0 & 0 & \cdots & -a_{n_1,n_1-1} & e_{n_1 n_1}s - a_{n_1 n_1} \end{vmatrix} = \\ = a_{21} a_{31} \cdots a_{n_1,n_1-1} = \overline{\alpha} \tag{17}$$

where $\overline{\alpha} = \alpha \det U \det V \det P_1 \det[E_0 s - A_0]^{-1}$.

The considerations can be easily extended for multi-input systems, $m > 1$. In this case the matrix P_1 of the orthogonal row operations is chosen so that all entries of the first row of $\overline{E}_1 = P_1 E_1$ are zero. By this reduction only the entries of $A_{1i}, i = 1, ..., k$ and B_{11} will be modified. The modified matrices will be denoted by $\overline{A}_{1i}, i = 1, ..., k$ and $\overline{B}_{11}$.

Let

$$\overline{K} = \overline{B}_1^{-1} \left\{\left[\overline{A}_{11}, \overline{A}_{12}, ..., \overline{A}_{1k}\right] + G\right\} \tag{18}$$

The matrix $G \in R^{m \times n}$ in (18) is chosen so that

$$\overline{E}_1 s - \overline{A}_1 + \overline{B}_1 \overline{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{l+1} h \\ \overline{a}_{21} & * & \cdots & * & * \\ 0 & \overline{a}_{32} & \cdots & * & * \\ \hdashline 0 & 0 & \cdots & \overline{a}_{l,l-1} & * \end{bmatrix} \tag{19}$$

(* denotes unimportant entries)

$$h = \frac{\alpha (-1)^{l+1}}{\overline{a}_{21} \overline{a}_{32} ... \overline{a}_{l,l-1} c}$$

and $c = \det U^{-1} \det V^{-1} \det P_1^{-1} \det[E_0 s - A_0]$.

Using (13), (18) and (19) it is easy to verify that

$$\det\left[Es - A + BK\right] = c \det\left[\overline{E}_1 s - \overline{A}_1 + \overline{B}_1 \overline{K}\right] = \alpha \tag{20}$$

□

Remark 2. Note that for $m>1$ some entries of the matrix G in (18) can be chosen arbitrarily. Therefore, the matrix $K=\overline{K}V^{-1}$ has a number of free parameters denoted by $k_1,k_2,...,k_l$.

The free parameters will be chosen so that the equation (10) has a solution F for given C and K.

It is well-known that the equation (10) has a solution if and only if

$$rank\, C = rank\begin{bmatrix} C \\ K \end{bmatrix} \qquad \text{(21a)}$$

or equivalently

$$\operatorname{Im} K^T \subset \operatorname{Im} C^T \text{ (T denotes the transpose)} \qquad \text{(21b)}$$

where Im denotes the image

The free parameters $k_1,k_2,...,k_l$ are chosen so that (21) holds.

Therefore, the following theorem has been proved.

Theorem 2. Let the conditions (2), (7), (8) and (12) be satisfied,

The problem has a solution, i.e. there exists F satisfying (5) if and only if the free parameters $k_1,k_2,...,k_l$ of K can be chosen so that the equation (10) has a solution F for given C and K.

From the condition (21) and (16) we have the following corollary.

Corollary 1. For $m=1$ problem has a solution if and only if the row $[\overline{a}_{11},\overline{a}_{12},...,\overline{a}_{1n_1-1}\overline{a}_{1n_1}-1]$ is proportional to the matrix C.

Remark 3. If the order of system is not high say $n\le 5$ the elementary row and column operations instead of the orthogonal operations can be used.

4. EXAMPLE

For the singular system (1) with

$$E=\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A=\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}, \qquad (22)$$

$$B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C=\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix}$$

find the gain matrix $F\in R^{2\times 2}$ such that the condition (5) is satisfied for $\alpha=1$.

In this case the pair (E,A) is regular since

$$\det[Es-A]=\begin{vmatrix} -1 & 2s+1 & s & -1 \\ 0 & s-1 & -s-2 & 2s \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix}=$$

$$=(3-s)(s-1)^2-(s+2)(s-1)+4s$$

The matrices (22) have already the desired forms (11) with $A_0=0, B_0=0$, $E_1=E, A_1=A, B_1=B$, $n_1=n=4, \overline{n}_1=2, \overline{n}_2=\overline{n}_3=1, m=2$ and

$$E_{11}=\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, E_{12}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{13}=\begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$E_{22}=[1], E_{23}=[-1], E_{33}=[1]$$

$$A_{11}=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, A_{12}=\begin{bmatrix} 0 \\ 2 \end{bmatrix}, A_{13}=\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$A_{21}=[0 \ \ -1], A_{22}=[1],$$

$$A_{23}=[-1], A_{32}=[2], A_{33}=[1], B_{11}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using the elementary row operations [6,7] we obtain

$$P_1=\begin{bmatrix} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\left[\overline{E}_1 s-\overline{A}_1, \overline{B}_1\right]=P_1\left[Es-A, B\right]=$$

$$=\begin{bmatrix} -1 & 0 & 5 & -5 & 1 & -2 \\ 0 & s & -1 & 2 & 0 & 1 \\ 0 & 1 & s-1 & 1-s & 0 & 0 \\ 0 & 0 & -2 & s-1 & 0 & 0 \end{bmatrix}$$

Taking into account that in this case

$$\left[\overline{A}_{11}, \overline{A}_{12}, \overline{A}_{13}\right]=\begin{bmatrix} 1 & 0 & -5 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix},$$

$$\overline{B}_1=\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, G=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & k_1 & k_2 & k_3 \end{bmatrix}$$

and using (18) we obtain

$$K=\overline{K}=\overline{B}_1^{-1}\left\{\left[\overline{A}_{11}, \overline{A}_{12}, \overline{A}_{13}\right]+G\right\}=$$

$$=\begin{bmatrix} 2 & 2k_1 & 2k_2-3 & 1+2k_3 \\ 0.5 & k_1 & k_2+1 & k_3-2 \end{bmatrix}$$

where k_1,k_2,k_3 are free parameters.

The free parameters are chosen so that the condition

$$rank\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} =$$

$$= rank\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \\ 2 & 2k_1 & 2k_2 - 3 & 1 + 2k_3 \\ 0.5 & k_1 & k_2 + 1 & k_3 - 2 \end{bmatrix} \qquad (23)$$

is satisfied.

The condition (23) is satisfied for $k_1 = 1, k_2 = 2, k_3 = 0$ and the equation

$$F\begin{bmatrix} 0.5 & 1 & 3 & -2 \\ 2.5 & 3 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0.5 & 1 & 3 & -2 \end{bmatrix}$$

has the solution

$$F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

It is easy to check that

$$\det[Es - A + BK] = \det P_1^{-1} \det[\overline{E}s - \overline{A} + \overline{B}K] =$$

$$= \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0.5 & s+1 & 2 & 0 \\ 0 & 1 & s-1 & 1-s \\ 0 & 0 & -2 & s-1 \end{vmatrix} = 1$$

5. CONCLUDING REMARKS.

The problem of infinite eigenvalue assignment by output feedbacks has been formulated and solved. Necessary and sufficient conditions for the existence of a solution to the problem have been established. Two steps procedure for computation of the output-feedback gain matrix has been derived and illustrated by a numerical example. With slight modifications the considerations can be extended for singular discrete-time linear systems. An extension of the considerations for two-dimensional linear systems (Kaczorek, 1993) is also possible but it is not trivial.

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