OUTPUT FEEDBACK MODEL PREDICTIVE CONTROL OF UNCERTAIN NORM-BOUNDED LINEAR SYSTEMS

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Abstract: An output feedback constrained MPC control scheme for uncertain LFR/Norm-Bounded discrete-time linear systems is discussed. The design procedure consists of an off-line step in which a state-feedback and an asymptotic observer (dynamic primal controller) are designed via BMI optimization and used to robustly stabilize a suitably augmented system. The on-line moving horizon procedure adds N free control moves to the action of the primal controller and its computation consists of solving an on-line LMI optimization problem whose numerical complexity grows up only *linearly* with the control horizon N. The effectiveness is illustrated by a numerical example.

Keywords: Predictive Control, Uncertain Linear Systems, Output Feedback, Convex Programming

1. INTRODUCTION

Model predictive control (MPC) is a standard control technique based on the on-line solution of a constrained optimization problem (see (Kothare *et al.*, 1996) and references therein). In this paper, we introduce an approach to design output feedback MPC controllers for LFR/Norm Bounded uncertain discrete-time linear systems. The main contribution is an extension of the full-state framework introduced in (Casavola *et al.*, 2004). It is shown that joint state estimation and minmax MPC can be cast into an optimization problem: the off-line step (controller/observer couple) can be reduced to a BMI (solvable using local search optimization algorithms (Kočvara and Stingl, 2003)) whereas the on-line step (receding horizon algorithm) can proved to be an LMI and solvable instead using standard semidefinite programming solvers.

Contributions on output feedback MPC ensuring stability for augmented systems (observer and moving horizon controller) were first established by (Michalska and Mayne, 1995), (Scokaert *et al.*, 1997). More recent contributions have been dealt by (Lee and Kouvaritakis, 2001) and (Wan and Kothare, 2002). In particular, the latter authors left unsolved how to take intro account the state estimation error for ensuring MPC solvability, especially in the presence of hard state-related constraints. Such analysis has been considered in the present work which also extends the results of (Wan and Kothare, 2002) in considering control horizons of arbitrary length N.

2. PROBLEM FORMULATION

Consider the following discrete-time linear system with uncertainties appearing in the feedback loop

$$
\begin{cases} x(t+1) &= \Phi x(t) + G u(t) + B_p\, p(t) \\ y(t) &= H x(t) + E u(t) \\ \|p(t)\|_2^2 &\le \left\| C_q x(t) + D_q u(t) \right\|_2^2 \end{cases} \tag{1}
$$

with $x \in \mathbb{R}^{n_x}$ denoting the state, $u \in \mathbb{R}^{n_u}$ the control input, $y \in \mathbb{R}^{n_y}$ the measured output and $p \in \mathbb{R}^{n_p}$ accounting for the uncertainty ($\|\cdot\|_2$ denotes the standard euclidean norm). It is further assumed that the plant input is subject to the following ellipsoidal constraint

$$
u(t) \in \Omega_u,\ \ \Omega_u \triangleq \left\{ u \in \mathbb{R}^{n_u} \,:\, u^T Q_u u \le \bar{u} \right\} \tag{2}
$$

The aim is to find a dynamic output-feedback regulation strategy $u(t) = g(y^t, u^{t-1})$ which possibly asymptotically stabilizes (1) subject to (2). In order to reconstruct the state, which is not directly measurable, a full state observer based on the nominal plant realization is proposed

$$
\tilde{x}(t+1) = \Phi \tilde{x}(t) + G u(t) + L\left(y(t) - \tilde{y}(t)\right) \tag{3}
$$

where $L \in \mathbb{R}^{n_x \times n_y}$ is the observer gain matrix and $\tilde{y}(t)$ is the output estimate. By defining the state estimation error as

$$
e(t) \triangleq x(t) - \tilde{x}(t) \tag{4}
$$

we assume that the uncertainty on the initial state satisfies

$$
e^T(0)\, W\, e(0) \le \bar{e}_0^2 \tag{5}
$$

Due to the presence of a state dependent signal $p(t)$ acting both on the state and the output, the separation principle does not hold true and conditions for the quadratic stability must be expressed in terms of the augmented state $[\tilde{x}^T(t)\ e^T(t)]^T$. Specifically, the family of systems plant/observer (1),(3) is quadratically stabilizable by a feedback control law based on the state estimate

$$
u(t) = K \tilde{x}(t) \tag{6}
$$

if there exists a controller/observer pair, such that, for all the initial states belonging to (5), all closed-loop augmented state trajectories will converge asymptotically to 0_x. Using the strategy (6), standard arguments (see (Wan and Kothare, 2002) for details) allows one to conclude that quadratic stability conditions can be guaranteed if the following matrix

$$
X_0 \triangleq \begin{bmatrix} \begin{bmatrix} \rho Q & 0 \\ 0 & \mu I_{n_p} \end{bmatrix} & \begin{bmatrix} \Phi_K^T & 0 \\ -(LH)^T & \Phi_L^T \\ 0 & B_p^T \end{bmatrix} Q & \begin{bmatrix} \mu C_{q,K}^T \\ -\mu C_q^T \\ 0 \end{bmatrix} \\ (*) & Q & 0 \\ (*) & 0 & I_{n_p} \end{bmatrix} \tag{7}
$$

is positive semidefinite. Notice that (7) is bilinear in $K \in \mathbb{R}^{n_u \times n_x}, L \in \mathbb{R}^{n_x \times n_y},\ Q = Q^T \in \mathbb{R}^{2n_x \times 2n_x} \ge 0$, $\mu \ge 0$ and $0 \le \rho < 1$ (the latter is a scalar affecting the convergence rate of the augmented state). Moreover, $\Phi_K \triangleq \Phi + GK$, $\Phi_L \triangleq \Phi + LH$, $C_{q,K} \triangleq C_q + D_q K$.

The control performance and the invariance properties need to be defined with respect to the true state, which can be regarded as a linear combination of the augmented state components ($x = \tilde{x} + e$). Then, we need to find conditions under which the control strategy (6) achieves a guaranteed cost

$$
J(x(0), u(\cdot)) \triangleq \max_{\substack{p(t) \in S_t \\ e^T(0)\, W\, e(0) \le \bar{e}_0^2}} \sum_{t=0}^{\infty} \left\{ \|x(t)\|_{R_x}^2 + \|u(t)\|_{R_u}^2 \right\} \tag{8}
$$

where $R_x = R_x^T \ge 0$, $R_u = R_u^T \ge 0$ are state and input weighting matrices ($\|\cdot\|_{R_x}$ and $\|\cdot\|_{R_u}$ denote matrix weighted euclidean norms) and the sets

$$
S_t \triangleq \left\{ p \mid \|p\|_2^2 \le \left\| (C_{q,K} + D_q K)\, \tilde{x}(t) + C_{q,K}\, e(t) \right\|_2^2 \right\} \tag{9}
$$

represent plant uncertainty domains at each time instant t. A bound on (8) is given by

$$
\max_{e^T(0)\, W\, e(0) \le \bar{e}_0^2} (\tilde{x}(0) + e(0))^T P\, (\tilde{x}(0) + e(0)) \tag{10}
$$

for a suitably chosen matrix $P = P^T > 0 \in \mathbb{R}^{n_x \times n_x}$. Moreover, if K, and L satisfy (7), the following ellipsoidal set

$$
\begin{aligned} C(P,\gamma) \triangleq \{ x \in \mathbb{R}^{n_x} \,\big|\, & (\tilde{x} + e)^T P\, (\tilde{x} + e) \le \gamma, \\ & \forall e^T W e \le \bar{e}_0^2 \} \end{aligned} \tag{11}
$$

can be proved to be a robust positively invariant region for the state evolutions of the closed-loop system, viz. $x(0) \subseteq\ C(P,\gamma)$ implies that

$$
\begin{aligned} x(t) \subseteq \Bigg\{ \begin{bmatrix} I_{n_x} & I_{n_x} \end{bmatrix} \begin{bmatrix} \Phi_K & -LH \\ 0 & \Phi_L \end{bmatrix}^t \begin{bmatrix} \tilde{x}(0) \\ e(0) \end{bmatrix}, & \\ \forall e^T(0)\, W\, e(0) \le \bar{e}_0^2 \Bigg\} \subseteq C(P,\gamma) & \end{aligned} \tag{12}
$$

for all t. Given the cost (8) and the upper bound (10) the following matrix

$$
X_1 \triangleq \begin{bmatrix} \tilde{P} & 0 & \tilde{\Phi}_{K,L}^T \begin{bmatrix} P \\ P \end{bmatrix} & \lambda \begin{bmatrix} C_{q,K}^T \\ -C_q^T \end{bmatrix} & \tilde{R}_x^{\frac{1}{2},T} & \begin{bmatrix} K^T R_u^{\frac{1}{2},T} \\ 0 \end{bmatrix} \\ 0 & \lambda I_{n_p} & B_p^T P & 0 & 0 & 0 \\ (*) & (*) & P & 0 & 0 & 0 \\ (*) & (*) & 0 & \lambda I_{n_p} & 0 & 0 \\ (*) & (*) & 0 & 0 & I_{2n_x} & 0 \\ (*) & (*) & 0 & 0 & 0 & I_{n_u} \end{bmatrix}, \tag{13}
$$

bilinear in the triplet (K, L, P) and in the scalar $\lambda \ge 0$, must be positive semidefinite where

$$
\tilde{\Phi}_{K,L} \triangleq \begin{bmatrix} \Phi_K & -LH \\ 0 & \Phi_L \end{bmatrix}, \tilde{P} \triangleq \begin{bmatrix} P & P \\ P & P \end{bmatrix}, \tilde{R}_x \triangleq \begin{bmatrix} R_x & R_x \\ R_x & R_x \end{bmatrix},
$$

The invariance condition $x(0) \subseteq C(P,\gamma)$ can equivalently be translated, via the S-procedure, into the requirement that the following matrix, linear in P, $\xi \ge 0$ and $\gamma > 0$

$$
X_2 \triangleq \begin{bmatrix} \xi W - P & -P\tilde{x}(0) \\ (*) & \gamma - \xi e_0^2 - \tilde{x}^T(0) P \tilde{x}(0) \end{bmatrix} \tag{14}
$$

be positive semidefinite. Finally, the input constraint

$$
u^T(t)\, Q_u\, u(t) \le \bar{u}^2
$$

with $u(t) = K\tilde{x}(t)$ is satisfied iff

$$
U \triangleq \begin{bmatrix} \bar{u}^2 I & Q_u^{\frac{1}{2}} K \\ K^T Q_u^{\frac{1}{2}T} & P \end{bmatrix} \tag{15}
$$

is positive semidefinite as well. All previous discussion can be summarized in the following Theorem 1.

Theorem 1. Let the triplet (P,K,L) be a solution of following BMI optimization problem

$$\begin{gathered}\min_{K,L,P,Q,\gamma,\mu,\lambda,\xi} \gamma \\ \text{s.t.} \\ P>0,\ Q>0 \\ X_0 \geq 0,\ X_1 \geq 0,\ X_2 \geq 0,\ U \geq 0 \\ \mu \geq 0,\ \lambda \geq 0,\ \xi \geq 0\end{gathered} \tag{16}$$

Then, provided that (5) is satisfied, the control law (6) with the state estimate computed via (3) satisfies the input constraints (2), ensures quadratic stability to the system (1) and achieves a guaranteed cost upper-bound γ to the quadratic index (8) with $\tilde{\Phi}_{K,L}x(t) \subseteq C(P,\gamma),\ \forall t$.

Proof - By collecting all the above discussion and exploiting standard results. □

In order to add predictive capability we consider the following family of virtual commands

$$u(\cdot|t) = \begin{cases} K\tilde{x}_k(t)+c_k(t), & k=0,1,\ldots,N-1, \\ K\tilde{x}_k(t) & k \geq N, \end{cases} \tag{17}$$

with $\tilde{x}_k(t) \triangleq \tilde{x}(t+k|t)$ and $c_k(t) \triangleq c(t+k|t)$. The latter vectors, over which the optimization takes place, provide N free perturbations over the action of the stabilizing and admissible primal dynamic controller defined by the (K,L) pair solving (16). Given the strategy (17), it is possible to consider the convex set-valued state predictions $x_k(t) = \tilde{x}_k(t)+e_k(t)$ which are the projections of the augmented state predictions

$$\begin{bmatrix} \tilde{x}_k(t) \\ e_k(t) \end{bmatrix} \triangleq \tilde{\Phi}_{K,L}^k \begin{bmatrix} \tilde{x}(t) \\ e(t) \end{bmatrix} + \sum_{i=0}^{k-1} \tilde{\Phi}_{K,L}^{k-1-i} \begin{bmatrix} Gc_i(t) \\ B_p p_i(t) \end{bmatrix} \tag{18}$$

along the plant state space, computed under the conditions $p_i(t) \triangleq p(t+i|t) \in S_i(t)$, $e(t)^T We(t) \leq \vec{e}_t^2$

$$S_i(t) \triangleq \left\{ p : \|p\|_2^2 \leq \max_{\tilde{x}_i(t),e_i(t)} \left\|C_{q,K}\tilde{x}_i(t)+C_q e_i(t)+D_q c_i(t)\right\|_2^2 \right\}, \tag{19}$$

$i=0,\ldots,k-1$ with $S_i(t)$ characterizing all admissible perturbations along the system trajectories corresponding to the virtual command sequences (17).

Remark 1 - One of the main features of the command family (17) is that it allows the setup of moving horizon strategies based on the notion of "closed-loop" state predictions. The rationale hinges upon the fact that, under (17) and given a couple (K,L) solution of (16), the augmented system state predictions from time t onwards $\tilde{x}_k(t) \triangleq \tilde{x}(t+k|t)$, $e_k(t) \triangleq e(t+k|t)$, $k=0,\ldots N-1$ are linear in terms of the input moves $c_k(t), k=0,\ldots N-1$. Moreover, the cost (20) can be equivalently rewritten as the following minmax quadratic index

$$\begin{aligned} V(\tilde{x}(t),P,c_k(t)) \triangleq \textstyle\sum_{k=0}^{N-1} & \left(\max_{\substack{p_k(t)\in S_k(t) \\ e(t)^T W e(t) \leq \vec{e}_t^2}} \|\tilde{x}_k(t)+e_k(t)\|_{R_x}^2 + \|c_k(t)\|_{R_u}^2 \right) \\ & + \max_{\substack{p_N(t)\in S_N(t) \\ e(t)^T W e(t) \leq \vec{e}_t^2}} \|\tilde{x}_N(t)+e_N(t)\|_P^2, \end{aligned} \tag{20}$$

to be minimized w.r.t. $c_k(t), k=0,\ldots N-1$.. Then, at each time instant t, our solution will consist of computing

$$\begin{gathered} c_k^*(t) \triangleq \arg\min_{c_k(t)} V(\tilde{x}(t),P,c_k(t)) \\ \text{s.t.} \\ K\tilde{x}_k(t)+c_k(t) \subset \Omega_u,\ k=0,1,\ldots,N-1, \\ \tilde{x}_N(t)+e_N(t) \subset C(P,\gamma), \\ Kz \in \Omega_u,\ \forall z \in C(P,\gamma) \end{gathered} \tag{21}$$

where $C(P,\gamma)$ is a robust invariant set under K, with (P,K,L,γ) solution of (16). It will be shown that the above optimization problem is solvable at each time t provided that it is solvable at time $t=0$. The resulting predictive control action, based on the state estimate provided by the observer, satisfies the constraints and stabilizes the plant. At the stage of the estimator design, the speed of the error dynamics is obviously influenced by the triplet (P,K,L), the N perturbations $c_k(t),\ k=0,\ldots,N-1$ and the output measurements $y(t)$. Starting from such a consideration and in order to improve the control performance of the proposed strategy, the error estimation bound

$$e(t)^T W e(t) \leq \vec{e}_t^2, \tag{22}$$

can be updated at each time instant. In the next section, along with the moving horizon strategy, sufficient LMI conditions will be derived in order to accomplish this requirement.

3. MOVING HORIZON SCHEME

3.1 Upper Bound Conditions

In this section we aim at determining a suitable upper-bound to (20) in terms of LMI feasibility conditions. We will suppose the generic time instant t equal to zero and denote $c_k = c_k(0)$, $p_k = p(k|0)$, $e_k = e(k|0)$, $\tilde{x}_k = \tilde{x}_k(0)$, $\tilde{x} = \tilde{x}(0)$ and $S_k = S_k(0)$ for $k = 0,1,\ldots,N-1$ for notational simplicity.

The simplest way to derive an easily computable upper-bound to the cost (20) is that of introducing non-negative reals $J_0, J_1, \ldots, J_{N-1}$ such that, for arbitrary P,K,L, and c_k, $k=0,1,\ldots,N-1$, the following inequalities

$$\max_{\substack{p_i\in S_i \\ i=0,\ldots,k \\ k=0,\ldots,N-2 \\ e^T(0)We(0)\leq \vec{e}_0^2}} (\tilde{x}_{k+1}+e_{k+1})^T R_x (\tilde{x}_{k+1}+e_{k+1}) + c_k^T R_u c_k \leq J_k \tag{23}$$

$$\max_{\substack{p_i\in S_i \\ i=0,\ldots,N-1 \\ e^T(0)We(0)\leq \vec{e}_0^2}} (\tilde{x}_N+e_N)^T P (\tilde{x}_N+e_N) + c_{N-1}^T R_u c_{N-1} \leq J_{N-1} \tag{24}$$

hold true. In such a case, in results that

$$V(\tilde{x},P,c_k(t)) \leq (\tilde{x}+e(0))^T R_x(\tilde{x}+e(0))+J_0+J_1+\cdots+J_{N-1}$$
$$\forall e^T(0)We(0) \leq \bar{e}_0^2 \tag{25}$$

Following the same procedure shown in (Casavola *et al.*, 2004) which makes an extensive use of the S-procedure, the upper bound conditions (23), (24), $k = 0,1,\ldots,N-1$ are satisfied if the following linear matrix inequalities in the variables J_k, $\underline{c}_k^T \triangleq \left[c_0^T\, c_1^T\, \ldots\, c_k^T\right]$

$$\Sigma_k \triangleq \begin{bmatrix} J_k - \tau_k^e \bar{e}_0^2 & -[\tilde{x}^T \;\; \underline{c}_k^T]\, L_k^T \\ (*) & I \end{bmatrix} \tag{26}$$

are positive definite, where L_k is the Choleski factor of

$$L_k^T L_k = \left(\tilde{E}_k + \sum_{i=0}^{k} \tau_{i,k}^p \Pi_i\right) +$$
$$\left(-\tilde{D}_k + \sum_{i=0}^{k} \tau_{i,k}^p \Psi_i\right)^T \left(-\begin{bmatrix} \bar{\Phi}_k^T R_x \bar{\Phi}_k & \bar{\Phi}_k^T R_x \bar{B}_k \\ \bar{B}_k^T R_x \bar{\Phi}_k & \bar{B}_k^T R_x \bar{B}_k \end{bmatrix} + \sum_{i=0}^{k} \tau_{i,k}^p \Xi_i + \tau_k^e \tilde{I}_k\right)^{-1}$$
$$\left(-\tilde{D}_k + \sum_{i=0}^{k} \tau_{i,k}^p \Psi_i\right)$$

$$\hat{\Phi}_k \triangleq \sum_{i=0}^{k-1} (\Phi_K^{k-1-i} LH \Phi_L^i),\; \bar{\Phi}_k \triangleq \hat{\Phi}_k + \Phi_L^k,\; \bar{G}_{k-1} \triangleq \begin{bmatrix} \Phi_K^{k-1} G & \Phi_K^{k-2} G & \ldots & G \end{bmatrix}$$

$$\tilde{\Phi}_{K,L,k-1} := \sum_{i=0}^{k-1} \Phi_K^{k-1-i} LH \Phi_L^i$$

$$\bar{B}_k \triangleq \begin{bmatrix} \bar{\Phi}_{k-1} B_p & \bar{\Phi}_{k-2} B_p & \ldots & \bar{\Phi}_1 B_p & B_p \end{bmatrix}$$

$$\hat{B}_k \triangleq \begin{bmatrix} \hat{\Phi}_{k-1} B_p & \hat{\Phi}_{k-2} B_p & \ldots & \hat{\Phi}_2 B_p & \hat{\Phi}_1 B_p \end{bmatrix}$$

$$\bar{E}_k \triangleq \begin{bmatrix} \Phi_K^{k,T} R_x \Phi_K^k & \Phi_K^{k,T} R_x \bar{G}_{k-1} \\ \bar{G}_{k-1}^T R_x \Phi_K^k & \bar{G}_{k-1}^T R_x \bar{G}_{k-1} + \begin{bmatrix} 0 & 0 \\ 0 & R_u \end{bmatrix} \end{bmatrix}$$

$$\tilde{D}_k \triangleq -\begin{bmatrix} \bar{\Phi}_k^T \\ \bar{B}_{k-1}^T \end{bmatrix} R_x \begin{bmatrix} \Phi_K^k & \bar{G}_{k-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \underline{c}_{k-1} \end{bmatrix} -$$

$$\Xi_i := -\begin{bmatrix} (C_q \bar{\Phi}_i + D_q K \tilde{\Phi}_{K,L,i-1})^T \\ (C_q \bar{B}_{i-1} + D_q [\bar{H}_{i-2}\, 0])^T \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} (C_q \bar{\Phi}_i + D_q K \tilde{\Phi}_{K,L,i-1})^T \\ (C_q \bar{B}_{i-1} + D_q [\bar{H}_{i-2}\, 0])^T \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Psi_i := \begin{bmatrix} (C_q \bar{\Phi}_i + D_q K \tilde{\Phi}_{K,L,i-1})^T \\ (C_q \bar{B}_{i-1} + D_q [\bar{H}_{i-2}\, 0])^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (C_q + D_q K)\Phi_K^i & (C_q + D_q K)\bar{G}_{i-1} & 0 \end{bmatrix}$$

$$\Pi_i := \begin{bmatrix} \Phi_K^{i,T}(C_q + D_q K)^T (C_q + D_q K)\Phi_K^i & \Phi_K^{i,T}(C_q + D_q K)^T (C_q + D_q K)\bar{G}_{i-1} & 0 \\ \bar{G}_{i-1}^T (C_q + D_q K)^T (C_q + D_q K)\Phi_K^i & \bar{G}_{i-1}^T (C_q + D_q K)^T (C_q + D_q K)\bar{G}_{i-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{I}_k := \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}$$

$k = 0,1,2,\ldots,N-1$, $i = 0,1,\ldots,N-1$, and τ_k^e, $\tau_{i,k}^p$ are positive scalars used in the S-procedure formulation of the upper bound condition related to J_k.

Remark 2 - The main difference with respect to the full-state feedback case stands in the presence of the error estimate $e(0)$ inside the argument of (23), which acts as an uncertainty and requires an additional condition in the S-procedure derivation. The consequence is that the coefficient $\tau_e^k \bar{e}_0^2$ appears now in (26) and affects its positive semidefiniteness, whereas this is not present in the state-feedback scheme of (Casavola *et al.*, 2004). Notice also that the greater the uncertainty level e_0, the higher the J_k which ensures positive semidefiniteness of (26) and the higher the upper-bound to the cost function. The same considerations apply to forthcoming derivation of all LMI Σ_k and Υ_k. □

3.2 *Input Constraints*

Next step is to find LMI conditions that allow one to enforce the quadratic input constraints (2) along the predictions for $k = 0,1,\ldots,N-1$. This consists of imposing that

$$(K\tilde{x} + c_0)^T Q_u (K\tilde{x} + c_0) \leq \bar{u} \tag{27}$$

$$(K\tilde{x}_k + c_k)^T Q_u (K\tilde{x}_k + c_k) \leq \bar{u}, \forall p_i \in S_i, \forall e(0)^T W e(0) \leq \bar{e}_0^2 \tag{28}$$

Condition (27) can be shown to be directly translated into the following LMI constraint

$$\Upsilon_0 \triangleq \begin{bmatrix} \bar{u} & -(K\tilde{x} + c_0)^T \\ (*) & Q_u^{-1} \end{bmatrix} \geq 0 \tag{29}$$

whereas the satisfaction of (28) is ensured if the following matrix inequalities

$$\Upsilon_k \triangleq \begin{bmatrix} \bar{u} - \theta_k^e \bar{e}_0^2 & -[\tilde{x}^T \;\; \underline{c}_k]\, T_k^T \\ (*) & I \end{bmatrix}, \tag{30}$$

linear in the triplet $(\tilde{x}, \underline{c}_k, \bar{u})$, is are positive semidefinite, where T_k is the choleski factor of

$$T_k^T T_k \triangleq \left(\tilde{N}_k + \sum_{i=0}^{k-2} \theta_{i,k}^p \Pi_i\right) + \left(\tilde{M}_k + \sum_{i=0}^{k-2} \theta_{i,k}^p \Psi_i\right)^T$$
$$\left(-\begin{bmatrix} \hat{\Phi}_k K^T Q_u K \hat{\Phi}_k & \hat{\Phi}_k K^T Q_u K \hat{B}_{k-2} \\ (*) & \hat{B}_{k-2}^T K^T Q_u K \hat{B}_{k-2} \end{bmatrix} + \sum_{i=0}^{k-2} \theta_{i,k}^p \Xi_i + \theta_k^e \tilde{I}_{k-2}\right)^{-1}$$
$$\left(\tilde{M}_k + \sum_{i=0}^{k-2} \theta_{i,k-2}^p \Psi_i\right)$$

where

$$\tilde{N}_k \triangleq \begin{bmatrix} \Phi_K^{k,T} K^T Q_u K \Phi_K^k & \Phi_K^{k,T} K^T Q_u K \bar{G}_{k-1} & \Phi_K^{k,T} K^T Q_u \\ (*) & \bar{G}_{k-1}^T K^T Q_u K \bar{G}_{k-1} & \bar{G}_{k-1}^T K^T Q_u \\ (*) & (*) & Q_u \end{bmatrix}$$

$$\tilde{M}_k \triangleq \begin{bmatrix} \tilde{\Phi}_{K,L,k-1}^T K^T Q_u \\ \bar{H}_{k-2}^T K^T Q_u \end{bmatrix} \begin{bmatrix} K\Phi_K^k & K\bar{G}_{k-1} & I \end{bmatrix}$$

$k = 0,\ldots,N-1$, $i = 0,1,\ldots,k$, and θ_k^e, $\theta_{i,k}^p$ are positive scalars used in the S-procedure formulation of the upper bound condition related to the input constraints at the k-th step prediction.

3.3 *Terminal Constraint*

It remains to satisfy the terminal penalty condition

$$(\tilde{x}_N + e_N)^T P(\tilde{x}_N + e_N) \leq \gamma,\; \forall e(0)^T W e(0) \leq \bar{e}_0^2 \tag{31}$$

This, for a given pair (P,γ), consists of imposing that all N-steps ahead state predictions $\tilde{x}_N + e_N$

$$\Phi_K^N \tilde{x} + \bar{G}_{N-1}\underline{c}_{N-1} + \bar{\Phi}_N e(0) + \bar{B}_{N-1}\underline{p}_{N-1},\;\; \forall p_i \in S_i,\;\; i = 0,\ldots,N-1$$

are contained in the positive invariance ellipsoidal $C(P,\gamma)$. By repeating the same arguments used in the derivation of LMIs (Σ_k), it is found that (31) is satisfied if

$$\Sigma_T := \begin{bmatrix} \gamma - \tau_\rho^e \bar{e}_0^2 & -[\tilde{x}^T \;\; \underline{c}_{N-1}]\, \tilde{L}_{N-1}^T \\ (*) & I \end{bmatrix} \geq 0 \tag{32}$$

where $\tilde{L}_{N-1}$ is obtained by using the same procedure for L_k, by not inserting R_u and using P in the place of R_x. Again, details can be found in (Casavola *et al.*, 2004).

3.4 State estimation error updating

Let us suppose that the MPC scheme will have a solution $c_0^*, c_1^*, \ldots, c_{N-1}^*$ for a generic instant time t. Then, the state estimation error is constrained to belong to the following uncertainty set

$$e(t)^T W e(t) \leq \vec{e}_t^2 \tag{33}$$

and the problem that we face in this section is how to update such a condition by taking into account the error contraction law. Given, at each time t, a (K,L) pair solving the off-line problem (16) and a sequence of free input moves $c_0^*, c_1^*, \ldots, c_{N-1}^*$, at the next time instant $t+1$ the state estimation error must satisfy the following equation

$$\begin{aligned}((\Phi+LH)e(t)+B_p p_0(t))^T W \\ ((\Phi+LH)e(t)+B_p p_0(t)) \leq \vec{e}_{t+1}^2,\end{aligned} \tag{34}$$

with the uncertainty radius $\vec{e}_{t+1}^2$ to be computed, $\forall e(t), p_0(t)$, s.t.

$$e(t)^T W e(t) \leq \vec{e}_t^2 \tag{35}$$

$$\|p_0(t)\|_2^2 \leq \left\|C_{q,K}\tilde{x}(t) + D_q c_0^* + C_q e(t)\right\|_2^2 \tag{36}$$

This can be done by means of the S-procedure. In fact, the following statement

(34) holds true for all $e(t), p_0(t)$ satisfying (35),(36)

holds true if there exist scalars $\varsigma_e, \varsigma_p \geq 0$ such that the following matrix

$$\Delta \triangleq \begin{bmatrix} \begin{bmatrix} -(\Phi+LH)^T W(\Phi+LH)+\varsigma_e W-\varsigma_p C_q^T C_q & -(\Phi+LH)^T W B_p \\ -B_p^T W(\Phi+LH) & -B_p^T W B_p+\varsigma_p I \end{bmatrix} & -\varsigma_p \begin{bmatrix} C_{q,K} & D_q \end{bmatrix} \begin{bmatrix} \tilde{x}(0) \\ c^*(0) \end{bmatrix} \\ (*) & \vec{e}_{t+1}^2-\varsigma_e \vec{e}_t^2-\varsigma_p \begin{bmatrix} \tilde{x}(t) \\ c_0^* \end{bmatrix}^T \begin{bmatrix} C_{q,K}^T C_{q,K} & C_{q,K}^T D_q \\ D_q^T C_{q,K} & D_q^T D_q \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ c_0^* \end{bmatrix} \end{bmatrix} \tag{37}$$

is positive semidefinite for each $\vec{e}_{t+1}^2$, $\varsigma_e \geq 0$, $\varsigma_p \geq 0$. Therefore, the uncertainty radius can be updated online solving the following LMI procedure

$$\begin{aligned} \min_{\vec{e}_{t+1}^2, \varsigma_e \geq 0, \varsigma_p \geq 0} \vec{e}_{t+1}^2, \quad \text{s.t.} \\ \Delta \geq 0 \end{aligned} \tag{38}$$

Remark 3 - The updating law (34) is contractive because the observer gain L is chosen such that is capable to asymptotically reconstruct the state of all plants belonging the LFR uncertainty structure. The consequence is that, given the family of sets

$$E_{t+1} \triangleq \{e(t+1) \,|\, e^T(t+1) W e(t+1) \leq \vec{e}_{t+1}^2\} \tag{39}$$

where $\vec{e}_{t+1}^2$ solves (38) ($\vec{e}_0^2$ is given) the following inclusions hold true: $E_{t+1} \subseteq E_t,\ t = 0, 1, \ldots$

All above developments allows one to write down a computable MPC scheme, hereafter denoted as *NB-Out-Frozen*, which consists of the following algorithm.

NB-Out-Frozen

0. (Initialization - offline) Given the initial state estimate $\tilde{x}(0)$ and the uncertainty interval on the error estimate (5), solve the BMI

$$[K_{\text{opt}}, L_{\text{opt}}, P_{\text{opt}}, Q_{\text{opt}}, \gamma_{\text{opt}}] \triangleq \arg \min_{K,L,P,Q,\gamma,\lambda,\mu,\xi} \gamma \tag{40}$$

subject to the constraints (7), (13), (14), (15). Compute the scalars $\tau_{i,k}^p$, τ_k^e, $i = 0, \ldots, k$, $k = 0, \ldots, N-1$,. Compute the scalars $\tau_p^{i,T}$, τ_e^T $i = 1, \ldots, N-1$. Compute the scalars $\theta_{i,k}^p, \theta_k^e$, $i = 0, \ldots, k-1$, $k = 1, .., N-1$;

1.1 (On-line) At each time instant $t \geq 0$, given $\tilde{x}(t)$, and for all $e^T(t) W e(t) \leq \vec{e}_t^2$ solve

$$[J_k^*(t), c_k^*(t)] \triangleq \arg \min_{J_k, c_k} \sum_{k=0}^{N-1} J_k$$

$$\text{s.t.}$$

$$\Sigma_k(t) \geq 0,\ \Upsilon_k(t) \geq 0,\ \ k = 0, 1, ..., N-1,$$

$$\Sigma_T(t) \geq 0$$

1.2 feed the plant with $u(t) = K\tilde{x}(t) + c_0^*(t)$;

1.3 from the measure of $y(t)$, evaluate the state estimate $\tilde{x}(t+1)$ by means of (3)

1.4 update the uncertainty interval on the error estimate by solving (38)

1.5 $t = t+1$ and go to step 1.1

where $\Sigma_k(t)$, $\Upsilon_k(t)$ and $\Sigma_T(t)$ denote the LMI computed according to (26), (29), (30) and (32) and evaluated for $\tilde{x} = \tilde{x}(t)$.

Theorem 2. Let the NB-Out-Frozen scheme have solution at time $t = 0$. Then, it has solution at each future time instant t, satisfies the input constraints and yields an asymptotically (quadratically) stable closed-loop augmented system.

Proof : It can be obtained by following similar arguments used in (Casavola *et al.*, 2004).

4. NUMERICAL EXPERIMENT

This example is adapted from an antenna positioning system and has been considered in (Kothare *et al.*, 1996). The LFR system matrices are

$$\Phi = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.495 \end{bmatrix}, B_p = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}, C_q = \begin{bmatrix} 0 & 4.95 \end{bmatrix}$$

$$D_q = 0,\ G = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \end{bmatrix}, E = 0$$

Since the main result is the introduction of a new output feedback receding horizon strategy, we want to study the impact of the horizon length on the overall control performance.

A saturation constraint on the input plant is equal to $\overline{u} = 0.4$. The goal is to regulate the output to zero. The

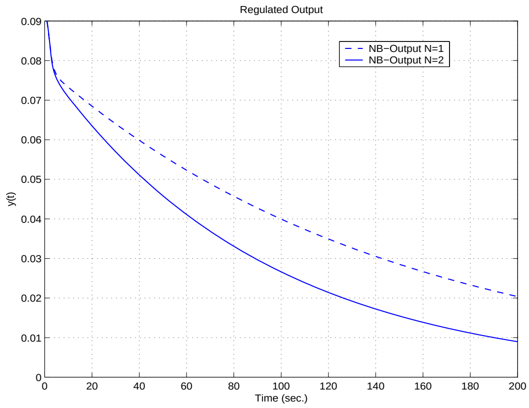

Fig. 1. Regulated output ($\bar{u} = 0.4$) for $N = 1,2$

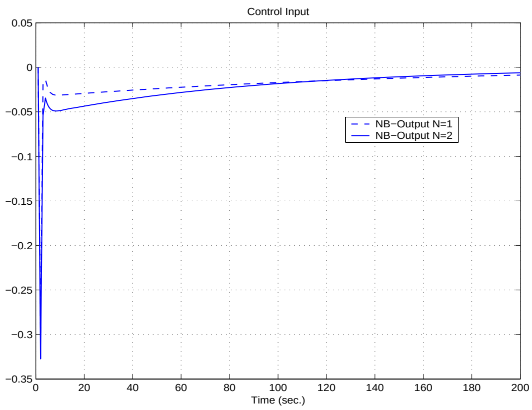

Fig. 2. Plant input ($\bar{u} = 0.4$) for $N = 1,2$

following weighting matrices have been chosen $R_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_u = 1$. By using the BMI solver PENBMI (available from http://www.penopt.com, (Kočvara and Stingl, 2003)) the following controller observer pair has been derived: $K = [-0.6144\ -5.9035]$ and $L = [-0.9793\ -0.3386]^T$. The on-line moving horizon part has been solved via standard LMI solvers for $N = 1$ and $N = 2$ over 200 sec. of simulation time, by supposing that the realization of the plant is equal to the nominal plant for all t, the initial state (not directly available and measurable) is equal to $x(0) = [0.1\ 0.1]^T$, the initial state estimate is equal to $\tilde{x} = 0_x$ and the initial state estimation error belongs to the ball of radius $\bar{e}_0 = 0.1$. As expected, the use of increasingly larger control horizons improves the control performance at expenses of a modest increment of the on-line computational burden (1876 flops per step for $N = 1$ and 4387 flops per step for $N = 2$; 0.2421 sec. average CPU time per step for $N = 1$ and 2.1753 sec. average CPU per step for $N = 2$)

5. CONCLUSIONS

In this paper, a novel output feedback MPC strategy has been presented for input-saturated LFR/Norm-Bounded uncertain systems. The novelty in this scheme relies on the simultaneous off-line design of an observer/controller pair, capable to cope with the model mismatch. Even if the joint selection of state-feedback/observer pair is a non-convex problem, a feasible couple w.r.t. a LQ performance measure can usually be found in a finite number of steps by means of a local algorithm. The on-line MPC strategy is based on minimizing, at each time instant, an upper bound on the worst-case cost for an augmented plant (state estimate/state estimation error), under the constraint that all future state prediction are robustly steered within N-steps into a feasible positively invariant set whose shape is fixed and derived in the off-line phase. The numerical experiences accomplished on a benchmark problem have shown the effectiveness of the approach.

REFERENCES

Boyd, S., L. El Ghaoui, E. Feron and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory*. Vol. 15 of *SIAM Studies in Applied Mathematics*. SIAM. London.

Casavola A., D. Famularo and G. Franzè (2004). “Robust Constrained Predictive Control of Uncertain Norm-Bounded Linear Systems”. *Automatica* **40**, 1865–1876.

El Ghaoui L. and V. Balakrishnan (2004). “Synthesis of fixed-structure controllers via numerical optimization”. *Proceedings of the 33rd Conference on Decision and Control*, Lake Buena Vista, FL, USA, 2678–2683.

Kothare, M. V., V. Balakrishnan and M. Morari (1996). Robust constrained model predictive control using linear matrix inequalities. *Automatica* **32**, 1361–1379.

Kočvara M. and M. Stingl (2003). “PENNON: a code for convex nonlinear and semidefinite programming ”. *Optimization Methods and Software*, **18**, 317–333.

Lee Y. I. and B. Kouvaritakis (2001). “Receding horizon output feedback control for linear systems with input saturation”. *IEE Procedings on Control Theory and Applications*, **148**, 109–115.

Mayne D. Q. , J. B. Rawlings, C. V. Rao and P. O. M. Scokaert (2000). “Constrained model predictive control: Stability and Optimality”. *Automatica*, **36**, 789–814.

Michalska H. and D. Q. Mayne (1995). “Moving horizon observers and observer-based control”. *IEEE Transactions on Automatic Control*, **40**, 995–1006.

Scokaert P. O. M., J. B. Rawlings and E. S. Meadows (1997). “Discrete-time stability with perturbations: Applications to model predictive control”. *Automatica*, **33**, 463–470.

Wan Z. and M. V. Kothare (2002). “Robust output feedback model predictive control using off-line linear matrix inequalities”. *Journal of Process Control*, **12**, 763–774.