AN ALGEBRAIC ANALYSIS APPROACH TO LINEAR TIME-VARYING BEHAVIORS

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Abstract: This paper introduces an algebraic analysis approach to time-varying systems given by linear ordinary differential equations with meromorphic coefficients. The analysis is carried out in a generic sense, i.e., the signals are considered to be smooth except for a discrete set of possible singularities. The algebra is based on a normal form for matrices over the resulting ring of differential operators, which is a non-commutative analogue of the Smith form. It is used to establish a duality between linear time-varying behaviors on the one hand, and modules over the ring of differential operators on the other. This correspondence leads to algebraic characterizations of the basic systems theoretic properties such as autonomy, controllability, and observability.

Keywords: Algebraic systems theory, Behaviour, Differential equations, Linear systems, Time-varying systems, Controllability, Observability.

1. INTRODUCTION

Algebraic analysis is concerned with the study of systems of linear differential equations using algebraic tools such as module theory and homological methods. Pioneering work in this area has been done, for instance, in (Malgrange, 1964) and (Palamodov, 1970).

The seminal paper (Oberst, 1990) established a link between the algebraic analysis approach and the behavioral approach to systems and control theory, introduced by Willems in the 1980s; see (Willems, 1991) for a survey. More precisely, Oberst introduced a categorical duality between the solution spaces of linear partial differential equations with constant coefficients and certain polynomial modules associated to them.

The key to this correspondence is a property of some signal spaces (e.g., the space of smooth functions or the space of distributions) when considered as a module over the ring of differential operators, namely, the injective cogenerator property. This property makes it possible to translate any statement on the solution spaces that can be expressed in terms of images and kernels, to an equivalent statement on the modules. Thus analytic properties can be identified with algebraic properties, and conversely, the results of manipulating the modules using (computer) algebra can be re-translated and interpreted using the language of systems theory. This duality is widely used in behavioral systems and control theory.

The generalization to varying coefficients is not at all straightforward (Wood, 2002). This is not so much due to the technicality that the rings of differential operators to be considered lose their commutativity when passing from the constant to the variable coefficient case. Rather, simple counter-examples show that in general, the space of distributions is not an injective cogenerator in the presence of varying coefficients, which implies that also no subspace of the distributions can have this property. In (Fröhler and Oberst, 1998), this problem could be fixed by resorting to the larger signal space of hyperfunctions. For showing that this space is even a large injective cogenerator, the coefficients had to be restricted to the case of rational functions without poles in the domain of interest.

The present paper also addresses the case of ordinary differential equations with rational (or meromorphic) coefficients, but uses a different setting to establish a module-behavior duality. A similar approach is proposed in the closely related paper (Ilchmann and Mehrmann, 2004), where comparable results are obtained, however, without exploiting the underlying algebraic machinery; see also (Ilchmann, 1989). A purely algebraic approach to time-varying systems with coefficients in a differential field can be found in (Fliess, 1990; Rudolph, 1996; Pommaret and Quadrat, 1998). This formal theory can also be applied to systems given by partial differential equations with variable coefficients (Pommaret and Quadrat, 1999; Chyzak *et al.*, 2003).

This paper is organized as follows: In Section 2, the ring $\mathcal{D}$ of linear ordinary differential operators with rational (meromorphic) coefficients is introduced, and its algebraic properties are discussed. Section 3 presents the signal space $\mathcal{A}$ of functions that are smooth everywhere except for a finite (discrete) set of points. Its properties are examined, and first conclusions are drawn concerning the solution spaces ("behaviors")

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$$

of linear systems $Rw = 0$, where $R \in \mathcal{D}^{g\times q}$ for some positive integers g, q. In particular, the $\mathcal{D}$-module $\mathcal{A}$ is an injective cogenerator. Section 4 exploits the resulting module-behavior duality and addresses the basic systems theoretic notions of autonomy, controllability etc. Finally, some of the peculiarities of state space systems are discussed. The results of this paper are proven in (Zerz, 2005*a*).

2. OPERATORS

Let $\mathcal{D} = K[\frac{d}{dt}]$, where K denotes either the field of real-rational functions, or the field of real-meromorphic functions. Thus $\mathcal{D}$ is the ring of linear ordinary differential operators with rational (meromorphic) coefficients.

Each $0 \neq d \in \mathcal{D}$ possesses a unique representation

$$d = a_n \tfrac{d^n}{dt^n} + \ldots + a_1 \tfrac{d}{dt} + a_0,$$

where $a_i \in K$ and $a_n \neq 0$. One calls n the degree of d.

The ring $\mathcal{D}$ is a domain (that is, there are no zero-divisors) but it is not commutative, because $\frac{d}{dt} ta = a + t\frac{d}{dt}a$ holds for all differentiable functions a, and thus one obtains the commutator rule

$$\tfrac{d}{dt}t - t\tfrac{d}{dt} = 1.$$

More generally, for $k \in K$, one has $\frac{d}{dt}k - k\frac{d}{dt} = k'$.

Theorem 1. The ring $\mathcal{D}$ is simple, that is, the only ideals that are both right and left ideals are the trivial ones, i.e., 0 and $\mathcal{D}$ itself. Moreover, $\mathcal{D}$ is a right and left principal ideal domain, that is, every left ideal and every right ideal can be generated by one single element (Goodearl and Warfield, 1989).

In fact, the ring $\mathcal{D}$ is even a right and left Euclidean domain (Cohn, 1971), which means that there exists a right and left "division with remainder," where the Euclidean function is given by the degree. Anyhow, Theorem 1 implies that $\mathcal{D}$ admits a skew field $\mathcal{K}$ of fractions containing elements of the form $d^{-1}n$ or nd^{-1}, where $0 \neq d \in \mathcal{D}$ and $n \in \mathcal{D}$. Thus, the rank of a matrix $R \in \mathcal{D}^{g\times q}$ is well-defined via (Lam, 2000)

$$\operatorname{rank}(R) = \dim_{\mathcal{K}} \mathcal{K}^{1\times g} R = \dim_{\mathcal{K}} R\mathcal{K}^q.$$

A matrix $R \in \mathcal{D}^{g\times q}$ is called right invertible if there exists a matrix $X \in \mathcal{D}^{q\times g}$ such that $RX = I_g$. Similarly, R is called left invertible if there exists $Y \in \mathcal{D}^{g\times q}$ such that $YR = I_q$. A matrix $U \in \mathcal{D}^{g\times g}$ is called unimodular if there exists a matrix $U^{-1} \in \mathcal{D}^{g\times g}$ with $UU^{-1} = U^{-1}U = I_g$.

Theorem 2. (Jacobson form). Let $R \in \mathcal{D}^{g\times q}$. Then there exist unimodular matrices $U \in \mathcal{D}^{g\times g}$ and $V \in \mathcal{D}^{q\times q}$ such that

$$URV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where $D = \operatorname{diag}(1, \ldots, 1, d) \in \mathcal{D}^{p\times p}$ for some $0 \neq d \in \mathcal{D}$, and $p := \operatorname{rank}(R)$ (Cohn, 1971).

Since $\mathcal{D}$ is Euclidean, the matrices U, V can be obtained by performing elementary row and column operations. The Jacobson form is also known as Teichmüller-Nakayama form.

Example: Consider

$$R = \begin{bmatrix} \frac{d}{dt} & -1 & \cos(t) \\ 1 & \frac{d}{dt} & -\sin(t) \end{bmatrix} \in K[\tfrac{d}{dt}]^{2\times 3} \qquad (1)$$

where K denotes the meromorphic functions. The Jacobson form is given by

$$URV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \end{bmatrix},$$

where $d = \cos(t) + \frac{d}{dt}\sin(t) = 2\cos(t) + \sin(t)\frac{d}{dt}$.

3. SIGNALS AND SYSTEMS

If K is the field of rational functions, let $\mathcal{A}$ denote the set of all functions that are smooth except for a finite number of points, that is, for each $a \in \mathcal{A}$ there exists a finite set $\mathbb{E}(a) \subset \mathbb{R}$ such that $a \in \mathcal{C}^\infty(\mathbb{R} \setminus \mathbb{E}(a), \mathbb{R})$. If K is the field of meromorphic functions, this definition is modified by admitting a discrete (rather than finite) set of exception points $\mathbb{E}(a) \subset \mathbb{R}$ for $a \in \mathcal{A}$. Functions whose values coincide almost everywhere will be identified. The set $\mathcal{A}$ is a real vector space and a left $\mathcal{D}$-module. A subspace $\mathcal{B} \subseteq \mathcal{A}^q$, where q is a positive integer, is called a linear behavior. A linear behavior is called K-differential if it can be written as the solution space of a system of linear ordinary differential equations with coefficients in K. Any linear K-differential behavior can be written as

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$$

where $R \in \mathcal{D}^{g\times q}$ for some positive integer g. One calls R a (kernel) representation of $\mathcal{B}$. In what follows, we deal exclusively with behaviors of this form.

The set $\mathcal{B}$ is an (additive) Abelian group. Let

$$URV = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

be the Jacobson form of R, and let $W := V^{-1} \in \mathcal{D}^{q\times q}$. Since $Rw = 0$ is equivalent to $URw = URVWw = 0$, there is an isomorphism of Abelian groups

$$\begin{array}{rcl} \mathcal{B} & \cong & \tilde{\mathcal{B}} := \{\tilde{w} \in \mathcal{A}^q \mid [D, 0]\tilde{w} = 0\} \\ w & \mapsto & \tilde{w} := Ww \end{array} \tag{2}$$

where

$$\tilde{\mathcal{B}} = \{\tilde{w} \in \mathcal{A}^q \mid \tilde{w}_1 = \ldots = \tilde{w}_{p-1} = 0, d\tilde{w}_p = 0\} \tag{3}$$

is fully decoupled, since $D = \mathrm{diag}(1, \ldots, 1, d)$.

Consider the left $\mathcal{D}$-module $\mathcal{M} = \mathcal{D}^{1\times q}/\mathcal{D}^{1\times g}R$. Its significance lies in the fact that there is an isomorphism of Abelian groups

$$\mathcal{B} \cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{A})$$

which is known as the Malgrange isomorphism. Explicitly, the Malgrange isomorphism assigns to each $w \in \mathcal{B}$ the $\mathcal{D}$-linear map $\phi_w : \mathcal{M} \to \mathcal{A}$ defined by $\phi_w([x]) := xw$, where $[x]$ denotes the residue class of $x \in \mathcal{D}^{1\times q}$ in $\mathcal{M}$. Conversely, for a $\mathcal{D}$-linear map $\phi : \mathcal{M} \to \mathcal{A}$, one defines $w_i := \phi([e_i])$, where e_i denotes the i-th natural basis vector of $\mathcal{D}^{1\times q}$.

Therefore, the module

$$\mathcal{M} = \mathcal{D}^{1\times q}/\mathcal{D}^{1\times g}R$$

will play an important role in the following considerations. According to the Jacobson form, there is an isomorphism of left $\mathcal{D}$-modules

$$\begin{array}{rcl} \mathcal{M} & \cong & \tilde{\mathcal{M}} := \mathcal{D}^{1\times q}/\mathcal{D}^{1\times p}[D, 0] \\ {[x]} & \mapsto & [xV] \end{array}$$

where $[\cdot]$ denotes the residue class of an element of $\mathcal{D}^{1\times q}$ in $\mathcal{M}$ or $\tilde{\mathcal{M}}$, respectively. Thus there is an isomorphism of left $\mathcal{D}$-modules

$$\mathcal{M} \cong \mathcal{D}/\mathcal{D}d \oplus \mathcal{D}^{1\times m} \tag{4}$$

where $m := q - p$ and $p = \mathrm{rank}(R)$. The module $\mathcal{D}/\mathcal{D}d$ is isomorphic to the torsion submodule

$$t\mathcal{M} = \{m \in \mathcal{M} \mid \exists 0 \neq \delta \in \mathcal{D} : \delta m = 0\}$$

of $\mathcal{M}$. The elements of $t\mathcal{M}$ are called torsion elements of $\mathcal{M}$. The module $\mathcal{M}$ is called torsion (module) if $t\mathcal{M} = \mathcal{M}$, and $\mathcal{M}$ is called torsion-free if $t\mathcal{M} = 0$. The module $\mathcal{M}/t\mathcal{M} \cong \mathcal{D}^{1\times m}$ is not only torsion-free, but even free (i.e., it possesses a basis).

The decomposition (4) induces an isomorphism of Abelian groups

$$\mathcal{B} \cong \{y \in \mathcal{A} \mid dy = 0\} \oplus \mathcal{A}^m, \tag{5}$$

because $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}d, \mathcal{A}) \cong \{y \in \mathcal{A} \mid dy = 0\}$ according to the Malgrange isomorphism, and $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{1\times m}, \mathcal{A}) \cong \mathcal{A}^m$, which follows from the fact that a $\mathcal{D}$-linear map defined on a free $\mathcal{D}$-module is uniquely determined by the image of a basis. Of course, the existence of the isomorphism (5) can also be seen directly from (2) and (3). The details of this decomposition will be investigated in Corollary 6 below.

Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$ be left $\mathcal{D}$-modules, and let $f : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{N} \to \mathcal{P}$ be $\mathcal{D}$-linear maps, i.e., left module homomorphisms. One says that

$$\mathcal{M} \xrightarrow{f} \mathcal{N} \xrightarrow{g} \mathcal{P} \tag{6}$$

is exact if $\mathrm{im}(f) = \ker(g)$. The same notion is used when $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are Abelian groups and f, g are group homomorphisms. A $\mathcal{D}$-module $\mathcal{A}$ is called an injective cogenerator if the exactness of the sequence (6) of left $\mathcal{D}$-modules is equivalent to the exactness of the induced sequence

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{A}) \leftarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{N}, \mathcal{A}) \leftarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}, \mathcal{A})$$

of Abelian groups. This property is an extremely powerful tool for systems theory.

Theorem 3. Let $\mathcal{D}$ and $\mathcal{A}$ be as defined above. The left $\mathcal{D}$-module $\mathcal{A}$ is an injective cogenerator.

The next section gives some systems theoretic consequences of the properties of $\mathcal{D}$ and $\mathcal{A}$ as described above.

4. SYSTEMS THEORY

4.1 Existence of full row rank representations

Corollary 1. Let $\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$ for some $R \in \mathcal{D}^{g\times q}$. Then $\mathcal{B}$ can be represented by a matrix with full row rank.

4.2 Equivalence of representations

Corollary 2. Let R_1, R_2 be two $\mathcal{D}$-matrices with the same number of columns, and let $\mathcal{B}_1, \mathcal{B}_2$ be the associated behaviors. One has $\mathcal{B}_1 \subseteq \mathcal{B}_2$ if and only if $R_2 = XR_1$ for some $\mathcal{D}$-matrix X. If $\mathcal{B}_1 = \mathcal{B}_2$, then R_1 and R_2 have the same rank. If R_1 and R_2 have full row rank, then $\mathcal{B}_1 = \mathcal{B}_2$ if and only if $R_2 = UR_1$ for some unimodular matrix U.

4.3 Elimination of latent variables

Corollary 3. Consider

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid \exists \ell \in \mathcal{A}^l : Rw = M\ell\}$$

where $R \in \mathcal{D}^{g\times q}$ and $M \in \mathcal{D}^{g\times l}$. Then there exists a kernel representation of $\mathcal{B}$.

4.4 Input-output structures and autonomy

Let $R \in \mathcal{D}^{p\times q}$ be a full row rank representation of $\mathcal{B}$. Then there exists a $p \times p$ submatrix P of R of full rank. Without loss of generality, arrange the columns of R such that $R = [-Q, P]$. Let $w = [u^T, y^T]^T$ be partitioned accordingly. Then the system law $Rw = 0$ takes the form $Py = Qu$. If $m = q - p > 0$, then this is called an input-output structure of $\mathcal{B}$ and $H = P^{-1}Q \in \mathcal{K}^{p\times m}$ is called its transfer matrix. The term input-output structure is justified by the fact that

$$\forall u \in \mathcal{A}^m \exists y \in \mathcal{A}^p : Py = Qu. \tag{7}$$

Indeed, the operator $P : \mathcal{A}^p \to \mathcal{A}^p$ is even surjective, i.e., for all $v \in \mathcal{A}^p$ there exists $y \in \mathcal{A}^p$ such that $Py = v$, and thus this is true, in particular, for $v = Qu$. If (7) holds, one says that u is a vector of free variables of $\mathcal{B}$. A system without free variables is called autonomous. This is formalized in the following definition.

Definition 1. The behavior $\mathcal{B}$ is called autonomous if there exists no $1 \le i \le q$ such that the projection onto the i-th component

$$\pi_i : \mathcal{B} \to \mathcal{A}, \quad w \mapsto w_i$$

is surjective.

Corollary 4. The following are equivalent:

(1) $\mathcal{B}$ is autonomous;
(2) $\mathcal{B}$ can be represented by a square matrix of full rank;
(3) $\mathcal{M}$ is torsion.

Theorem 4. The following are equivalent:

(1) $\mathcal{B}$ is autonomous;
(2) there exists a finite (discrete) set $\mathbb{E} \subset \mathbb{R}$ such that for all open intervals $I \subset \mathbb{R} \setminus \mathbb{E}$, and all $w \in \mathcal{B}$ that are smooth on I, it holds that

$$w|_J = 0 \quad \Rightarrow \quad w|_I = 0$$

for all open intervals $J \subseteq I$.

Examples:

- Consider $R = \frac{d}{dt} + \frac{1}{t}$. Put $\mathbb{E} = \{0\}$. On every interval $I \subset \mathbb{R} \setminus \mathbb{E}$ on which w is smooth, one has $w(t) = \frac{c}{t}$ for some $c \in \mathbb{R}$. In spite of its singularity at zero, the function $w(t) = \frac{1}{t}$ can be interpreted as a distribution on $\mathbb{R}$.
- Consider $R = \frac{d}{dt} + \frac{1}{t^3}$. Set $\mathbb{E} = \{0\}$. The solutions take the form $w(t) = ce^{\frac{1}{2t^2}}$. In contrast to the previous example, it is known that there exists no distribution that coincides with the regular distribution generated by $w(t) = e^{\frac{1}{2t^2}}$ on $\mathbb{R} \setminus \{0\}$. This shows that the set of distributions is not an injective cogenerator as a $K[\frac{d}{dt}]$-module.
- Consider $R = \frac{d}{dt} - \frac{1}{t}$. Again one puts $\mathbb{E} = \{0\}$. The solutions are $w(t) = ct$. Here there exist solutions that are smooth on all of $\mathbb{R}$, but also any function of the form

$$w(t) = \begin{cases} c_1 t & \text{for } t < 0 \\ c_2 t & \text{for } t > 0 \end{cases}$$

  where $c_1, c_2 \in \mathbb{R}$ is a solution.
- Consider $R = \frac{d}{dt} - \frac{1}{t^3}$. Set $\mathbb{E} = \{0\}$. The solutions are given by $w(t) = ce^{-\frac{1}{2t^2}}$. Putting $w(0) := 0$, these solutions are smooth on all of $\mathbb{R}$, even if one selects different values of the constant c for $t > 0$ and $t < 0$.
- Consider $R = \frac{d}{dt} + \frac{2t}{(1-t^2)^2}$. Put $\mathbb{E} = \{\pm 1\}$. A solution is given by

$$w(t) = \begin{cases} e^{-\frac{1}{1-t^2}} & \text{for } -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

  which is smooth on all of $\mathbb{R}$. This example shows that an autonomous equation may possess non-zero solutions of compact support.

4.5 Image representations and controllability

One says that the behavior $\mathcal{B}$ admits an image representation if there exists a matrix $M \in \mathcal{D}^{q\times l}$ such that

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid \exists \ell \in \mathcal{A}^l : w = M\ell\}. \tag{8}$$

Corollary 5. The following are equivalent:

(1) $\mathcal{B}$ admits an image representation;
(2) any kernel representation matrix of $\mathcal{B}$ that has full row rank is right invertible;
(3) $\mathcal{M}$ is free.

Definition 2. The system $\mathcal{B}$ is called controllable if for all $w_1, w_2 \in \mathcal{B}$ and almost all $t_0 \in \mathbb{R}$, there exists $w \in \mathcal{B}$, an open interval $t_0 \in I \subseteq \mathbb{R}$ such that w_1, w_2, w are smooth on I, and $\tau > 0$ with $t_0 + \tau \in I$ such that

$$w(t) = \begin{cases} w_1(t) & \text{if } t < t_0 \\ w_2(t) & \text{if } t > t_0 + \tau \end{cases}$$

for all $t \in I$.

Theorem 5. $\mathcal{B}$ is controllable if and only if it admits an image representation.

Corollary 6. There exists a largest controllable subsystem $\mathcal{B}^c$ of $\mathcal{B}$, and $\mathcal{B}$ can be decomposed into a direct sum $\mathcal{B} = \mathcal{B}^a \oplus \mathcal{B}^c$, where $\mathcal{B}^a$ is autonomous.

This decomposition corresponds to (4). Note that

$$\begin{aligned}\mathcal{B}^a &\cong \mathrm{Hom}_{\mathcal{D}}(t\mathcal{M}, \mathcal{A}) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}d, \mathcal{A}) \;\cong\; \{y \in \mathcal{A} \mid dy = 0\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}^c &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}/t\mathcal{M}, \mathcal{A}) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{1\times m}, \mathcal{A}) \;\cong\; \mathcal{A}^m.\end{aligned}$$

4.6 Observability

Let $R = [R_1, R_2]$ and let $w = [w_1^T, w_2^T]^T$ be partitioned accordingly. One says that w_1 is observable from w_2 in $R_1w_1 + R_2w_2 = 0$ if w_1 is uniquely determined by w_2. Due to linearity, this is equivalent to

$$\mathcal{B}_1 := \{w_1 \in \mathcal{A}^{q_1} \mid R_1w_1 = 0\} = \{0\}.$$

Corollary 7. Let $\mathcal{B}$ be given by $Rw = R_1w_1 + R_2w_2 = 0$. Then w_1 is observable from w_2 if and only if R_1 is left invertible.

An image representation (8) is called observable if ℓ is observable from w in $w = M\ell$. If $\mathcal{B}$ possesses an image representation at all, then it also admits an observable image representation.

Corollary 8. $\mathcal{B}$ is controllable if and only if it possesses an observable image representation.

4.7 State space systems

For state space systems, one recovers the well-known generalizations of the Kalman controllability and observability criteria to time-varying systems. For brevity, only the controllability results are presented; the observability side is analogous.

Corollary 9. A state space system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where $A \in K^{n\times n}$ and $B \in K^{n\times m}$, is controllable if and only if

$$\mathfrak{K} = \left[\, B,\ (A - \tfrac{d}{dt})B,\ \ldots,\ (A - \tfrac{d}{dt})^{n-1}B \,\right] \in K^{n\times nm}$$

has full row rank.

This follows from the fact that the right invertibility of

$$R = \left[\, \tfrac{d}{dt}I - A,\ -B \,\right]$$

over $\mathcal{D} = K[\frac{d}{dt}]$ is equivalent to the right invertibility of $\mathfrak{K}$ over K.

Similarly, there exists a generalization of the Kalman controllability decomposition. Note that $\dot{x} = Ax + Bu$ becomes

$$\dot{z} = T^{-1}(AT - \dot{T})z + T^{-1}Bu$$

if one sets $x = Tz$ for some non-singular $T \in K^{n\times n}$.

Corollary 10. Let a state space system be given by $A \in K^{n\times n}$ and $B \in K^{n\times m}$. There exists a non-singular matrix $T \in K^{n\times n}$ such that

$$T^{-1}(AT - \dot{T}) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \text{ and } T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where the state space system given by $A_1 \in K^{r\times r}$, $B_1 \in K^{r\times m}$ is controllable.

Example: Returning to the matrix R from (1), consider the behavior $\mathcal{B}$ that consists of all $[x_1, x_2, u]^T$ with $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } B(t) = \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}.$$

In view of the Jacobson form computed earlier, this behavior is not controllable. This is in accordance with the Kalman-like controllability test, because in the present example, $(A - \frac{d}{dt}I)B = 0$, and hence $\operatorname{rank}(\mathfrak{K}) = 1$. With the transformation $z = Tx$, where

$$T = \begin{bmatrix} -\cos(t) & \sin(t) \\ \sin(t) & \cos(t) \end{bmatrix},$$

one obtains the Kalman decomposition

$$\dot{z}_1 = u, \quad \dot{z}_2 = 0.$$

The equation $\dot{z}_2 = 0$ implies that

$$\sin(t)x_1(t) + \cos(t)x_2(t) = \text{const.}$$

along any trajectory of the system, making it intuitively clear that the system cannot be controllable. Algebraically speaking, $[(\sin(t), \cos(t), 0)]$ is a torsion element of $\mathcal{M}$. In other words, $\sin(t)x_1(t) + \cos(t)x_2(t)$ is an autonomous observable (Chyzak *et al.*, 2003; Pommaret and Quadrat, 1999) of the system. On the other hand, note that

$$R(t_0) = \begin{bmatrix} \frac{d}{dt} & -1 & \cos(t_0) \\ 1 & \frac{d}{dt} & -\sin(t_0) \end{bmatrix} \in \mathbb{R}[\tfrac{d}{dt}]^{2\times 3}$$

is right invertible for all $t_0 \in \mathbb{R}$, which corresponds to the observation that for all t_0, the matrix pair $A(t_0) \in \mathbb{R}^{2\times 2}$, $B(t_0) \in \mathbb{R}^{2\times 1}$ is controllable.

The controllable part of $\mathcal{B}$ is given by the equations $\dot{z}_1 = u$, $z_2 = 0$, where z_1, z_2 are as defined above. This corresponds to setting the autonomous observable z_2 to zero.

CONCLUSION AND FUTURE WORK

In this paper, the foundations of a new algebraic analysis approach to the control theory of linear time-varying systems have been laid. Based on a normal form for matrices over the ring of linear ordinary differential operators with coefficients in a differential field such as the rational or meromorphic functions, the basic systems theoretic properties have been characterized, and the results are strikingly similar, to a large extent, to the ones that are well-known for linear ordinary differential equations with constant coefficients.

The author would like to thank an anonymous referee for pointing out that the theory of state representations and Markovian systems carries over, to some extent, to the time-varying setting treated in this paper. However, a detailed treatment of these issues requires several concepts that have not been addressed here, such as row-proper representations, input-output structures with proper transfer matrices, and realization theory. Since the coverage of these topics would go beyond the space limitations of this article, these questions will be addressed in a separate paper (Zerz, 2005*b*).

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