DYNAMIC OBSERVERS FOR NONLINEAR LIPSCHITZ SYSTEMS

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Abstract: The problem of observer design for Lipschitz nonlinear systems is considered. A new dynamic framework which is a generalization of previously used Lipschitz observers is introduced. The correct necessary and sufficient condition on the dynamic gain that ensures asymptotic convergence of the new observer is presented. The equivalence between this condition and an H_∞ optimal control problem which satisfies the standard regularity assumptions in H_∞ optimization theory is shown. A design procedure solvable using commercially available software is presented and a simulation example is given to illustrate the proposed design.

Keywords: Observers, Nonlinear systems, H_∞ , Dynamic filters, Regularization.

1. INTRODUCTION

Nonlinear state observer design has been an area of constant research for the last three decades and, despite important progress, many outstanding problems still remain unsolved. Reference (Nijmeijer and Fossen, 1999) provides a good account of recent research on this subject covering both theory and applications. A class of nonlinear systems that has seen much attention in the literature is the class of *L*ipschitz systems:

$$\dot{x}(t) = Ax(t) + \Gamma(y, u, t) + \Phi(x, u, t) \qquad (1)$$

$$y(t) = Cx(t), \; A \in \mathbb{R}^{n\times n}, \; C \in \mathbb{R}^{p\times n} \qquad (2)$$

and where the function $\Phi(x, u, t)$ satisfies a uniform Lipschitz condition globally in x, i.e,

$$\| \Phi(x_1, u, t) - \Phi(x_2, u, t) \| \leq \; \alpha \| x_1 - x_2 \| \quad (3)$$

for all $u \in \mathbb{R}^m$ and $t \in \mathbb{R}$ and for all x_1 and $x_2 \in \mathbb{R}^n$. Here $\alpha \in \mathbb{R}$ is referred to as the *Lipschitz constant* and is independent of x, u and t.

Lipschitz systems constitute a very important class. Any nonlinear system $\dot{x} = f(x, u)$ can be expressed in the form of (1) as long as $f(x, u)$ is continuously differentiable with respect to x. Many nonlinearities satisfy (3) at least locally. Examples include trigonometric nonlinearities occurring in robotic systems, the nonlinearities which are square or cubic in nature, etc. The function $\Phi(x, u, t)$ can also be considered as a perturbation affecting the system as in (Schreier *et al.*, 1997).

Observer design for Lipschitz systems was first considered by Thau in his seminal paper (Thau, 1973) where he obtained a sufficient condition to ensure the asymptotic stability of the observer. Thau's condition is a very useful *analysis tool* but does not address the fundamental *design* problem. Encouraged by Thau's result, several authors studied observer design for Lipschitz systems. In (Raghavan, 1992; Raghavan and Hedrick, 1994), Raghavan formulated a procedure to tackle the design problem. His algorithm is based on solving an algebraic Riccati equation to obtain the static observer gain. Raghavan's technique was later extended by Garg and Hedrick, (Garg and Hedrick, 1996), to study fault detection and identification in Lipschitz systems. Unfortunately, Raghavan's algorithm often fails to succeed

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even when the matrices (A, C) satisfy the usual observability assumptions. Raghavan showed that the observer design might still be tractable using state transformations. Another shortcoming of his algorithm is that it does not provide insight into what conditions must be satisfied by the observer gain to ensure stability. A rather complete solution of these problems was later presented by (Rajamani, 1998). Rajamani obtained the necessary and sufficient condition on the observer matrix that ensures asymptotic stability of the observer and formulated a design procedure, based on the use of a gradient based optimization method. He also discussed the equivalence between the stability condition and the minimization of the H_∞ norm of a system in the standard form. However, he pointed out that the design problem is not solvable as a standard H_∞ optimization problem since the regularity assumptions required in the H_∞ framework are not satisfied.

In this paper, we show that the condition introduced in (Rajamani, 1998) is related to a modified H_∞ problem satisfying all of the regularity assumptions. Based on this result, we propose a new observer design for Lipschitz nonlinear systems. The observer synthesis is carried out using H_∞ optimization and can therefore be done using commercially available software packages. Our formulation employs the input-output observer framework introduced in (Marquez and Riaz, 2003) in which the static gain used in the classical observers is replaced with a dynamical filter. The paper is organized as follows: section 2 introduces some background results and notations. In section 3, we introduce our dynamic generalization of previously used Lipschitz observers and provide the necessary and sufficient condition for the stability of the new observer. In section 4, we present the main result of this paper, where we formulate the observer design for Lipschitz nonlinear systems as a regular H_∞ problem proving that its solution is necessary and sufficient for observer stability. Simulation results are shown in section 5 and some conclusions are drawn in section 6.

2. BACKGROUND RESULTS AND NOTATION

In this section we summarize some preliminary results on observer design for systems of the form (1)-(3) and where the pair (A, C) is detectable. In all the literature available for this class of nonlinear systems, the observer proposed falls in the class of *Luenberger-like* observers, namely:

$$\dot{\hat{x}} = A\hat{x} + \Gamma(y, u, t) + \Phi(\hat{x}, u, t) + L(y - \hat{y}) \quad (4)$$

$$\hat{y} = C\hat{x} \quad (5)$$

The observer error dynamics is then given by

$$\dot{e} = (A - LC)\ e + \Phi(x, u, t) - \Phi(\hat{x}, u, t) \quad (6)$$

where $e = x - \hat{x}$. Thau was the first to introduce a sufficient condition for the asymptotic stability of the error in (6). His result was as follows:

Theorem 1. (Thau, 1973) If the gain L is chosen s.t $\alpha < \frac{\lambda_{min}(Q)}{2\ \lambda_{max}(P)}$ with the Lyapunov equation $(A - LC)^T\ P + P(A - LC) = -Q$, then the estimation error in (6) is asymptotically stable.

Theorem 1 provides a very important sufficient condition for the existence of an observer, but provides no insight into how to design the observer. Raghavan proposed a design algorithm based on the following theorem:

Theorem 2. (Raghavan and Hedrick, 1994) If there exists an $\varepsilon > 0$ such that the Algebraic Riccati Equation (ARE) in (7) has a symmetric positive definite solution P, then the observer gain $L = \frac{1}{2\varepsilon}PC^T$ stabilizes the error dynamics in (6) for all Φ with a Lipschitz constant α.

$$AP + PA^T + P(\alpha^2 I - \frac{1}{\varepsilon}C^T C)P + I + \varepsilon I = 0 \quad (7)$$

According to this result, Raghavan proposed an iterative binary search procedure over ε, to obtain the observer gain. However, given a particular system of the form (1)-(3) with a specific Lipschitz constant α^*, this procedure may fail even if the pair of matrices (A, C) is observable. Moreover, Theorem 2 provides no insight into what conditions the matrix $(A - LC)$ must satisfy to ensure observer stability. The answer to this puzzle was provided by Rajamani in the following theorem:

Theorem 3. (Rajamani, 1998) The error dynamics in (6) is asymptotically stable for all Φ with a Lipschitz constant α if and only if L is chosen so as to ensure that $(A-LC)$ is stable and such that

$$min_{\omega \in \mathbb{R}^+}\ \sigma_{min}(A - LC - j\omega I) > \alpha \quad (8)$$

The beauty of this result is that it presents necessary and sufficient conditions for observer stability as a condition on the *observer matrix*. Rajamani also related his result it to the H_∞ theory by rewriting (8) as:

$$\|\ [sI - (A - LC)]^{-1}\ \|_\infty < \frac{1}{\alpha} \quad (9)$$

where the left hand side of (9) is equivalent to the H_∞ norm of the transfer function between ω and ζ in the following so-called standard form:

$$\dot{z} = \begin{bmatrix} A \end{bmatrix} z + \begin{bmatrix} I_n & -I_n \end{bmatrix} \begin{bmatrix} \omega \\ \nu \end{bmatrix} \quad (10)$$

$$\begin{bmatrix} \zeta \\ \varphi \end{bmatrix} = \begin{bmatrix} I_n \\ C \end{bmatrix} z + \begin{bmatrix} 0_n & 0_n \\ 0_{pn} & 0_{pn} \end{bmatrix} \begin{bmatrix} \omega \\ \nu \end{bmatrix} \quad (11)$$

where:

$$\begin{aligned} \omega &= \tilde{\phi} = \Phi(x, u, t) - \Phi(\hat{x}, u, t) \\ \nu &= L\ (y - \hat{y}) \\ \zeta &= e = x - \hat{x} \\ \varphi &= y - \hat{y} \end{aligned} \quad (12)$$

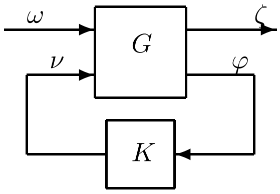

Fig. 1. Standard setup.

which can also be represented by Fig. 1 where the plant G has the state space representation in (13) with the matrices defined in (10)-(11) and where the controller K is the static observer gain L.

$$\hat{g}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \quad (13)$$

Unfortunately, observer synthesis cannot be carried out directly using the standard H_∞ solution since the standard form in (10)-(11) does not satisfy all of the regularity assumptions in the H_∞ framework, summarized in Section 4.1 (Notice that $D_{12}^T D_{12}$ and $D_{21} D_{21}^T$ are both singular). Therefore, in (Rajamani, 1998), Rajamani considered using a gradient based optimization method to continuously change the locations of the closed loop eigenvalues to minimize a performance index related to (8). Moreover, he considered the special case of A being Hurwitz in (Rajamani and Cho, 1998), introducing an analytical solution when a certain sufficient condition on the so called *distance to unobservability* of (A, C) is satisfied.

In this paper, we generalize the necessary and sufficient condition in (8) to a more general dynamic framework. We then prove that the new condition is equivalent to a standard H_∞ problem satisfying all the regularity assumptions (unlike (10)-(11)). Based on these results, we present a systematic procedure to compute the observer gain within the H_∞ framework. The following definitions and notations will be used throughout the paper:

Definition 1. ($\mathcal{L}_2$ space) The space $\mathcal{L}_2$ consists of all measurable functions $u : \mathbb{R}^+ \to \mathbb{R}^q$, satisfying

$$\|u\|_{\mathcal{L}_2} \triangleq \sqrt{\int_0^\infty \| u(t) \|^2 \; dt} < \infty. \quad (14)$$

The norm $\|u\|_{\mathcal{L}_2}$ defined in (14) is the so-called $\mathcal{L}_2$ norm of the function u. Consider now a system $H : \mathcal{L}_2 \to \mathcal{L}_2$. We will represent by $\gamma(H)$ the $\mathcal{L}_2$ gain of H defined by $\gamma(H) = \sup_u \frac{\|Hu\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}}$. It is well known that, for a linear system $H : \mathcal{L}_2 \to \mathcal{L}_2$ with a transfer matrix $\hat{H}(s)$, $\gamma(H)$ is equivalent to the H-infinity norm of $\hat{H}(s)$ defined as follows:

$$\gamma(H) \equiv \| \hat{H}(s) \|_\infty \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\hat{H}(j\omega))$$

where $\sigma_{\max}$ represents the maximum singular value of $\hat{H}(j\omega)$. The matrices I_n, 0_n and 0_{nm} represent the identity matrix of order n, the zero square matrix of order n and the zero n by m matrix respectively. The symbol $\hat{T}_{yu}$ represents the transfer matrix from input u to output y. The partitioned matrix $K = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right]$ (when used as an operator from y to u, i.e, $u = Ky$) represents the state space representation:

$$\dot{\xi} = A_L \xi + B_L y$$
$$u = C_L \xi + D_L y$$

3. GENERALIZATION TO DYNAMIC FRAMEWORK

In this paper, following the approach in (Marquez and Riaz, 2003; Marquez and Riaz, 2005), we will make use of dynamical observers of the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \Gamma(y, u, t) + \Phi(\hat{x}, u, t) + \eta(t) \quad (15)$$
$$\hat{y}(t) = C\hat{x}(t) \quad (16)$$

where $\eta(t)$ is obtained by applying a dynamical compensator K of order n on the output estimation error. In other words $\eta(t)$ is given from

$$\dot{\xi} = A_L \xi + B_L (y - \hat{y}) \quad (17)$$
$$\eta = C_L \xi + D_L (y - \hat{y}). \quad (18)$$

We will also write

$$K = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \quad (19)$$

to represent the compensator in (17)-(18). It is straightforward to see that this observer structure reduces to the usual observer in (4)-(5) in the special case where the gain K is the constant gain given by $K = \left[\begin{array}{c|c} 0_n & 0_{np} \\ \hline 0_n & L \end{array}\right]$. The additional dynamics brings additional degrees of freedom in the design, something that will be exploited in the proposed H_∞ procedure. In this section, we generalize Theorem 3 to the dynamic framework as follows. First, note that the observer error dynamics in (6) is now given by

$$\dot{e} = A\; e + \Phi(x, u, t) - \Phi(\hat{x}, u, t) - \eta \quad (20)$$

which can also be represented by the setup in Fig. 1 where G has the state space representation in (13) with the same matrices defined in (10)-(11) and with the same variables in (12) except for ν which is now given by

$$\nu = \eta = K(y - \hat{y}) \quad (21)$$

We denote by $\hat{T}_{\zeta\omega}$ the transfer function between ω and ζ for this setup. The following theorem is then the generalization of Theorem 3:

Theorem 4. Given the Lipschitz system of equations (1)-(2), the state $\hat{x}$ of the observer (15)-(19) globally asymptotically converges to the system state x for all $\Phi(\cdot,\cdot,\cdot)$ satisfying (3) with a Lipschitz constant α if and only if K is chosen s.t:

$$sup_{\omega \in \mathbb{R}} \; \sigma_{max}[\hat{T}_{\zeta\omega}(j\omega)] < \frac{1}{\alpha} \quad (22)$$

Proof: (Sufficiency) Using the variable definitions in (12) along with ν in (21) and the matrices in (10), (11) and (19), $\hat{T}_{\zeta\omega}$ can be represented as:

$$\hat{T}_{\zeta\omega} = \hat{T}_{e\tilde{\phi}} = \left[\begin{array}{cc|c} A - D_L C & -C_L & I_n \\ B_L C & A_L & 0_n \\ \hline I_n & 0_n & 0_n \end{array}\right] \quad (23)$$

and is such that $\gamma(\hat{T}_{e\tilde{\phi}}) = \| \hat{T}_{e\tilde{\phi}} \|_\infty < \frac{1}{\alpha}$ according to (22). The proof for sufficiency follows from noting that the estimation error e is given from the feedback interconnection of $\hat{T}_{e\tilde{\phi}}$ and Δ as shown in Fig. 2 where Δ is the static nonlinear time-varying operator defined as follows:

$$\begin{aligned} \Delta(t) : e \rightarrow \tilde{\phi} &= \Phi(x, u, t) - \Phi(\hat{x}, u, t) \\ &= \Phi(e + \hat{x}(t), u(t), t) - \Phi(\hat{x}(t), u(t), t) \end{aligned}$$

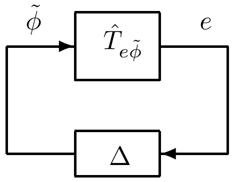

Fig. 2. Feedback interconnection.

In this feedback loop, $\gamma(\hat{T}_{e\tilde{\phi}}) < \frac{1}{\alpha}$ as mentioned earlier and, although an exact expression for Δ is not available, we have $\gamma(\Delta) \leq \alpha$ because from the Lipschitz condition in (3), it follows that

$$\gamma(\Delta) \leq \frac{\sqrt{\int_0^\infty \alpha^2 \| x - \hat{x} \|^2 \ \mathrm{d}t}}{\sqrt{\int_0^\infty \| x - \hat{x} \|^2 \ \mathrm{d}t}} \leq \alpha$$

Using the bounds on the $\mathcal{L}_2$ gains of the operators $\hat{T}_{e\tilde{\phi}}$ and Δ, we will make use of a dissipativity argument by noting that the following properties are satisfied for the feedback loop in Fig. 2:

(a) Δ is a static nonlinearity (no internal states) and $\hat{T}_{e\tilde{\phi}}$ is the dynamic LTI system in (23).

(b) The mappings $\hat{T}_{e\tilde{\phi}} : \tilde{\phi} \rightarrow e$ and $\Delta : e \rightarrow \tilde{\phi}$ have finite $\mathcal{L}_2$ gains $\gamma(\hat{T}_{e\tilde{\phi}})$ and $\gamma(\Delta)$, and moreover they satisfy $\gamma(\hat{T}_{e\tilde{\phi}}).\gamma(\Delta) < 1$.

(c) $\hat{T}_{e\tilde{\phi}}$ and Δ are dissipative with the supply rates $\omega_1 = -e^T e + \gamma(\hat{T}_{e\tilde{\phi}})^2 \tilde{\phi}^T \tilde{\phi}$ and $\omega_2 = -\tilde{\phi}^T \tilde{\phi} + \alpha^2 e^T e$ respectively. We will denote by S_1 and S_2 the storage functions associated with these supply rates.

It is an straightforward application of Corollary 1 in (Hill and Moylan, 1977) that $S_1 + aS_2$, $a > 0$, is a Lyapunov function for the feedback system of Fig. 2 and that this system is asymptotically stable. This implies that $e \rightarrow 0$ as $t \rightarrow \infty$.
(Necessity) This is a direct result of the small gain theorem for LTI systems (see proof of Theorem 4 in (Rajamani, 1998) for more details) which implies that if $\gamma(\hat{T}_{e\tilde{\phi}}) \geq \frac{1}{\alpha}$ in Fig. 2, then there exists $\Phi(x, u, t) = M(t)x$ with $\| \hat{M}(s) \|_\infty \leq \alpha$ s.t the closed loop system in Fig. 2 is unstable. $\triangle$

Corollary 1. Under the conditions of Theorem 4, if condition (3) holds locally, then local asymptotic convergence of the observer is guaranteed.

4. A NEW H_∞ OBSERVER DESIGN

We herein present our main results by proving that the condition in (22) (and (8) as a special case) is actually equivalent to a standard H_∞ problem satisfying all the regularity assumptions.

4.1 Problem regularization

By adding a "weighted" disturbance term in the output equation (2), now we tackle the problem of designing an observer for the following system

$$\dot{x}(t) = Ax(t) + \Gamma(y, u, t) + \Phi(x, u, t) \quad (24)$$
$$y(t) = Cx(t) + \epsilon \ d(t), \qquad \epsilon > 0 \quad (25)$$

where the function $\Phi(x, u, t)$ satisfies the Lipschitz condition. Using the same observer defined by (15)-(19), it can be seen that the standard form in (10)-(11) has now the following form

$$\dot{z} = \begin{bmatrix} A \end{bmatrix} z + \begin{bmatrix} \begin{bmatrix} I_n & 0_{np} \end{bmatrix} & -I_n \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ d(t) \end{bmatrix} \\ \nu \end{bmatrix} \quad (26)$$

$$\begin{bmatrix} \zeta \\ \varphi \end{bmatrix} = \begin{bmatrix} I_n \\ C \end{bmatrix} z + \begin{bmatrix} \begin{bmatrix} 0_n & 0_{np} \end{bmatrix} & 0_n \\ \begin{bmatrix} 0_{pn} & \epsilon I_p \end{bmatrix} & 0_{pn} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ d(t) \end{bmatrix} \\ \nu \end{bmatrix} \quad (27)$$

This can also be represented by the standard setup in Fig. 1, except for redefining the matrices of $\hat{g}(s)$ in (13) and replacing ω by $\bar{\omega}$ defined as:

$$\bar{\omega} \triangleq \begin{bmatrix} \omega & d(t) \end{bmatrix}^T \quad (28)$$

This standard form, however, still does not satisfy the regularity assumptions since $D_{12}^T D_{12}$ is singular. Fortunately, regularization can be done by extending the external output ζ to include the "weighted" vector $\beta\nu$. This adds another change in Fig. 1 consisting of replacing ζ by $\bar{\zeta}$ defined as:

$$\bar{\zeta} = \begin{bmatrix} \zeta & \beta\nu \end{bmatrix}^T \quad (29)$$

The entries of $\hat{g}(s)$ in (13) are then given by:

$$\dot{z} = \begin{bmatrix} A \end{bmatrix} z + \begin{bmatrix} \begin{bmatrix} I_n & 0_{np} \end{bmatrix} & -I_n \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ d \end{bmatrix} \\ \nu \end{bmatrix} \quad (30)$$

$$\begin{bmatrix} \begin{bmatrix} \zeta \\ \beta\nu \end{bmatrix} \\ \varphi \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I_n \\ 0_n \end{bmatrix} \\ C \end{bmatrix} z + \begin{bmatrix} \begin{bmatrix} 0_n & 0_{np} \\ 0_n & 0_{np} \end{bmatrix} & \begin{bmatrix} 0_n \\ \beta I_n \end{bmatrix} \\ \begin{bmatrix} 0_{pn} & \epsilon I_p \end{bmatrix} & 0_{pn} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \omega \\ d \end{bmatrix} \\ \nu \end{bmatrix} \quad (31)$$

All the regularity assumptions summarized below (Zhou and Doyle, 1998) are now satisfied iff $(A,\ C)$ is detectable (with no new design restrictions):

(1) (A,B_2) is stabilizable (for any matrix A).
(C_2,A) is detectable (iff (A, C) is detectable).
(2) $D_{21}D_{21}^T = \epsilon^2 I_{p\times p}$, which is nonsingular.
$D_{12}^T D_{12} = \beta^2 I_{n\times n}$, which is nonsingular.
(3) rank $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = 2n$ = full col. rank.
rank $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n+p$ = full row rank.
(4) $D_{22} = 0$.

4.2 Proof of equivalence

Let T_1 be the setup in Fig. 1 associated with (10)-(11), T_2 the one associated with (26)-(27) and T_3 the one associated with (30)-(31) where the three share the same controller K in (19). And let $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ be their corresponding transfer matrices. The following two lemmas demonstrate a certain equivalence relationships among these setups (see Appendix A for detailed proofs).

Lemma 1. Consider a stabilizing controller K for the setups T_1 and T_2, then $\| \hat{T}_1 \|_\infty < \gamma$ if and only if $\exists\ \epsilon > 0$ such that $\| \hat{T}_2 \|_\infty < \gamma$.
Lemma 2. Given $\epsilon > 0$ and a stabilizing controller K for the setups T_2 and T_3, then $\| \hat{T}_2 \|_\infty < \gamma$ if and only if $\exists\ \beta > 0$ such that $\| \hat{T}_3 \|_\infty < \gamma$.

We now present our main result in the form of a theorem showing that the observer gain K needed to stabilize the error dynamics in the initially proposed design problem must solve a regular H_∞ control problem. To this end, we define the regular continuous H_∞ problem "Problem 1" as follows:
Problem 1: *Given $\epsilon > 0$ and $\beta > 0$, find $\mathcal{S}$, the set of admissible controllers K satisfying $\| \hat{T}_{\bar{\zeta}\bar{\omega}} \|_\infty < \gamma$ for the setup in Fig. 1 with G having the state space representation in (13) along with the matrices in (30)-(31).*

The main result is summarized as follows:
Theorem 5. Given the Lipschitz system of equations (1)-(2), the state $\hat{x}$ of the observer (15)-(19) globally asymptotically converges to the system state x for all Φ satisfying (3) with a Lipschitz constant α if and only if $\exists\ \epsilon,\ \beta > 0$ s.t $K \in \mathcal{S}^*$ (the set of controllers solving "Problem 1" defined above with $\gamma = \frac{1}{\alpha}$).
Proof: A direct result of Theorem 4, Lemmas 1 and 2. △

4.3 A new H_∞ design procedure

The following iterative "binary search" procedure is then proposed to evaluate the observer gain K:
Design procedure:
Step 1 Set ϵ, $\beta > 0$ and $\gamma \leftarrow \frac{1}{\alpha}$.
Step 2 Test solvability of Problem 1. If test fails then go to Step 3 ; otherwise solve the problem (using available software packages) and any $K \in \mathcal{S}$ is a candidate observer gain that globally stabilizes the error dynamics.
Step 3 Set $\epsilon \leftarrow \frac{\epsilon}{2}$ and $\beta \leftarrow \frac{\beta}{2}$. If ϵ or $\beta <$ a threshold value then *stop* ; otherwise go to Step 2.

Comments
- When the optimization problem can not be solved due to its infeasibility or due to limitations of the used software, one can increase γ which corresponds to a smaller Lipschitz constant α. The word *stop* in step 3 can then be replaced by: *decrease α and go to step 1.* The algorithm is then guaranteed to work as $\alpha \to 0$. However, the region of convergence is decreased unlike if original α is used.
- Same design can be used when the output is disturbed as in (25). The small gain theorem guarantees that the estimation error $e(t) \in \mathcal{L}_2$ if $d(t) \in \mathcal{L}_2$.
- Design of the H_∞ observer can also be done by including appropriate weightings to the original problem to emphasize the performance requirements of the observer over specific frequency ranges.

5. SIMULATION RESULTS

We consider an illustrative example of a 2^{nd} order system of the form (1)-(2) with $A = \begin{bmatrix} -2 & 3 \\ 3 & 1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $\Phi = \begin{bmatrix} 0 \\ 1.5\ sin(x_1) \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. The Lipschitz constant is $\alpha = 1.5$. The system initial condition $x(0) = \begin{bmatrix} 1.6 & 2 \end{bmatrix}$. This example is not meant to be realistic. It has been designed to show the proposed observer design. Therefore, in our simulation we use a state feedback control law to stabilize the equilibrium point at the origin, assuming both states to be available for feedback. The gain matrix L needed to use the observer (4)-(5) was obtained in (Raghavan and Hedrick, 1994) as $L = \begin{bmatrix} 2 & 4 \end{bmatrix}^T$. Using the H_∞ design of sec. 4.3, the matrices A_L, B_L, C_L and D_L for the observer (15)-(19) are obtained as follows (with $\epsilon = \beta = 1$):

$$A_L = \begin{bmatrix} -2.2092 & -376.4306 \\ -0.4894 & -95.3217 \end{bmatrix}, B_L = \begin{bmatrix} 159.2165 \\ 35.5614 \end{bmatrix}$$
$$C_L = \begin{bmatrix} 0.0726 & 0.3088 \\ 2.3163 & 19.3184 \end{bmatrix}, D_L = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Fig. 3 and 4 show the observers performance, where the initial condition for both observers is taken as $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

6. CONCLUSION

A new H_∞ observer design for Lipschitz nonlinear systems is proposed. It is first shown that the classical "Luenberger-like" observers are special cases of a more general dynamic framework, one that shows promise given the additional degrees of freedom. The equivalence between the observer design problem and a standard H_∞ control problem that satisfies all of the regularity assumptions is shown. A systematic design procedure that can be carried out using commercially available software products is also presented.

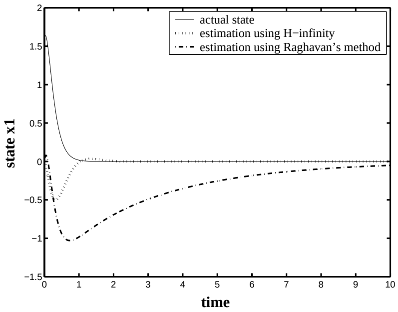

Fig. 3. Estimation of state x_1.

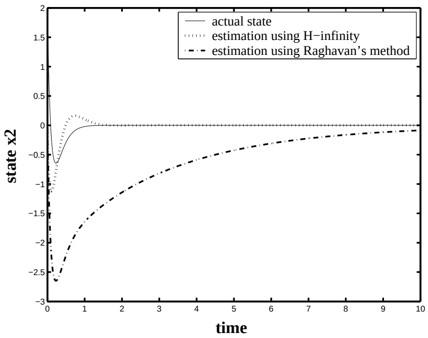

Fig. 4. Estimation of state x_2.

Appendix A.

Using the definitions in the beginning of sec. 4.2 along with the definition of ζ, ω, $\bar{\zeta}$ and $\bar{\omega}$ in (12), (28) and (29), the transfer matrix $\hat{T}_1$ is given from (23), while $\hat{T}_2$ and $\hat{T}_3$ are given from:

$$\hat{T}_2 = \hat{T}_{\zeta\bar{\omega}} = \left[\begin{array}{cc|cc} A - D_L C & -C_L & I_n & -\epsilon D_L \\ B_L C & A_L & 0_n & \epsilon B_L \\ \hline I_n & 0_n & 0_n & 0_{np} \end{array}\right]$$

$$\hat{T}_3 = \hat{T}_{\bar{\zeta}\bar{\omega}} = \left[\begin{array}{cc|cc} A - D_L C & -C_L & I_n & -\epsilon D_L \\ B_L C & A_L & 0_n & \epsilon B_L \\ \hline I_n & 0_n & 0_n & 0_{np} \\ \beta D_L C & \beta C_L & 0_n & \epsilon\beta D_L \end{array}\right]$$

We will refer to their common state transition matrix as $\hat{H} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix}$ and we hence have:

$$\begin{aligned} \hat{T}_1 &= \hat{H}_{11} \\ \hat{T}_2 &= \left[\hat{H}_{11} \;\; -\epsilon \hat{H}_{11} D_L + \epsilon \hat{H}_{12} B_L\right] \\ \hat{T}_3 &= \begin{bmatrix} \hat{H}_{11} & -\epsilon \hat{H}_{11} D_L + \epsilon \hat{H}_{12} B_L \\ \beta \hat{N}_1 & \beta \hat{N}_2 \end{bmatrix} \end{aligned} \qquad \text{(A.1)}$$

where $\hat{N}_1 = D_L C \hat{H}_{11} + C_L \hat{H}_{21}$ and $\hat{N}_2 = \epsilon\left(-\hat{N}_1 D_L + D_L C \hat{H}_{12} B_L + C_L \hat{H}_{22} B_L + D_L\right)$.

Proof of Lemma 1 (Sufficiency) For the "two input/one output" standard setup T_2, let $\exists \epsilon > 0$ and a stabilizing controller K s.t $\| \hat{T}_2 \|_\infty < \gamma$. But from (A.1) we have $\| \hat{T}_2 \|_\infty = \max\Big(\| \hat{H}_{11} \|_\infty, \| -\epsilon \hat{H}_{11} D_L + \epsilon \hat{H}_{12} B_L \|_\infty\Big)$. Hence, $\| \hat{T}_1 \|_\infty < \gamma$.
(Necessity) Let $\exists$ a controller K such that $\| \hat{T}_1 \|_\infty < \gamma$. It follows that $\| \hat{H}_{11} \|_\infty = \sigma < \gamma$.
$\therefore \| \hat{T}_2 \|_\infty = \max\Big(\sigma, \epsilon \| -\hat{H}_{11} D_L + \hat{H}_{12} B_L \|_\infty\Big)$.
But since K is a stabilizing controller, then $\| -\hat{H}_{11} D_L + \hat{H}_{12} B_L \|_\infty = \rho$ (where ρ is a finite number). Hence, $0 < \epsilon < \frac{\gamma}{\rho} \Rightarrow \| \hat{T}_2 \|_\infty < \gamma$. $\triangle$

Proof of Lemma 2 (Sufficiency) For the "two input/two output" setup T_3, let $\exists \epsilon > 0$, $\beta > 0$ and a controller K s.t $\| \hat{T}_3 \|_\infty < \gamma$. But $\| \hat{T}_3 \|_\infty = \max\Big(\| \hat{H}_{11} \|_\infty, \| -\epsilon \hat{H}_{11} D_L + \epsilon \hat{H}_{12} B_L \|_\infty, \| \beta \hat{N}_1 \|_\infty, \| \beta \hat{N}_2 \|_\infty\Big)$. Therefore, $\| \hat{T}_2 \|_\infty = \max\Big(\| \hat{H}_{11} \|_\infty, \| -\epsilon \hat{H}_{11} D_L + \epsilon \hat{H}_{12} B_L \|_\infty\Big) < \gamma$.
(Necessity) Let $\exists \epsilon > 0$, K s.t $\| \hat{T}_2 \|_\infty = \sigma < \gamma$.
$\therefore \max\Big(\| \hat{H}_{11} \|_\infty, \epsilon \| \hat{H}_{11} D_L - \hat{H}_{12} B_L \|_\infty\Big) = \sigma$.
$\therefore \| \hat{T}_3(z) \|_\infty = \max\Big(\| \beta \hat{N}_1 \|_\infty, \| \beta \hat{N}_2 \|_\infty, \sigma\Big)$.
But since K is a stabilizing controller, then $\| \hat{N}_1 \|_\infty = \rho_1$ and $\| \hat{N}_2 \|_\infty = \rho_2$ (where ρ_1 and ρ_2 are finite numbers).
$\therefore 0 < \beta < \frac{\gamma}{\max(\rho_1, \rho_2)} \Rightarrow \| \hat{T}_3 \|_\infty < \gamma$. $\triangle$

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