IMPROVING LOCAL ANTI-WINDUP PERFORMANCE: PRELIMINARY RESULTS ON A TWO-STAGE APPROACH

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Abstract: To obtain improved local performance while retaining the property of global stability, this paper proposes a new two-stage approach to anti-windup compensation for systems containing input saturation. The method ensures that (i) if no saturation occurs, then linear operation continues unhindered; (ii) if *mild* saturation occurs, an aggressive local anti-windup compensator becomes active and; (iii) if *severe* saturation occurs a globally stabilising anti-windup algorithm enters the fray. A notable feature of the scheme is that, for stable plants, a two-stage compensator will always exist. Another interesting aside, is that simple, possibly *ad hoc* techniques can be combined to yield a good two-stage compensator which may perform better, locally, than many 'optimal' schemes.

Keywords: Anti-windup, constrained control

1. INTRODUCTION

It is well known that local and global analyses of the stability and performance properties of nonlinear systems can be vastly different. At its most severe, a local analysis of a nonlinear system will yield a linear system, which, in many cases can yield important information about the system's behaviour. However, if the system state evolves to some point far beyond that neighbourhood in which the linearisation is valid, results may be misleading. In contrast, a global analysis, where possible, may yield results which, locally are not as sharp as one would like and may give unduly conservative estimates of a system's performance properties and such like.

Systems which are linear, apart from a saturation element at the plant input, are an important class of nonlinear systems and demonstrate the above argument perfectly. In fact, rather than conduct a global nonlinear analysis, the common approach to controller design for such systems is to design a controller which behaves in a desirable manner while the saturation is behaving as the identity operator, and then, if necessary, some other add-on compensator - *the anti-windup compensator* (AWC) - is designed to become active once saturation occurs.

Over the years many researchers have sought to address the stability properties of AWC's and a number of approaches have emerged. One of the first methods was proposed by Glattfelder and Schaufelberger (1988) who advocated the use of classical absolute stability theory tools such as the Circle and Popov Criteria. These approaches have prevailed until today and much of the modern literature on stability of anti-windup (AW) schemes is based on the algebraic form (often as matrix inequalities) of these criteria (see, for example Mulder *et al.* (2001)). Other methods are also available and, most notably for unstable systems, various other, local, stability criteria have been proposed by Hu *et al.* (2001), Hippe (2003) and Gomes da Silva Jr and Tarbouriech (1999).

One of the main obstacles to the performance of compensator design techniques suggested in Mulder *et al.* (2001), Turner and Postlethwaite (2004) and Grimm *et al.* (2001) is that *global* stability is enforced. This is, from one perspective, advantageous, as these algorithms allow simultaneous controller synthesis and global stability guarantees. However, these optimisation based schemes tend to loose some of the intuition which practitioners find important and therefore techniques such as the Hanus (Hanus *et al.* 1987) and high-gain techniques still retain great favour in industry. Moreover, these *ad hoc* techniques often outperform optimisation-based techniques, at least locally, although they satisfy no formal stability criteria.

There is often a trade-off in AWC design between designing a compensator which performs very well locally, but which cannot be proved globally stabilising; and a compensator which is inferior locally but is guaranteed to be globally stabilising. The purpose of this paper is to propose a type of AWC which can achieve local results which are as good as the ad-hoc anti-windup compensators, but which also retain global stability. This naturally leads to a two-stage AW approach where the first stage of AW is an aggressive local design which, if used alone, would not guarantee global stability. This first stage is active during what we shall call *mild* actuator saturation (defined later). The second stage supersedes the first when actuator saturation becomes *severe* (defined later) and here a less aggressive but globally stabilising compensator dominates the closed-loop behaviour. Incidentally, this approach may be preferable from another perspective: for mild saturation AW objectives, such as retention of linear behaviour as much as possible, might be desirable; if actuator saturation is severe anti-windup objectives, such as fast de-saturation of an actuator might be preferable. Other material focused on a similar problem to that discussed here has recently appeared in Zaccarian and Teel (2004).

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2. ANTI-WINDUP: SINGLE STAGE

2.1 Preliminaries

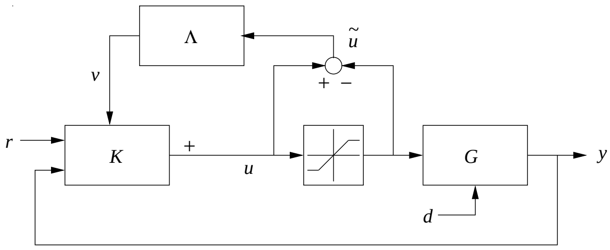

Fig. 1. General one-stage anti-windup configuration

Consider the system in Figure 1 where a single stage of AW compensation is shown. The saturation and deadzone nonlinearities are defined as:

$$\text{Sat}_{\bar{u}}(u) = \begin{bmatrix} \text{sat}_{\bar{u}_1}(u_1) \\ \vdots \\ \text{sat}_{\bar{u}_m}(u_m) \end{bmatrix} \quad \text{Dz}_{\bar{u}}(u) = \begin{bmatrix} \text{Dz}_{\bar{u}_1}(u_1) \\ \vdots \\ \text{Dz}_{\bar{u}_m}(u_m) \end{bmatrix} \quad (1)$$

where $\text{sat}_{\bar{u}_i} = \text{sign}(u_i)\min\{|u_i|, \bar{u}_i\}$ and $\text{Dz}_{\bar{u}_i} = \text{sign}(u_i)\max\{0, |u_i| - \bar{u}_i\}$ and $\bar{u}_i > 0 \quad \forall i$. The saturation and deadzone functions are related through the identity

$$\text{Sat}_{\bar{u}}(u) + \text{Dz}_{\bar{u}}(u) = u, \quad \forall u \in \mathbb{R}^m \quad (2)$$

The plant has a state-space realisation

$$[\,G_1 \;\; G_2\,] \sim \begin{cases} \dot{x}_p = A_p x_p + B_{pd} d + B_p u_m \\ y = C_p x_p + D_{pd} d + D_p u_m \end{cases} \quad (3)$$

where $y \in \mathbb{R}^{n_p}$ is the output, $d \in \mathbb{R}^{n_d}$ is the disturbance, $x_p \in \mathbb{R}^{n_p}$ is the plant state, and $u_m = \text{Sat}_{\bar{u}}(u) \in \mathbb{R}^m$ is the plant input. The controller is implemented as

$$[K_1 \;\; K_2 \;\; K_{aw1} \;\; K_{aw2}] \sim \begin{cases} \dot{x}_c = A_c x_c + B_{cr} r + B_{cy} y + v_1 \\ u = C_c x_c + D_{cr} r + D_{cy} y + v_2 \end{cases} \quad (4)$$

where $u \in \mathbb{R}^m$ is the control signal, $r \in \mathbb{R}^{n_r}$ is the reference and $x_c \in \mathbb{R}^{n_c}$ is the controller state. The vector $v := [v_1' \quad v_2']' \in \mathbb{R}^{n_c+m}$ is a signal generated by the anti-windup compensator (if active) and is used to enforce stability, and improve performance, during actuator saturation. Due to its linearity, the controller can be partitioned as

$$K_1 \sim (A_c, B_{cr}, C_c, D_{cr}) \quad (5)$$

$$K_2 \sim (A_c, B_{cy}, C_c, D_{cy}) \quad (6)$$

$$K_{aw1} \sim (A_c, I_{n_c}, C_c, 0_{m\times n_c}) \quad (7)$$

$$K_{aw2} \sim (0_{0\times 0}, 0_{0\times m}, 0_{m\times 0}, I_m) \quad (8)$$

If $u \preceq \bar{u}$ (this notation means $u_i \le \bar{u}_i \forall i$), the system behaves as the *nominal linear system*, that is as the system were the saturation replaced by the identity operator. During this period the AWC is not active, and K_{aw1} and K_{aw2} do not play a role in the system. We make the following assumption on the nominal linear system (Turner and Postlethwaite (2004)).

Assumption 1. The plant and the nominal linear closed loop are both (internally) stable and well posed. Equivalently:

- The poles of $G(s)$ have strictly negative real parts
- The transfer function matrix

$$\begin{bmatrix} I & -K_2 \\ -G_2 & I \end{bmatrix} \quad (9)$$

is invertible in $\mathscr{RH}_\infty$

The second part of the assumption ensures that the AW problem makes sense. The first part ensures that global stability results can be obtained and is a necessary condition for the existence of an AWC which can ensure robust global stability of the closed-loop.

Standard AW is described in terms of a linear transfer function matrix, $\Lambda(s)$, which implies that:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1(s) \\ \Lambda_2(s) \end{bmatrix} \tilde{u} = \Lambda \tilde{u}, \quad \tilde{u} = \text{Dz}_{\bar{u}}(u) \quad (10)$$

This type of configuration is very general and one which is used in the work of Grimm *et al.* (2001) and Mulder *et al.* (2001). If we set $r = 0$ and $d = 0$,

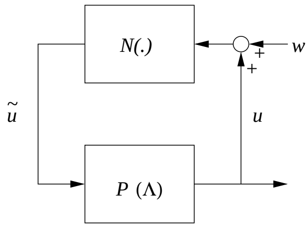

Fig. 2. Equivalent representation of one-stage anti-windup

using the identity $\text{Dz}_{\bar{u}}(u) = u - \text{Sat}_{\bar{u}}(u)$ we can re-draw Figure 1 as Figure 2 (as in Mulder *et al.* (2001) and Grimm *et al.* (2001)). Equivalently, we have

$$u = P(\Lambda)\tilde{u} \quad (11)$$

$$\tilde{u} = \mathscr{N}(u + w) \quad (12)$$

$$z = u \quad (13)$$

where

$$\mathscr{N}(.) = \text{Dz}_{\bar{u}}(.) \quad (14)$$

$$P(\Lambda) := (I - K_2 G_2)^{-1}(-K_2 G_2 + K_{aw2}\Lambda_2 + K_{aw1}\Lambda_1) \quad (15)$$

A state-space realisation for $P(\Lambda)$ is readily calculated from the state-space matrices given above for the plant, controller and AWC. The signals w and z will be useful in the next section and we denote the operator mapping w to z as $\mathscr{T}(\Lambda) : \mathbb{R}^m \mapsto \mathbb{R}^m$.

At this stage it would be normal to consider the deadzone as a Sector$[0, I]$ nonlinearity and use the Circle Criterion to synthesise an AWC which would provide global stability. In fact, if the plant is stable there always exists such a compensator (see Grimm *et al.* (2001) for example). However we shall take a different approach.

2.2 Local anti-windup

It is well known that the deadzone nonlinearity inhabits the Sector$[0, I]$ that is, if

$$\tilde{u}_1 = \mathscr{N}(u) := \text{Dz}_{\bar{u}}(u) \quad (16)$$

then we have, for some diagonal $W > 0$, that (see (Khalil 1996), Chapter 10)

$$\tilde{u}_1' W (u - \tilde{u}_1) \ge 0 \quad (17)$$

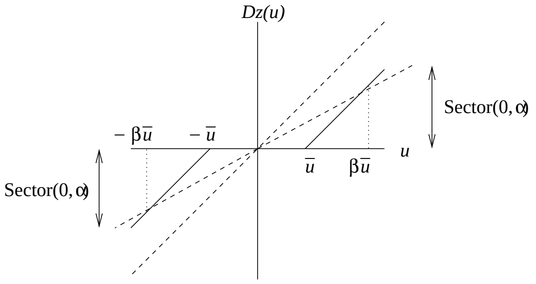

Fig. 3. The deadzone and Sector bounds

From Figure 3 we can see that for any u, the deadzone nonlinearity remains in the Sector$[0,1]$ (for the scalar case). However, let us suppose that u_i never becomes greater that $\beta_i \bar{u}_i \quad \forall i$ where $\beta_i > 1, \quad \forall i$, then locally we can see that the deadzone actually inhabits a narrower sector, Sector$[0,\mathscr{A}]$ where $\mathscr{A} := \text{diag}(\alpha_1,\ldots,\alpha_m)$ and $\alpha_i := \frac{\beta_i - 1}{\beta_i} \in (0,1)$. Again, this can be seen for the scalar case in Figure 3. Note that $\mathscr{A} < I$.

The approach we take now is to synthesise an AWC which stabilises the system in Figure 2 for all nonlinearities $\mathscr{N} \in$ Sector$[0,\mathscr{A}]$. An AWC which is globally stabilising for all such sector nonlinearities will, then, be *locally stabilising* for the system in Figure 2 when $\mathscr{N}(.) = \text{Dz}_{\bar{u}}(.)$, providing the magnitude of u_i does not exceed $\beta_i \bar{u}_i$. Thus two approaches can be taken

(1) An existing, ad-hoc, AWC can be analysed to find a suitable sector bound, $\mathscr{A}$, for which it is guaranteed to be (globally) stabilising; or
(2) Given a set of β_i, or equivalently a sector matrix $\mathscr{A}$, an AWC can be synthesised which guarantees (global) stability for all nonlinearities in the Sector$[0,\mathscr{A}]$.

It is immediate from the results of Grimm *et al.* (2001), that, as there exists a globally stabilising AWC for all $\mathscr{N} \in$ Sector$[0,I]$, there will always exist a globally stabilising AWC for all $\mathscr{N} \in$ Sector$[0,\mathscr{A}]$ because $\mathscr{A} < I$. However, typically we are able to use much more aggressive AW techniques when the sector is narrower and also there always exists a sufficiently 'small' $\mathscr{A}$ such that a specific type of locally stabilising AWC is globally stabilising for all $\mathscr{N} \in$ Sector$[0,\mathscr{A}]$. Appendix A gives details on how to synthesise a static AWC which is globally stabilising for all $\mathscr{N} \in$ Sector$[0,\mathscr{A}]$. Our second assumption is thus:

Assumption 2. With $\Lambda = \Theta$, the following condition holds: The interconnection of $P_\Theta = P(\Theta)$ and $\mathscr{N} \in$ Sector$[0,\mathscr{A}]$ through equations (11) - (13) is globally exponentially stable (when $w = 0$) and the operator $\mathscr{T}_\Theta : w \mapsto z$ is well-defined.

As the interconnection of P_Θ and $\mathscr{N} \in$ Sector$[0,\mathscr{A}]$ is globally exponentially stable, this ensures that the map $\mathscr{T}_\Theta$ is actually finite gain $\mathscr{L}_p$ stable. However, by itself, this means simply that the system in Figure 1 will be locally asymptotically stable when the AWC $\Lambda = \Theta$ is used (as the deadzone nonlinearity is actually in Sector$[0,I]$ globally); it gives no guarantees of global stability.

2.3 Global anti-windup

Consider Figure 2 again. Assumption 1 is a necessary and sufficient condition to guarantee the existence of an AWC, which we call $\Phi(s)$, which globally stabilises the origin of the closed-loop system. The performance of such a compensator is not guaranteed to be locally optimal, but obviously the property of global, or at least large-signal, stability is desirable. This motivates our third assumption.

Assumption 3. With $\Lambda = \Phi$, the following condition holds: The interconnection of $P_\Phi = P(\Phi)$ and $\mathscr{N} \in$ Sector$[0,I]$ through equations (11) - (13) is globally exponentially stable (when $w = 0$) and the operator $\mathscr{T}_\Phi : w \mapsto z$ is well-defined.

We do not give details on how to construct such an AWC, but there are several methods available from which one can construct a suitable globally stabilising AWC (Grimm *et al.* (2001),Wada and Saeki (1999)). The next section will demonstrate how the locally stabilising AWC, $\Theta(s)$ and the globally stabilising AWC, $\Phi(s)$ can be combined.

3. ANTI-WINDUP: SECOND STAGE

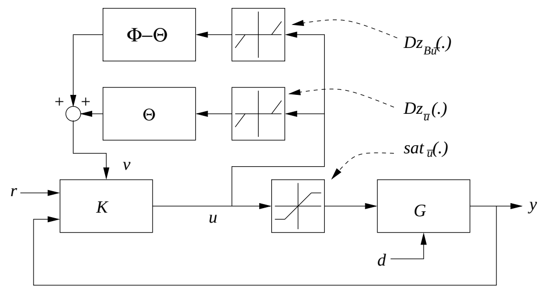

Fig. 4. Two-stage anti-windup configuration

Consider Figure 4 where what we shall call *two stage anti-windup* is shown. The AWC, Γ is defined as

$$\Gamma : \begin{cases} v_1 = \Theta_1 \text{Dz}_{\bar{u}}(u) + (\Phi_1 - \Theta_1)\text{Dz}_{\mathscr{B}\bar{u}}(u) \\ v_2 = \Theta_2 \text{Dz}_{\bar{u}}(u) + (\Phi_2 - \Theta_2)\text{Dz}_{\mathscr{B}\bar{u}}(u) \end{cases} \quad (18)$$

where $\mathscr{B} := \text{diag}(\beta_1,\ldots,\beta_m)$. Note that if the $\beta_i = \infty$, or the block $\Phi(s) - \Theta(s)$ is disconnected, we recover the standard one-stage AWC. The operation of the AWC, Γ, is roughly as follows. When no saturation occurs $u \preceq \bar{u}$, the system will continue to operate linearly, as $Dz(u) = 0 \quad \forall u \preceq \bar{u}$. However when *mild* saturation in at least one channel occurs, that is $\bar{u}_i \le |u_i| < \beta_i \bar{u}_i$ for some i the first stage of AW becomes active i.e. $\text{Dz}_{\bar{u}} \neq 0$, but $\text{Dz}_{\mathscr{B}\bar{u}}(u) = 0$. Then if *severe* saturation occurs, that is $|u_i| \ge \beta_i \bar{u}_i$ for some i, both stages of AW will become active as now $\text{Dz}_{\mathscr{B}\bar{u}}(u) \neq 0$ also. To summarise, Γ gives good local performance, but then 'takes out' this compensator when the saturation becomes sufficiently large and the global AWC takes the more prominent role.

It is assumed that Θ has been designed such that it gives desirable local stability (Assumption 2) and performance in some sense, but is not necessarily globally stabilising. Φ is assumed (by Assumption 3) to simply be globally stabilising. The attractive feature of Γ is that it blends together two types of AW behaviour in a continuous fashion. The remainder of this section will demonstrate that the compensator Γ globally stabilises the system in Figure 4.

Fact 4. (1) If $\mathrm{Dz}_{\bar{u_i}}(u_i) \neq 0$ and $\mathrm{Dz}_{\beta_i \bar{u_i}}(u_i) \neq 0$ then

$$\mathrm{sign}(\mathrm{Dz}_{\bar{u_i}}(u_i)) = \mathrm{sign}(\mathrm{Dz}_{\beta_i \bar{u_i}}(u_i))\ \forall i,\ \forall \beta > 1 \quad (19)$$

$$(2) \quad \mathrm{Dz}_{\bar{u_i}}(u_i) = 0 \Rightarrow \mathrm{Dz}_{\beta_i \bar{u_i}}(u_i)) = 0\ \forall i,\ \forall \beta > 1 \quad (20)$$

Lemma 5.

$$\tilde{\mathscr{N}}(u) := \mathrm{Dz}_{\bar{u}}(u) - \mathrm{Dz}_{\mathscr{B}\bar{u}}(u) \in \mathrm{Sector}[0, \mathscr{A}] \quad (21)$$

Proof: We prove that

$$\tilde{\mathscr{N}_i}(u_i) := \mathrm{Dz}_{\bar{u_i}}(u_i) - \mathrm{Dz}_{\beta_i \bar{u_i}}(u_i) \in \mathrm{Sector}[0, \alpha_i] \quad (22)$$

for all i. Using Fact 4, it follows that we need to consider only three cases (for each i):

(1) If $\mathrm{Dz}_{\bar{u_i}}(u_i) = 0$ (which implies $\mathrm{Dz}_{\beta_i \bar{u_i}}(u_i) = 0$ as $\beta > 1$), then $\tilde{\mathscr{N}_i}(u_i) = 0$. This corresponds to $u_i < \bar{u_i}$.

(2) If $\mathrm{Dz}_{\bar{u_i}}(u_i) \neq 0$, but $\mathrm{Dz}_{\beta_i \bar{u_i}}(u_i) = 0$ (as $\beta > 1$ this is possible), then $\tilde{\mathscr{N}_i}(u_i) = \mathrm{Dz}_{\bar{u_i}}(u_i)$. This corresponds to the case when $u_i \geq \bar{u_i}$, but less than $\beta_i u_i$.

(3) If $u_i \geq \beta_i \bar{u_i}$, then $\mathrm{Dz}_{\bar{u_i}}(u_i)$ and $\mathrm{Dz}_{\beta_i \bar{u_i}}(u_i)$ are both non-zero, but of the same sign, by Fact 1. Hence we have

$$\tilde{\mathscr{N}_i}(u_i) = \mathrm{sign}(u_i)[|u_i| - \bar{u_i} - (|u_i| - \beta_i \bar{u_i})] \quad (23)$$

$$= \mathrm{sign}(u_i) \underbrace{[(\beta_i - 1)\bar{u_i}]}_{>0} \quad (24)$$

Combining these three different cases gives the result for each i. The lemma then follows as a result of this. □□.

The following is the main result of the paper.

Theorem 6. Define $\mathscr{T}_\Theta : w \mapsto z$ to be the (finite gain stable) operator formed from the interconnection of equations (11) - (13) with $\Lambda = \Theta$. Define also $\mathscr{T}_\Phi : w \mapsto z$ to be the (finite gain stable) operator formed from the interconnection of equations (11) - (13) with $\Lambda = \Phi$

If Assumptions 1,2 and 3 are satisfied then the compensator Γ ensures global stability of the system in Figure 4 if

$$\|\mathscr{T}_\Phi(.)\|_{i,2} < \frac{1}{\|\mathscr{T}_\Theta(.)\|_{i,2}} \quad (25)$$

Proof: First note that if $r = 0, d = 0$, then Figure 4 can be represented by the equations

$$u = M[\tilde{u}_1' \quad \tilde{u}_2']' \quad (26)$$

where

$$\tilde{u}_1 := \mathscr{N}_1(u) = \mathrm{Dz}_{\bar{u}}(u) \quad (27)$$

$$\tilde{u}_2 := \mathscr{N}_2(u) = \mathrm{Dz}_{\mathscr{B}\bar{u}}(u) \quad (28)$$

and

$$M := (I - K_2 G_2)^{-1}[-K_2\ K_{aw1}\ K_{aw2}] \begin{bmatrix} G_2 & 0 \\ \Theta_1 & \Phi_1 - \Theta_1 \\ \Theta_2 & \Phi_2 - \Theta_2 \end{bmatrix} \quad (29)$$

Adding and subtracting $-K_2 G_2 \tilde{u}_2$ to eq (26) we obtain

$$u = u_1 + u_2 \quad (30)$$

where

$$u_1 = \underbrace{(I - K_2 G_2)^{-1}(-K_2 G_2 + K_{aw1}\Theta_1 + K_{aw2}\Theta_2)}_{P(\Theta)} \tilde{u}_{1,2} \quad (31)$$

$$u_2 = \underbrace{(I - K_2 G_2)^{-1}(-K_2 G_2 + K_{aw1}\Phi_1 + K_{aw2}\Phi_2)}_{P(\Phi)} \tilde{u}_2 \quad (32)$$

and

$$\tilde{u}_{1,2} := \mathscr{N}_3(u) = \tilde{u}_1 - \tilde{u}_2 = \mathrm{Dz}_{\bar{u}}(u) - \mathrm{Dz}_{B\bar{u}}(u) \quad (33)$$

Note that equations (30),(31) and (33) are in the form of equations (11) - (13), so we can define the operator

$$u_1 = \mathscr{T}_\Theta(u_2) \quad (34)$$

By Assumption 2, $\mathscr{T}_\Theta$ is well-defined and finite-gain stable because, by Lemma 1 $\mathscr{N}(.) \in \mathrm{Sector}[0, \mathscr{A}]$. Similarly, equations (30), (32) and (28) are in the form of equations (11) - (13), so we can define the operator

$$u_2 = \mathscr{T}_\Phi(u_1) \quad (35)$$

By Assumption 3, $\mathscr{T}_\Phi$ is well-defined and finite-gain stable. By Assumption 1, there always exists compensators satisfying Assumptions 2 and 3. The results follow by applying the Small Gain Theorem to equations (34) and (35). □□.

Corollary 7. There always exists a Γ such that Theorem 6 holds.

Proof: It is sufficient to prove that there always exists an AWC which satisfies Assumption 3 with $\mathscr{T}_\Phi$ having small enough $\mathscr{L}_2$ gain. To see this choose $\Phi_1 = B_{cy} G_2(s)$ and $\Phi_2 = D_{cy} G_2(s)$, then we have

$$P(\Phi) = (I - K_2 G_2)^{-1}(-K_2 G_2 + K_2 G_2) = 0 \quad (36)$$

Hence $\|\mathscr{T}_\Phi\|_{i,2} = 0$ and therefore Theorem 1 holds. □□.

Remark The above choice makes the second stage of AW compensation equivalent to internal model control (IMC) AW. This ensures no expensive computational optimisation is required, although it is possible that the transient response may suffer. □

Corollary 8. There always exists a static first stage AWC (Θ) which solves Theorem 1.

Proof: From the appendix, it is always possible to choose the sector narrow enough (i.e. β_i and α_i small enough) to ensure that a static AWC solves Lemma 1. By corollary 1, it follows that there exists always a choice of Φ such that $\|\mathscr{T}_2\|_{i,2} = 0$. □□

4. EXAMPLE

Consider the example used in Romanchuk (1999) and Turner and Postlethwaite (2004). Figure 5 shows the response of the system due to a 3-2-1 type pulse input both channels without saturation; the system has a fast, well-damped and decoupled response. When input saturation with saturation limits of ± 8 in both channels is introduced as shown in Figure 6 this response deteriorates and unacceptable behaviour in one channel results. To address this problem, we use

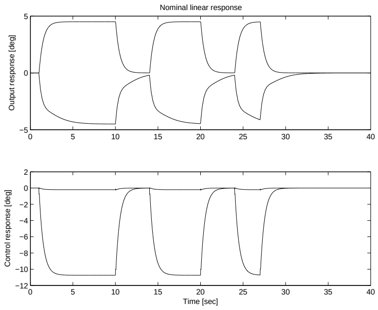

Fig. 5. 3-2-1 pulse response: linear system

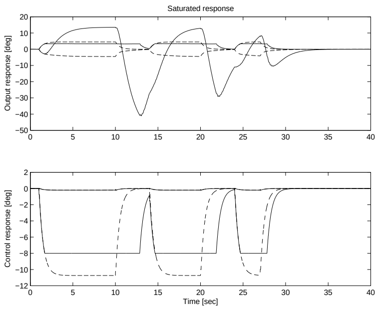

Fig. 6. 3-2-1 pulse response: $-$ saturated system; $--$ linear system

an 'optimal' static AWC as suggested in Turner and Postlethwaite (2004), but only require this to stabilise the system in the Sector$[0,0.2]\times I_2$, allowing it to be more aggressive than originally reported in Turner and Postlethwaite (2004). We also use an optimal full-order AWC as suggested in Turner *et al.* (2004) but stipulate that this must be globally stabilising; that is for all values of the deadzone in Sector$[0,I]$. It transpires that as both of these AWC's are based on the representation given in (Weston and Postlethwaite 2000) they have a very compact implementation [2]. Table 1 shows the $\mathscr{L}_2$ gains of the operator $\mathscr{T}_p$ which governs the deviation of saturated performance from nominal linear performance (see Turner and Postlethwaite (2004) for more details). Table 1 suggests that

Compensator	$\mathscr{L}_2$ gain	Number of states
Static compensator, $\alpha = 0.5$	≈ 0.1	0
Full order compensator, $\alpha = 1$,	≈ 376	3

Table 1. Comparison of local/global $\mathscr{L}_2$ gains

locally, we can expect much better performance from our static AWC, providing our control signals do not exceed $\bar{u}/(1-\alpha) = 8/(1-0.2) = 10$ in each channel. However, we are not guaranteed of any performance - or even stability - once the control signals cross this boundary. Conversely, our full-order AWC is globally stabilising although its $\mathscr{L}_2$ gain is substantially more than the locally performing compensator. From this we might expect that a two-stage AWC constructed from the combination of the two above compensators might perform better locally than the full-order AWC, while retaining its global stability and performance properties. Figures 7, 8 and 9 show the responses

[2] This will be reported elsewhere due to space limitations

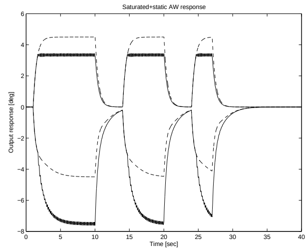

Fig. 7. 3-2-1 pulse response: $-$ w/static AW; $--$ linear system

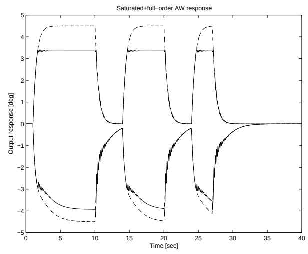

Fig. 8. 3-2-1 pulse response: $-$ w/full-order AW; $--$ linear system

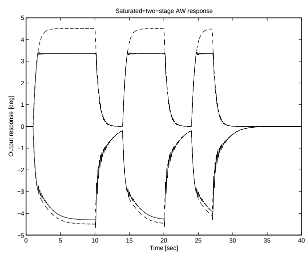

Fig. 9. 3-2-1 pulse response: $-$ w/two-stage AW; $--$ linear system

obtained from our example system using the locally optimal static AWC, the globally optimal full-order AWC, and the two-stage AWC constructed from combining the two. The reference demand is a step input of amplitude 4.5 in each channel. This is enough to make the control signal saturate and exceed 10 degrees in one channel, meaning that our locally optimal static AWC is not guaranteed to perform well. Notice that this is indeed the case with the static compensated closed-loop's response resulting in large overshoots in the responses of the second channel. As expected the full order AWC is less aggressive and tracking in the second channel is not quite achieved. However, using the two-stage AWC, an improvement in tracking is maintained. Further simulation results show that, as the input demands get larger, the two-stage AWC's response converges to that of the full order AWC as expected.

Remark: In many simple examples in the literature, it is difficult to demonstrate that a two-stage AWC shows significant performance improvement over a one-stage AWC; often static AWC's, with only local stability and performance *guarantees*, actually perform very well far beyond the region in which this behaviour is guaranteed. Alternatively, full-order AWC's may perform so well locally, that a locally optimal AWC is not required (interestingly this transpires to be the case for the example given in Zaccarian and Teel (2004) where a full-order AWC synthesised according to the results of Turner *et al.* (2004) performs as well as the *nonlinear* compensator given there). However we believe that AWC's such as the two-stage proposal here, or the nonlinear one developed in Zaccarian and Teel (2004) are likely to show their value when considering complex, high-order systems.

5. CONCLUSION

This paper has laid the foundations for a two-stage approach to AW. The results allow an aggressive, locally stabilising, AWC to be combined with a globally stabilising second-stage AWC. The application of a suitable second stage design is dependent on a small-gain inequality holding, although the IMC compensator always satisfies this inequality. The results potentially allow popular AW techniques (which lack stability guarantees) to be applied with confidence.

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Appendix A. EXISTENCE OF A LOCALLY STABILISING STATIC AWC

In this section we shall prove that there always exists a locally stabilising static AWC, for small enough sector i.e. for α_i small enough. As we consider only stability we have that $r = 0$ and $d = 0$. Therefore, $P_\Theta = P(\Theta)$ has state-space realisation

$$P(\Theta) \sim \left\{ \begin{array}{rcl} \dot{\tilde{x}} &=& \tilde{A}\tilde{x} + (B_0 + \tilde{B}\Theta)\tilde{u}_1 \\ u &=& \tilde{C}\tilde{x} + (D_0 + \tilde{D}\Theta)\tilde{u}_1 \end{array} \right\} \qquad \text{(A.1)}$$

where the state-space matrices of P_1 are defined as

$$\tilde{x} = \begin{bmatrix} x_p \\ x_c \end{bmatrix} \quad \Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$$
$$\tilde{A} = \begin{bmatrix} A_p + B_p\Delta D_{cy}C_p & B_p\Delta C_c \\ B_{cy}\tilde{\Delta}C_c & A_c + B_c\tilde{\Delta}D_pC_c \end{bmatrix}$$
$$B_0 = \begin{bmatrix} -B_p\Delta \\ -B_c\tilde{\Delta}D_p \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0 & B_p\Delta \\ I & B_c\tilde{\Delta}D_p \end{bmatrix}$$
$$\tilde{C} = \begin{bmatrix} \Delta D_cC_p & \Delta C_c \end{bmatrix} \quad D_0 = -\Delta D_{cy}D_p \quad \tilde{D} = \begin{bmatrix} 0 & \Delta \end{bmatrix}$$

where $\Delta = (I - D_{cy}D_p)^{-1}$ and $\tilde{\Delta} = (I - D_pD_{cy})^{-1}$ exist due to Assumption 1. In this case Θ_1, Θ_2 are static matrices. Next, as $\mathscr{N} \in \text{Sector}[0, \mathscr{A}]$, we have that

$$2\tilde{u}_1 W(\mathscr{A}u - \tilde{u}_1) \geq 0 \qquad \text{(A.2)}$$

for some diagonal matrix $W > 0$ (note that $\mathscr{A} = \text{diag}(\alpha_1, \ldots, \alpha_m) > 0$ as well). Using the multivariable form of the Circle Criterion we obtain that the system is asymptotically stable if the following matrix inequality holds

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} & PB_0 + P\tilde{B}\Theta + \tilde{C}'\mathscr{A}W \\ \star & -2W + W\mathscr{A}(D_0 + \tilde{D}\Theta) + (D_0' + \Theta'\tilde{D}')\mathscr{A}W \end{bmatrix} < 0 \qquad \text{(A.3)}$$

for some $P > 0$, diagonal $W > 0$ and matrix Θ. Using the congruence transformation $\text{diag}(P^{-1}, W^{-1}) = \text{diag}(Q, U)$ we obtain the matrix inequality

$$\begin{bmatrix} Q\tilde{A}' + \tilde{A}Q & B_0U + \tilde{B}\Theta U + Q\tilde{C}\mathscr{A} \\ \star & -2U + \mathscr{A}D_0U + UD_0'\mathscr{A} + \mathscr{A}\tilde{D}\Theta U + U\Theta'\tilde{D}'\mathscr{A} \end{bmatrix} < 0 \qquad \text{(A.4)}$$

If we choose $\mathscr{A} = U = \alpha_{max}I$ and use the Schur complement, then inequality (A.4) holds iff

$$Q\tilde{A}' + \tilde{A}Q - \alpha_{max}^2(B_0 + \tilde{B}\Theta + Q\tilde{C}') \times$$
$$(-2I + \alpha_{max}D_0 + \alpha_{max}D_0' + \alpha_{max}\tilde{D}\Theta + \alpha_{max}\Theta'\tilde{D}')^{-1}$$
$$\times (B_0 + \tilde{B}\Theta + Q\tilde{C}') < 0 \qquad \text{(A.5)}$$

By Assumption 1, the nominal system is asymptotically stable, implying that $\tilde{A}$ is Hurwitz and consequently that $Q\tilde{A}' + \tilde{A}Q < 0$. Hence, by choosing α_{max} small enough, the above inequality can always be ensured to hold.

The above result mirrors the one in (Kapoor *et al.* 1998) where it is proved that all AWC's are at least locally stabilising. Of course, such a solution is not necessarily optimum and a better approach might be to solve inequality (A.4) for Q, U, Θ for a given $\mathscr{A}$. Inequality (A.4) is bilinear but defining $L = \Theta U$ yields the linear matrix inequality

$$\begin{bmatrix} Q\tilde{A}' + \tilde{A}Q & B_0U + \tilde{B}L + Q\tilde{C}\mathscr{A} \\ \star & -2U + \mathscr{A}D_0U + UD_0'\mathscr{A} + \mathscr{A}\tilde{D}L + L'\tilde{D}'\mathscr{A} \end{bmatrix} < 0 \qquad \text{(A.6)}$$

Such a problem is not guaranteed to be feasible for arbitrary $\mathscr{A}$, but as proved above, there will exist an $\mathscr{A}$ such that (A.6) is feasible.

An alternative approach is to solve (A.4) for Q, U and $\mathscr{A}$ for given Θ. Again (A.4) is bilinear, but reversing the congruency transformation yields

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} & PB_0 + P\tilde{B}\Theta + \tilde{C}'Y \\ \star & -2W + YD_0 + D_0'Y + Y\tilde{D}\Theta + \Theta'\tilde{D}'Y \end{bmatrix} < 0 \qquad \text{(A.7)}$$

which is linear in $P > 0$, diagonal $W > 0$ and diagonal $Y > 0$, where $Y = \mathscr{A}W = W\mathscr{A}$. Hence $\mathscr{A}$ can be determined as $\mathscr{A} = YW^{-1}$.