NONLINEAR DISCRETE-TIME ROBUST OUTPUT REGULATION PROBLEM

Weiyao Lan Jie Huang

Department of Automation and Computer-Aided Engineering, Chinese University of Hong Kong, Hong Kong, China

Abstract: In this paper, we will establish a framework that can convert a robust output regulation problem for discrete-time nonlinear systems into a robust stabilization problem for an appropriately augmented system consisting of the given plant and a specific dynamic system called internal model. We then apply this framework to solve the local robust output regulation problem for a general class of discrete-time nonlinear systems. The results of this paper gives a discrete-time counterpart of the recent results on the continuous-time robust output regulation problem.

Keywords: discrete-time system, nonlinear system, output regulation, stabilization, internal model.

1. INTRODUCTION

The robust output regulation problem aims to design a feedback control law for an uncertain plant such that the output of the plant can asymptotically track a class of reference inputs and/or reject a class of disturbances while maintaining the closed-loop stability. Here both the class of reference inputs and the class of the disturbances are generated by autonomous differential equations. For continuous-time systems, the problem has been extensively studied since 1970s for linear systems (Davison (1976), Francis and Wonham (1976), Francis (1977) et al.) and since 1990s for nonlinear systems (Isidori and Byrnes (1990), Huang and Rugh (1990), Huang and Rugh (1992), Huang and Lin (1993*a*), Huang and Lin (1994*a*), Khalil (1994), Byrnes *et al.* (1997),Khalil (2000), and Huang (2001), to name just a few). Recently, a new framework for studying the robust output regulation problem was proposed in Huang and Chen (2004). Under this new framework, the robust output regulation problem for a given plant can be systematically converted into a robust stabilization problem for an appropriately augmented plant, thus setting a stage for solving the robust output regulation problem using a variety of stabilization techniques.

In spite of extensive research on the continuous-time nonlinear robust output regulation problem for over a decade, the research on the discrete-time nonlinear output regulation problem has been limited only to the special case where the plant is assumed to contain no uncertainty, e.g., Castillo and Gennaro (1991), Castillo *et al.* (1993), Huang and Lin (1993*b*), and Huang and Lin (1994*b*). The objective of this paper is to study the robust output regulation problem for a general class of discrete-time uncertain nonlinear systems in a spirt similar to the recent work on continuous-time nonlinear systems given in Huang and Chen (2004).

Due to the page limit, the proof of the most results of this paper is omitted. A full version of this paper can be found in Lan and Huang (2005),

⋆ The work described in this paper was partially supported by the Hong Kong Research Grant Council under Grant CUHK4316/02E, and the National Natural Science Foundation of China under Grant 60374038.

2. PROBLEM DESCRIPTION

Consider a discrete-time nonlinear system described by

$$\begin{aligned} x(t+1) &= f(x(t),u(t),v(t),w),\ x(0)=x_0, \\ e(t) &= h(x(t),u(t),v(t),w),\ t=0,1,\cdots \end{aligned} \quad (1)$$

where $x(t) \in \Re^n$ is the plant state, $u(t) \in \Re^m$ the plant input, $e(t) \in \Re^p$ the plant output representing the tracking error, $w \in \Re^N$ the plant uncertain parameters, and $v(t) \in \Re^q$ the exogenous signal representing the disturbance and/or the reference input. It is assumed that $v(t)$ is generated by an autonomous system

$$v(t+1) = A_1 v(t),\ \ v(0)=v_0,\ \ t=0,1,\cdots \quad (2)$$

where all the eigenvalues of A_1 are simple and located on the unit circle. Without lost of generality, assume A_1 is given by

$$A_1 = \text{diag}\{S_0, S_1, \cdots, S_k\} \quad (3)$$

where $S_0 = 1$, $2k+1 = n$ and

$$S_i = \begin{bmatrix} \cos\omega_i & \sin\omega_i \\ -\sin\omega_i & \cos\omega_i \end{bmatrix},\quad \omega_i > 0,\ \ i=1,2,\cdots,k.$$

We will consider the class of dynamic state feedback control laws of the form

$$\begin{aligned} u(t) &= k(x(t), z(t)) \\ z(t+1) &= g(x(t), z(t), e(t)) \end{aligned} \quad (4)$$

where $z(t)$ is the compensator state vector of dimension n_c to be specified later. The dynamic output feedback controller can be viewed as a special case of (4) when $x(t)$ does not explicitly appear in (4). Letting $x_c = \text{col}(x,z)$, the resulting closed-loop system can be written as

$$\begin{aligned} x_c(t+1) &= f_c(x_c(t), v(t), w),\ x_c(0) = x_{c0} \\ e(t) &= h_c(x_c(t), v(t), w) \end{aligned} \quad (5)$$

where

$$\begin{aligned} f_c(x_c, v, w) &= \begin{bmatrix} f(x, k(x,z), v, w) \\ g(x, z, h(x, k(x,z), v, w)) \end{bmatrix} \\ h_c(x_c, v, w) &= h(x, k(x,z), v, w) \end{aligned}$$

For simplicity, all the functions involved in this setup are assumed to be sufficiently smooth and defined globally on the appropriate Euclidean spaces, with the value zero at the respective origins. We also assume that 0 is the nominal value of the uncertain parameter w, and $f(0,0,0,w) = 0$, $h(0,0,0,w) = 0$ for all $w \in \Re^N$.

Discrete-time Robust Output Regulation Problem. Design a feedback control law of the form (4) such that,
R1: the matrix $\frac{\partial f_c(0,0,0)}{\partial x_c}$ is a Schur matrix, i.e., the modulus of all the eigenvalues of the matrix $\frac{\partial f_c(0,0,0)}{\partial x_c}$ are smaller than 1.
R2: the solution of the closed-loop system (5) is such that $\lim_{t\to\infty} e(t) = 0$. ∎

Remark 2.1. Using the center manifold theory for map (Carr, 1981), it can be shown that condition R1 implies that, for all sufficiently small x_{c0}, v_0 and w, the solution of the closed-loop system (5) is bounded for all $t = 0,1,2,\cdots$. It can also be seen that a special case of the above problem where the plant does not contain the uncertain parameter w was solved in Castillo and Gennaro (1991) and Huang and Lin (1993*b*). From Theorem 2.2 of Huang and Lin (1993*b*), it can be deduced that the robust output regulation problem defined above is solvable only if the following assumption holds.

A1: there exist sufficiently smooth functions $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ defined in $V \times W$ where V and W are some open neighborhoods of the origins of $\Re^q$ and $\Re^N$, respectively, such that $\mathbf{x}(0,0) = 0$, $\mathbf{u}(0,0) = 0$, and

$$\mathbf{x}(A_1 v, w) = f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \quad (6)$$

$$0 = h(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \quad (7)$$

Equations (6) and (7) are called discrete regulator equations for uncertain nonlinear systems. However, like the continuous-time case, solvability of the discrete regulator equations is not sufficient for the solvability of the robust output regulation problem. Additional assumption has to be imposed on the solution of the discrete regulator equations. This additional assumption is made to guarantee the existence of the steady state generator introduced in the next section.
∎

3. PROBLEM CONVERSION

In this section, we will establish a framework for tackling the robust output regulation problem for discrete-time nonlinear systems which is a discrete analog of what was obtained in Huang and Chen (2004). Similar to the continuous case, this framework will be established in two steps. First, introduce the concepts of discrete-time steady state generator and discrete-time internal model, and establish the existence conditions of the discrete-time steady state generator, and the internal model. The internal model and the given plant together defines an augmented system. Second, show that, in a suitable set of coordinate and input transformations, the solvability of the robust stabilization problem for the augmented system will lead to the solvability of the robust output regulation problem of the original plant. As a result, we can convert the robust output regulation problem of the given plant into a robust stabilization problem of the augmented system.

Discrete-time Steady State Generator. Let $g_o : \Re^{n+m} \to \Re^l$ be a mapping for some positive integer $m \le l \le n+m$. Under assumption A1, the discrete-time nonlinear system (1) and (2) is said to have a discrete-time steady state generator with output g_o if there exists a triple $\{\theta, \alpha, \beta\}$, where $\theta : \Re^{q+n} \to \Re^s$, $\alpha : \Re^s \to \Re^s$, and $\beta : \Re^s \to \Re^l$ for some integer s are sufficiently smooth functions vanishing at the origin, such that

$$\begin{aligned} \theta(v(t+1), w) &= \alpha(\theta(v(t), w)) \\ g_o(\mathbf{x}(v(t), w), \mathbf{u}(v(t), w)) &= \beta(\theta(v(t), w)) \end{aligned} \tag{8}$$

If, in addition, the linearization of the pair $(\beta(\theta), \alpha(\theta))$ at the origin is observable, then $\{\theta, \alpha, \beta\}$ is called a linearly observable discrete-time steady state generator with output g_o. ∎

Remark 3.1. (i) The above definition is clearly a discrete analog of the continuous steady state generator introduced in Huang and Chen (2004). However, unlike the continuous case where the steady state generator is a set of nonlinear differential equations, here the steady state generator is a set of nonlinear difference equations.

(ii) In the sequel, we will always assume that $g_o(x, u) = \mathrm{col}(x_1, \cdots, x_d, u)$ for some integer $0 \le d \le n$. In this case, the steady state generator is a dynamic system that can generate the partial (when $d < n$) or whole (when $d = n$) solution of the discrete regulator equations. It is known that the information provided by the solution of the discrete regulator equations is necessary for designing a feedback control law to solve the output regulation problem. But since the solution of the discrete regulator equations relies on the uncertain parameter, it cannot be directly used in the feedback control design. In contrast, the steady state generator is independent of v and w, the information provided by the steady state generator can be used for feedback design. ∎

As will be seen later, the concept of the steady state generator will lead to a general characterization of the internal model. At this stage, let us first establish the existence conditions for the discrete-time steady state generator which is parallel to the one given in Lemma 3.1 of Huang and Chen (2004) for continuous-time case. To this end, let us introduce a property of the polynomial functions. Let $\pi(v, w)$ be a polynomial in v with coefficients depending on w, we have the following equivalent conditions.

Lemma 3.1. Assume $\pi(v, w)$ is an m-dimensional analytic function of v. Along the trajectory of (2), the following are equivalent.

(i) There exist some set of r real numbers $a_1, a_2, \cdots, a_r$ such that

$$\begin{aligned} &\pi(v(t+r), w) - a_1\pi(v(t), w) - a_2\pi(v(t+1), w) \\ &- \cdots - a_r\pi(v(t+r-1), w) = 0 \end{aligned} \tag{9}$$

(ii) Let $\Omega = \{\omega \mid \omega = l_1\omega_1 + \cdots + l_k\omega_k \ge 0, \quad l_1, \cdots, l_k = 0, \pm 1, \pm 2, \cdots\}$. Then there exist $\hat{\omega}_0 = 0$, and $\hat{\omega}_1, \cdots, \hat{\omega}_{n_k} \in \Omega$ for some finite integer n_k such that

$$\pi(v(t), w) = \sum_{l=-n_k}^{n_k} C_l(w, v_0) e^{j\hat{\omega}_l t} \tag{10}$$

where $j = \sqrt{-1}$, and for $l = 0, \pm 1, \cdots, \pm n_k$, C_l are m-dimensional column vectors, for $l \ne 0$, $\hat{\omega}_l = -\hat{\omega}_{-l}$, and $C_l^* = C_{-l}$ where C_l^* is the conjugate complex of C_l.

(iii) There exist some integer s and functions $\psi_l(w, v_0) \in \Re^m$, $l = 1, \cdots, s$, such that

$$\pi(v(t), w) = \sum_{l=1}^{s} \psi_l(w, v_0) v^{[l]}(t) \tag{11}$$

where

$$\begin{aligned} v^{[l]} = [&v_1^l,\ v_1^{l-1}v_2,\ \cdots,\ v_1^{l-1}v_q, \\ &v_1^{l-2}v_2^2,\ v_1^{l-2}v_2v_3,\ \cdots,\ v_1^{l-2}v_2v_q,\ \cdots,\ v_q^l]. \end{aligned}$$

∎

Remark 3.2. By Lemma 3.1, if $\pi(v, w)$ is a polynomial function of v or a trigonometric polynomial function of t along the trajectory of (2), then there exist an integer r and real numbers $a_1, \cdots, a_r$ such that $\pi(v, w)$ satisfies a difference equation of the form (9). We will call the monic polynomial $P(\lambda) = \lambda^r - a_r\lambda^{r-1} - \cdots - a_2\lambda - a_1$ a zeroing polynomial of $\pi(v, w)$ if $\pi(v, w)$ satisfies (9). $P(\lambda)$ is called a minimal zeroing polynomial of $\pi(v, w)$ if $P(\lambda)$ is a zeroing polynomial of $\pi(v, w)$ of least degree. Let $\pi_i(v, w)$, $i = 1, \cdots, I$, for some positive integer I, be I polynomials in v. They are called pairwise coprime if their minimal zeroing polynomials $P_1(\lambda), \cdots, P_I(\lambda)$ are pairwise coprime. Without lost of generality, assume $C_l \ne 0$, $l = 0, 1, \cdots, n_k$ in (10). Then from (10), a minimal zeroing polynomial of $\pi(v, w)$ can be defined as $P(\lambda) = (\lambda - 1)\prod_{l=1}^{n_k}(\lambda - e^{j\hat{\omega}_l})(\lambda - e^{-j\hat{\omega}_l})$. Clearly, all the zeros of $P(\lambda)$ are simple and lie on the unit circle. ∎

We are now ready to give the sufficient conditions of the existence of a discrete-time steady state generator.

Lemma 3.2. Under assumption A1, and assume there exist I polynomials $\pi_i(v, w)$ in v with their minimal zeroing polynomials $P_i(\lambda) = \lambda^{r_i} - a_{ir_i}\lambda^{r_i - 1} - \cdots - a_{i2}\lambda - a_{i1}$, $i = 1, \cdots, I$, and a sufficiently smooth function $\Gamma : \Re^{r_1 + \cdots + r_I} \to \Re^{d+m}$ vanishing at the origin such that for all $v(t)$ of the exosystem and all $w \in \Re^N$,

$$
\begin{aligned}
&g_o(\mathbf{x}(v,w),\mathbf{u}(v,w)) \\
&=\Gamma(\pi_1(v,w),\cdots,\pi_1(A_1^{r_1-1}v,w),\cdots, \\
&\qquad \pi_I(v,w),\cdots,\pi_I(A_1^{r_I-1}v,w))
\end{aligned} \tag{12}
$$

then, the system (1) and (2) has a discrete-time steady state generator with output $g_o(x,u)$ as follows,

$$
\begin{aligned}
\theta(v,w) &= T\mathrm{col}(\theta_1(v,w),\cdots,\theta_I(v,w)) \\
\alpha(\theta) &= T\Phi T^{-1}\theta \\
\beta(\theta) &= \Gamma(T^{-1}\theta)
\end{aligned} \tag{13}
$$

where T is any nonsingular matrix of dimension $r_{\mathrm{sum}} = r_1 + r_2 + \cdots + r_I$, and for $i = 1, 2, \cdots, I$, $\theta_i = \mathrm{col}(\pi_i(v,w), \pi_i(A_1 v, w), \cdots, \pi_i(A_1^{r_i-1}v,w)$, and $\Phi = \mathrm{diag}(\Phi_1,\cdots,\Phi_I)$ where Φ_i are the companion matrix of $P_i(\lambda)$. Moreover, let $E = (E_1,\cdots,E_I)$ be the jacobian matrix of Γ at the origin of $\Re^{(d+m)\times r_{\mathrm{sum}}}$ where $E_i \in \Re^{(d+m)\times r_i}$. If $\pi_i(v,w)$, $i = 1,\cdots,I$, are pairwise coprime, and the pair $\{E_i,\Phi_i\}$ is observable for all $i = 1,\cdots,I$, then the generator (13) is linearly observable. ∎

Based on the discrete-time steady state generator, the discrete-time internal model is defined as follows.

Discrete-time Internal Model. Assume, for the nonlinear system (1) and (2), (θ,α,β) is a discrete-time steady state generator with output $g_o(x,u)$, the following system

$$
\eta(t+1) = \gamma(\eta(t), x(t), u(t)) \tag{14}
$$

is called a discrete-time internal model with output $g_o(x,u)$ if

$$
\gamma(\theta(v,w),\mathbf{x}(v,w),\mathbf{u}(v,w)) = \alpha(\theta(v,w)) \tag{15}
$$

∎

Attaching the internal model (14) to the given plant (1) yields an augmented system as follows

$$
\begin{aligned}
x(t+1) &= f(x(t),u(t),v(t),w) \\
\eta(t+1) &= \gamma(\eta(t),x(t),u(t)) \\
e(t) &= h(x(t),u(t),v(t),w)
\end{aligned} \tag{16}
$$

Performing coordinate and input transformation

$$
\begin{aligned}
\bar{\eta} &= \eta - \theta(v,w) \\
\bar{x}_i &= \begin{cases} x_i - \beta_i(\eta), & i = 1,\cdots,d \\ x_i - \mathbf{x}_i(v,w), & i = d+1,\cdots,n \end{cases} \\
\bar{u} &= u - \beta_u(\eta) = u - [\beta_{d+1}(\eta),\cdots,\beta_{d+m}(\eta)]^T
\end{aligned} \tag{17}
$$

on the augmented system (16) gives the following system

$$
\begin{aligned}
\bar{x}(t+1) &= \bar{f}(\bar{\eta}(t),\bar{x}(t),\bar{u}(t),v(t),w) \\
\bar{\eta}(t+1) &= \bar{\gamma}(\bar{\eta}(t),\bar{x}(t),\bar{u}(t),v(t),w) \\
e(t) &= \bar{h}(\bar{\eta}(t),\bar{x}(t),\bar{u}(t),v(t),w)
\end{aligned} \tag{18}
$$

Theorem 3.1. Under assumption A1, and suppose that the system (1) and (2) has a discrete-time steady state generator (θ,α,β) with output $g_o(x,u) = \mathrm{col}(x_1,\cdots,x_d,u)$ and a discrete-time internal model described by (14). Then, in the new coordinates and input (17), the augmented system (18) has the property that, for all trajectories $v(t)$ of the exosystem, and all $w \in \Re^N$,

$$
\begin{aligned}
\bar{f}(0,0,0,v(t),w) &= 0 \\
\bar{\gamma}(0,0,0,v(t),w) &= 0 \\
\bar{h}(0,0,0,v(t),w) &= 0
\end{aligned} \tag{19}
$$

∎

Remark 3.3. Theorem 3.1 states that, for all trajectories $v(t)$ of the exosystem, and any $w \in \Re^N$, the origin $(\bar{\eta},\bar{x}) = (0,0)$ is the equilibrium point of the unforced augmented system, and the error output equation is identically zero at $(\bar{\eta},\bar{x},\bar{u}) = (0,0,0)$. Thus, if a controller of the form

$$
\begin{aligned}
\bar{u}(t) &= k(\bar{x}_1(t),\cdots,\bar{x}_d(t),\xi(t),e(t)) \\
\xi(t+1) &= \zeta(\bar{x}_1(t),\cdots,\bar{x}_d(t),\xi(t),e(t))
\end{aligned} \tag{20}
$$

where $\xi \in \Re^Z$, $k(0,\cdots,0) = 0$, and $\zeta(0,\cdots,0) = 0$, robustly stabilizes the equilibrium point of the closed-loop system of (18) locally or globally for all $v(t) \subset V$ and $w \in W$, then the following controller

$$
\begin{aligned}
u(t) &= \beta_u(\eta(t)) + k\Big(x_1(t) - \beta_1(\eta(t)), \\
&\qquad \cdots, x_d(t) - \beta_d(\eta(t)), \xi(t), e(t)\Big) \\
\eta(t+1) &= \gamma(\eta(t),x(t),u(t)) \\
\xi(t+1) &= \zeta\Big(x_1(t) - \beta_1(\eta(t)), \\
&\qquad \cdots, x_d(t) - \beta_d(\eta(t)), \xi(t), e(t)\Big)
\end{aligned} \tag{21}
$$

solves the robust output regulation problem for the original system (1) and (2) locally or globally. Therefore, Theorem 3.1 has converted the discrete-time robust output regulation problem for the given plant into a stabilization problem of the augmented system (18). ∎

4. SOLVABILITY OF (LOCAL) ROBUST OUTPUT REGULATION PROBLEM

In this section, we will solve the local robust output regulation problem for the plant (1) by stabilizing the corresponding augmented system. For this purpose, let

$$
\begin{aligned}
A &= \frac{\partial f}{\partial x}(0,0,0,0), \quad B = \frac{\partial f}{\partial u}(0,0,0,0) \\
C &= \frac{\partial h}{\partial x}(0,0,0,0), \quad D = \frac{\partial h}{\partial u}(0,0,0,0)
\end{aligned}
$$

Theorem 4.1. Under assumption A1, and suppose that (A,B) is stabilizable, (A,C) is detectable. If the system (1) and (2) has a linearly observable discrete-time steady state generator (θ, α, β) of the form (13) with output $g_o(x,u) = u$, and for all λ such that $P_i(\lambda) = 0, i = 1, \cdots, I$,

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + m \tag{22}$$

Then the discrete-time robust output regulation problem is solvable by an output feedback control law. ∎

Proof: Since the system (1) and (2) has a linearly observable discrete-time steady state generator (θ, α, β) of the form (13) with output $g_o(x,u) = u$, letting (M, N) be some controllable pair where M is a Schur matrix with appropriate dimensions, and T a nonsingular matrix such that $T\Phi - MT = NE$ gives a discrete-time internal model of the form

$$\eta(t+1) = M\eta(t) + N(u(t) - \beta(\eta(t)) + ET^{-1}\eta(t)) \tag{23}$$

with output $g_o(x,u) = u$, where Φ and E are as defined in Lemma 3.2. Then there exists a coordinate and input transformation of the form (17) with $d = 0$,

$$\bar{\eta} = \eta - \theta(v,w), \bar{x} = x - \mathbf{x}(v,w), \bar{u} = u - \beta(\eta)$$

This transformation converts the augmented system (16) into the form of (18) with

$$\begin{aligned} &\bar{f}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \\ &= f(x,u,v,w) - f(\mathbf{x}, \mathbf{u}, v, w) \\ &= f(\bar{x} + \mathbf{x}, \bar{u} + \beta(\eta), v, w) - f(\mathbf{x}, \mathbf{u}, v, w) \\ &= f(\bar{x} + \mathbf{x}, \bar{u} + \beta(\bar{\eta} + \theta(v,w)), v, w) - f(\mathbf{x}, \mathbf{u}, v, w) \end{aligned}$$

$$\begin{aligned} &\bar{\gamma}(\bar{\eta}, \bar{x}, \bar{u}, v, w) \\ &= M\eta + N(u - \beta(\eta) + ET^{-1}\eta) - \alpha(\theta) \\ &= M(\bar{\eta} + \theta) + N(\bar{u} + ET^{-1}(\bar{\eta} + \theta)) - \alpha(\theta) \\ &= (M + NET^{-1})\bar{\eta} + N\bar{u} + M\theta + NET^{-1}\theta - \alpha(\theta) \\ &= (M + NET^{-1})\bar{\eta} + N\bar{u} \end{aligned}$$

$$\begin{aligned} \bar{h}(\bar{\eta}, \bar{x}, \bar{u}, v, w) &= h(x,u,v,w) \\ &= h(\bar{x} + \mathbf{x}, \bar{u} + \beta(\bar{\eta} + \theta(v,w)), v, w) \end{aligned}$$

By Remark 3.3, it suffices to (locally) stabilize the equilibrium point at the origin of (18) with $v = 0$ and $w = 0$. To this end, linearizing (18) at the origin $(\bar{x} = 0, \bar{\eta} = 0, \bar{u} = 0)$ with v and w being set to zero gives

$$\begin{aligned} \bar{x}(t+1) &= A\bar{x}(t) + BET^{-1}\bar{\eta}(t) + B\bar{u}(t) \\ \bar{\eta}(t+1) &= (M + NET^{-1})\bar{\eta}(t) + N\bar{u}(t) \\ e(t) &= C\bar{x}(t) + DET^{-1}\bar{\eta}(t) + D\bar{u}(t) \end{aligned} \tag{24}$$

It can be shown that (24) is stabilizable and detectable. In fact, noting that (A, B) is stabilizable and M is a Schur Matrix, it is easy to conclude that (24) is stabilizable by the decomposition

$$\begin{aligned} &\begin{bmatrix} A - \lambda I & BET^{-1} & B \\ 0 & M + NET^{-1} - \lambda I & N \end{bmatrix} \\ &= \begin{bmatrix} A - \lambda I & 0 & B \\ 0 & M - \lambda I & N \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & ET^{-1} & I \end{bmatrix} \end{aligned}$$

To show that (24) is detectable, first note that (A, C) is detectable and $M + NET^{-1} = T\Phi T^{-1}$, the following matrix

$$\begin{bmatrix} A - \lambda I & BET^{-1} \\ 0 & M + NET^{-1} - \lambda I \\ C & DET^{-1} \end{bmatrix} \tag{25}$$

has full rank for all $\lambda \notin \sigma(\Phi)$ and $|\lambda| \geq 1$. For the case $\lambda \in \sigma(\Phi)$, by Remark 3.2, λ is on the unit circle. Thus, the following decomposition

$$\begin{aligned} &\begin{bmatrix} A - \lambda I & BET^{-1} \\ 0 & M + NET^{-1} - \lambda I \\ C & DET^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A - \lambda I & 0 & B \\ 0 & M - \lambda I & N \\ C & 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & ET^{-1} \end{bmatrix} \end{aligned}$$

shows that the matrix (25) also has full rank for $\lambda \in \sigma(\Phi)$ and $|\lambda| \geq 1$ by the condition (22) and M is a Schur matrix. Thus, (25) has full rank for all $|\lambda| \geq 1$, i.e., (24) is detectable.

As a result, system (24) can be stabilized by a dynamic linear output control law of the form

$$\begin{aligned} \bar{u}(t) &= -K\xi(t) \\ \xi(t+1) &= Q\xi(t) + Le(t) \end{aligned}$$

Finally, the following controller

$$\begin{aligned} u(t) &= -K\xi(t) + \beta(\eta(t)) \\ \eta(t+1) &= M\eta(t) + N(u(t) - \beta(\eta(t)) + ET^{-1}\eta(t)) \\ \xi(t+1) &= Q\xi(t) + Le(t) \end{aligned}$$

solves the discrete-time robust output regulation problem for the original system (1) and (2). ∎

5. CONCLUSIONS

In this paper, we have established a framework that can convert the robust output regulation problem for a class of discrete-time uncertain nonlinear plant into a robust stabilization problem for an augmented system. Due to this framework, similar to the continuous case, the discrete-time robust output regulation problem can always be tackled in two steps. The first step is to form the augmented system which consists of the given plant and the internal model. Whether or not the first step can be accomplished depends on the existence of the internal model, or what is the same, the steady state generator. We have shown that the steady state generator hence the internal model always exists if the solution of the discrete regulator equations satisfies conditions given in Lemma 3.2. The second step is to robustly stabilize the augmented system either locally or globally, which will in turn lead to the solution of the local or global robust output regulation problem.

We have given the solvability conditions of the local robust output regulation problem for a general class of nonlinear systems.

Technically, the local version of the discrete-time robust output regulation problem does not pose more specific difficulty than its continuous counterpart because the same eigenvalue placement technique works for both the discrete-time and continuous-time systems. However, the global version of the discrete-time robust output regulation problem seems to be more challenging than its continuous counterpart because the global robust stabilization problem for discrete-time nonlinear systems is more difficult than that for continuous-time systems. A simple example may be motivating. Consider a scalar continuous-time linear system $\dot{x} = ax + u$ where the uncertain parameter a belongs to any fixed compact set. Clearly, a high gain state feedback control $u = -Kx$ with K sufficiently large robustly stabilizes the system. However, for a scalar discrete-time linear system $x(t+1) = ax(t) + u(t)$, there exists no linear state feedback controller that can robustly stabilizes the system when the uncertain parameter a belongs to an interval whose length is greater than 2.

Therefore, in order to handle the global robust output regulation problem for discrete-time nonlinear systems, more research on the robust stabilization of the discrete-time nonlinear systems has to be carried out first. This is indeed one of our current research topics.

REFERENCES

Byrnes, C. I., F. Delli Priscoli, A. Isidori and W. Kang (1997). Structurally stable output regulation of nonlinear systems. *Automatica* **33**, 369–385.

Carr, J. (1981). *Applications of the Center Manifold Theory*. Springer-Verlag. New York.

Castillo, B. and S. D. Gennaro (1991). Asymptotic output tracking for siso nonlinear discrete time systems. In: *Proc. the 30th IEEE Conf. on Decision and Control*. pp. 1802–1806.

Castillo, B., S. D. Gannaro, S. Monaco and D. Normand-Cyrot (1993). Nonlinear regulation for a class of discrete-time systems. *Systems & Control Letters* **20**, 57–66.

Davison, E. J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Trans. Automat. Contr.* **21**, 25–34.

Francis, B. A. (1977). The linear multivariable regulation problem. *SIAM J. Contr. and Optimiz.* **15**, 486–505.

Francis, B. A. and W. M. Wonham (1976). The internal model principle of control theory. *Automatica* **12**, 457–465.

Huang, J. (2001). Remarks on the robust output regulation problem for nonlinear systems. *IEEE Trans. Automat. Contr.* **46**(12), 2028–2031.

Huang, J. and C. F. Lin (1993*a*). Internal model principle and robust control of nonlinear systems. In: *Proc. 32nd IEEE Conf. on Decision and Control*. pp. 1501–1506.

Huang, J. and C. F. Lin (1993*b*). On the discrete-time nonlinear servomechanism problem. In: *Proc. Amer. Contr. Conf.*. San Francisco, CA. pp. 844–848.

Huang, J. and C. F. Lin (1994*a*). On a robust nonlinear servomechanism problem. *IEEE Trans. Automat. Contr.* **39**, 1510–1513.

Huang, J. and C. F. Lin (1994*b*). A stability property and its application to discrete-time nonlinear system control. *IEEE Trans. Automat. Contr.* **39**(11), 2307–2311.

Huang, J. and W. J. Rugh (1990). On a nonlinear multivariable servomechanism problem. *Automatica* **26**, 963–972.

Huang, J. and W. J. Rugh (1992). Stabilization on zero-error manifolds and the nonlinear servomechanism problem. *IEEE Trans. on Automat. Contr.* **37**(7), 1009–1013.

Huang, J. and Z. Chen (2004). A general framework for tackling the output regulation problem. *IEEE Trans. Automat. Contr.* **49**, 2203–2218.

Isidori, A. and C. I. Byrnes (1990). Output regulation of nonlinear systems. *IEEE Trans. Automat. Contr.* **35**(2), 131–140.

Khalil, H. (1994). Robust servomechanism output feedback controllers for feedback linearizable systems. *Automatica* **30**, 1587–1589.

Khalil, H. (2000). On the robust servomechanisms for minimum phase nonlinear systems. *Int. J. of Robust and Nonlinear Control* **10**, 339–361.

Lan, W. and J. Huang (2005). Robust output regulation for discrete-time nonlinear systems. *International Journal of Robust and Nonlinear Control* **15**, 63–81.