A FLYWHEEL TO STABILIZE A TWO-LINK PENDULUM

Yannick Aoustin * **Alexander Formal'sky** ** **Yuri Martynenko** **

* *IRCCyN, Unité mixte CNRS 6957 - Université de Nantes Ecole Centrale de Nantes - Ecole Supérieure des Techniques Industrielles et des Mines de Nantes BP 92101, 1 rue de la Noë, 44321 Nantes cedex 03, France*
{Yannick.Aoustin}@irccyn.ec-nantes.fr
** *Moscow Lomonosov State University, 1, Michurinski Prospect, Moscow 119192 Russia.*
{formal,martynenko}@imec.msu.ru

Abstract: Control laws are designed to stabilize three unstable equilibriums of a planar double link pendulum. A flywheel, actuated by an electrical drive equips the first link. The control signal - voltage is applied to this drive. The limits imposed on the voltage are explicitly taken into account. Our under actuated system is controllable. The number of unstable modes, one or two, depends on the considered unstable state. The control laws are designed to suppress the unstable modes using the linear models of the motion near each unstable state. The numerical investigations of the nonlinear model with the designed nonlinear control laws are presented *Copyright© 2005 IFAC.*

Keywords: inverted pendulum, flywheel, stabilization, limited control.

1. INTRODUCTION

Many mechanical systems have fewer independent control actuators than degrees of freedom. The stabilization of under actuated processes is difficult and there has been major interest in developing stabilizing algorithms which use linear and non linear control laws, energy based controllers, *etc.* For example inverted pendulums are considered in (Akensson, 2000; Aström and Furuta, 2000; Beznos *et al.*, 2003; Chung and Hauser, 1995; Fantoni *et al.*, 2001; Grishin *et al.*, 2002). The challenge is to deal with an unstable equilibrium point. The control law has to "suppress" the instability. Another difficulty with actuated mechanical systems is to take into account the limit on the control, as for example in (Aoustin and Formal'sky, 2005; Aström and Brufani, 1997; Beznos *et al.*, 2003; Gilbert *et al.*, 1994; Grishin *et al.*, 2002; Martynenko *et al.*, 2004). In (Beznos *et al.*, 2003; Fantoni *et al.*, 2001; Grishin *et al.*, 2002), one-link inverted pendulums with a flywheel are considered. It is shown that one inverted pendulum can be stabilized, using a flywheel with an energy-based control and linear feedback designed with the Jordan form of the motion equations. These results are expanded by experiments in (Beznos *et al.*, 2003; Grishin *et al.*, 2002).

This paper is devoted to the stabilization of unstable equilibriums of a two-link pendulum with a flywheel. The control law design is based on a pole placement method and/or on the Jordan form of the motion equations. The limits of the input voltage are explicitly taken into account.

The organization of the paper is the following. Section 2 is devoted to the model of the two-link pendulum with a flywheel. The linear model of the double-link mechanism motion around each unstable equilibrium is presented in Section 3. The statement of the problem is defined in Section 4. Section 5 considers the stabilization problem of two equilibriums with only one link inverted. Section 6 presents the stabilization problem of the third equilibrium with the two inverted links. Section 7 presents our conclusion and perspectives.

2. MODEL DESCRIPTION OF THE TWO-LINK PENDULUM WITH FLYWHEEL

2.1 Motion equations

We consider a two-link planar mechanism with a flywheel (see its diagram in Figure 1). The stator of the electrical actuator is connected to link 1, the output shaft is connected to the flywheel. Thus, the actuator rotates the flywheel with respect to link 1. Let m_i $(i = 1, 3)$ be the masses of the links.

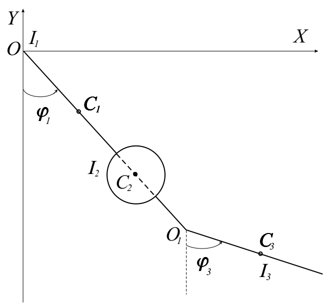

Fig. 1. Diagram of the two-link pendulum with a flywheel.

The inertia moment of link 1 around point O is denoted I_1. Let $OO_1 = b$ and $OC_1 = a_1$ be the length of link 1 and the distance between joint O and the center of mass C_1 respectively. The mass and the inertia moment around the center of mass C_2 of the flywheel are denoted m_2 and I_2 respectively. The distance between joint O and the center of mass C_2 of the flywheel is equal to $OC_2 = a_2$. The inertia moment of the second link around its center of mass C_3 is denoted I_3. The distance between joint O_1 and the center of mass C_3 of link 2 is $O_1C_3 = a_3$. The variable φ_1 (φ_3) is the angle between the vertical axis Y and the first (second) link, φ_2 is the absolute flywheel rotation angle. The generalized coordinates φ_1, φ_2, φ_3 and the three associated velocities $\dot{\varphi}_1$, $\dot{\varphi}_2$, $\dot{\varphi}_3$ characterize the behavior of our system. The generalized force is the torque Γ due to the electrical actuator of the flywheel. This torque is directly proportional to the electrical current in the armature winding. By neglecting the armature inductance, this torque can be written in the form (Gorinevsky *et al.*, 1997):

$$\Gamma = c_u U \; - \; c_v(\dot{\varphi}_2 - \dot{\varphi}_1) \tag{1}$$

where c_u, c_v are positive constants, U is the voltage supplied to the motor

$$|U| \leq U_0, \;\; U_0 \; = \; const \tag{2}$$

The difference $\dot{\varphi}_2 - \dot{\varphi}_1$ is the velocity of the flywheel with respect to link 1.
The expressions for the kinetic energy E_c and potential energy Π of the mechanism are the following (g is the gravity acceleration):

$$\begin{array}{c} 2E_c = a_{11}\dot{\varphi}_1^2 + a_{22}\dot{\varphi}_2^2 + a_{33}\dot{\varphi}_3^2 + ... \\ 2a_{13}\dot{\varphi}_1\dot{\varphi}_3 cos(\varphi_1 - \varphi_3) \end{array} \tag{3}$$

$$\Pi = -c_1 cos\varphi_1 - c_3 cos\varphi_3 \tag{4}$$

where $a_{11} = I_1 + m_2a_2^2 + m_3b^2$, $a_{22} = I_2$, $a_{13} = m_3a_3b$, $a_{33} = I_3 + m_3a_3^2$, $c_1 = (m_1a_1 + m_2a_2 + m_3b)g$, $c_3 = m_3a_3g$ and $u_0 = c_u/c_1U_0$.
The motion equations can be derived using Lagrange's method and expressions (1), (3), (4):

$$\begin{array}{c} a_{11}\ddot{\varphi}_1 \; + \; a_{13}cos(\varphi_1 - \varphi_3)\ddot{\varphi}_3 \; + \; ... \\ a_{13}sin(\varphi_1 - \varphi_3)\dot{\varphi}_3^2 \; + \; c_1 sin\varphi_1 \\ = \; -c_uU + c_v(\dot{\varphi}_2 - \dot{\varphi}_1) \end{array} \tag{5}$$

$$a_{22}\ddot{\varphi}_2 \; = \; c_uU - c_v(\dot{\varphi}_2 - \dot{\varphi}_1) \tag{6}$$

$$\begin{array}{c} a_{13}cos(\varphi_1 - \varphi_3)\ddot{\varphi}_1 + a_{33}\ddot{\varphi}_3 \; - \; ... \\ a_{13}sin(\varphi_1 - \varphi_3)\dot{\varphi}_1^2 \; + \; c_3 sin\varphi_3 = 0 \end{array} \tag{7}$$

The motion equations (5) - (7) do not depend on angle φ_2. This cyclic variable is not important for the problem of the pendulum stabilization. System (5) - (7) has three unstable equilibriums:

$$\varphi_1 = 0, \quad \varphi_3 = \pi \tag{8}$$

$$\varphi_1 = \pi, \quad \varphi_3 = 0 \tag{9}$$

$$\varphi_1 = \pi, \quad \varphi_3 = \pi \tag{10}$$

The determinant of the controllability matrix (Kalman *et al.*, 1969) of the model (5) - (7), linearized around each instable equilibrium is:

$$\frac{a_{13}^2c_1^2c_3^4c_4^6}{a_{22}(a_{11}a_{33} - a_{13}^2)^6}.$$

Then the linear model is controllable near each unstable equilibrium, if $a_3 \neq 0$, $b \neq 0$, $c_u \neq 0$ and $m_1a_1 + m_2a_2 + m_3b \neq 0$.

2.2 Dimensionless motion equations

Let us introduce the dimensionless variables τ, α_{22} α_{13}, α_{33}, β, c and u:

$$\tau = t/T \quad (T = \sqrt{a_{11}/c_1}), \quad \alpha_{22} = a_{22}/a_{11},$$
$$\alpha_{13} = a_{13}/a_{11}, \quad \alpha_{33} = a_{33}/a_{11}, \quad c = c_3/c_1,$$
$$\beta = c_v/\sqrt{a_{11}c_1}, \quad u = c_u U/c_1, \quad |u| \leq u_0,$$

Substituting the dimensionless quantities in (5)-(7), we rewrite these non linear equations in a simpler form with only five parameters ($' \equiv d/d\tau$):

$$\begin{aligned} &\varphi_1^{''} + \alpha_{13}cos(\varphi_1 - \varphi_3)\varphi_3^{''} + sin\varphi_1 + ... \\ &\alpha_{13}sin(\varphi_1 - \varphi_3)\varphi_3^{'2} = -u + \beta(\varphi_2^{'} - \varphi_1^{'}) \end{aligned} \tag{11}$$

$$\alpha_{22}\varphi_2^{''} = u - \beta(\varphi_2^{'} - \varphi_1^{'}) \tag{12}$$

$$\begin{aligned} &\alpha_{13}cos(\varphi_1 - \varphi_3)\varphi_1^{''} + \alpha_{33}\varphi_3^{''} - ... \\ &\alpha_{13}sin(\varphi_1 - \varphi_3)\varphi_1^{'2} + csin\varphi_3 = 0 \end{aligned} \tag{13}$$

3. LINEAR MODEL OF THE PENDULUM WITH FLYWHEEL

Let us linearize (11)-(13) near equlibriums (8)-(10). If x is (5x1) state vector, then linear state model has the form

$$x' = Ax + Bu \tag{14}$$

with

$$A = \begin{bmatrix} 0_{2\times2} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ D^{-1}E & D^{-1}\begin{pmatrix} -\beta & \beta & 0 \\ \beta & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{bmatrix},$$

$$B = \begin{bmatrix} 0_{2\times1} \\ D^{-1}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \end{bmatrix}$$

The matrices D, E and vector x are

(1) for $\varphi_1 = 0$, $\varphi_3 = \pi$:
$$D = \begin{pmatrix} 1 & 0 & -\alpha_{13} \\ 0 & \alpha_{22} & 0 \\ -\alpha_{13} & 0 & \alpha_{33} \end{pmatrix}, E = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & c \end{pmatrix},$$
$$x = (\varphi_1, (\varphi_3 - \pi), \dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3)^T$$

(2) for $\varphi_1 = \pi$, $\varphi_3 = 0$:
$$D = \begin{pmatrix} 1 & 0 & -\alpha_{13} \\ 0 & \alpha_{22} & 0 \\ -\alpha_{13} & 0 & \alpha_{33} \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -c \end{pmatrix},$$
$$x = ((\varphi_1 - \pi), \varphi_3, \dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3)^T$$

(3) for $\varphi_1 = \pi$, $\varphi_3 = \pi$:
$$D = \begin{pmatrix} 1 & 0 & \alpha_{13} \\ 0 & \alpha_{22} & 0 \\ \alpha_{13} & 0 & \alpha_{33} \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & c \end{pmatrix},$$
$$x = ((\varphi_1 - \pi), (\varphi_3 - \pi), \dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3)^T$$

Introducing a nondegenerate linear transformation $x = Sy$, with a constant matrix S, we obtain the well-known Jordan form of equation (14):

$$y' = \Lambda y + du \tag{15}$$

where, with the eigenvalues of A, λ_i $(i = 1, ..., 5)$

$$\Lambda = S^{-1}AS = diag(\lambda_i), \quad d = S^{-1}B = [d_i]^T. \tag{16}$$

Let the eigenvalues with positive real part have the smaller subscripts. For the equilibriums (8) and (9), we will obtain $\lambda_1 > 0$, $Re\lambda_i < 0$ $(i = 2-5)$. For the equilibrium (10) $\lambda_1 > 0$, $\lambda_2 > 0$, $Re\lambda_i < 0$ $(i = 3, 4, 5)$.

4. PROBLEM STATEMENT

Let $x = x_e = 0$ be the desired equilibrium state of (14). Let us design the feedback control to stabilize this equilibrium $x_e = 0$, under constraints (2). Then, the objective is to design an admissible (satisfying inequality (2)) feedback control $u = u(x)$ $(|u(x)| \leq u_0)$ to ensure the asymptotic stability of the desired state $x_e = 0$. Let W be the set of piecewise continuous functions of time $u(t)$, satisfying inequalities (2). Let Q be the set of the initial states $x(0)$, from which the origin $x_e = 0$ can be reached, using an admissible control functions of time $u(t) \in W$. In other words, system (14) can reach the origin $x_e = 0$ with the control $u(t) \in W$, only when starting from the initial states $x(0) \in Q$. The set Q is called controllability region. If A has eigenvalues with positive real part and the control variable u is limited, then Q is an open subset of the phase space X for (14) (Formal'sky, 1974). Let us consider system (15), (16). If, for example, only one eigenvalue λ_1 is positive, then we can extract from system (15), (16) first scalar equation corresponding to one unstable mode

$$y_1' = \lambda_1 y_1 + d_1 u \tag{17}$$

Here, the controllability region Q is described by

$$|y_1| < |d_1| \, u_0/\lambda_1 \tag{18}$$

When two eigenvalues have positive real part, the origin $x_e = 0$ can be reached, only if both initial values $y_1(0)$ and $y_2(0)$ are bounded (see Subsection 6.1). For any admissible feedback control $u = u(x)$ $(|u(x)| \leq u_0)$ the corresponding region of attraction $V \subset Q$. Here, as usual, V is the set of initial states $x(0)$, from which system (14), with the feedback $u = u(x)$ asymptotically tends to the origin point $x_e = 0$ as $t \to \infty$.

Some eigenvalues of matrix A are located in the left half of the complex plane. The other eigenvalues of A are in the right half. We will design a control law which "transfers" the latter eigenvalues to the left half.

The structure and the properties of this control law depend on the unstable equilibrium that we stabilize. These different cases will be now detailed.

5. STABILIZATION OF THE EQUILIBRIUMS WITH ONE LINK INVERTED

5.1 Control design for $\varphi_1 = 0$, $\varphi_3 = \pi$ (link 2 inverted) and $\varphi_1 = \pi$, $\varphi_3 = 0$ (link 1 inverted)

Matrix A for both equilibriums has one real positive eigenvalue $\lambda_1 > 0$. In this case, our system contains only one unstable mode. Equation (17) corresponds to this mode. This unstable mode can be suppressed using control

$$u = \gamma y_1 \tag{19}$$

with $\lambda_1 + d_1\gamma < 0$ (Grishin *et al.*, 2002). For (14) in closed loop with feedback (19), only pole λ_1 is replaced with a negative pole $\lambda_1 + d_1\gamma$. Poles $\lambda_{2,3,4,5}$ do not change. Under feedback control (19) with saturation (2)

$$u(x) = \begin{cases} u_0,\ if\ \gamma y_1 \geq u_0 \\ \gamma y_1,\ if\ |\gamma y_1| \leq u_0 \\ -u_0,\ if\ \gamma y_1 \leq -u_0 \end{cases} \tag{20}$$

we obtain the largest possible region of attraction V for (14) (or (15)), (2) (Grishin *et al.*, 2002). The region of attraction under control (20) is $V = Q$ and consequently it is described by inequality (18).

Note that variable y depends on the original variable x because $x = Sy$. Due to this, (20) defines a control feedback, which depends on the original variables x_i $(i = 1 - 5)$.

We can "suppress" the instability due to $\lambda_1 > 0$ also by using a pole placement design with a linear feedback control,

$$u = -Kx \tag{21}$$

To find the gains of feedback control (21), we use Ackermann's formula:

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-\infty} \alpha_c(A)$$

where, $\alpha_c(A)$ represents the matrix polynomial formed with the coefficients of the desired characteristic polynomial $\alpha_c(s)$. For (14) in closed loop, any poles can be prescribed, because this system is Kalman controllable.

If (2) is taken into account, we obtain from (21) a linear feedback control with saturation,

$$u(x) = \begin{cases} u_0,\ if\ -Kx \geq u_0 \\ -Kx,\ if\ |Kx| \leq u_0 \\ -u_0,\ if\ -Kx \leq -u_0 \end{cases} \tag{22}$$

According to Lyapounov's theorem (Slotine and Li, 1991) the equilibrium point $x = 0$ of the nonlinear system is asymptotically stable under control (22) with some region of attraction.

5.2 Numerical results with link 2 inverted

The following numerical values of the parameters of our mechanism are considered:
$m_1 = 0.04\,kg$, $m_2 = 0.313\,kg$, $m_3 = 0.03\,kg$,
$a_1 = 0.15\,m$, $a_2 = b = 0.3\,m$, $a_3 = 0.12\,m$,
$I_1 = 12.10^{-4}\,kg{\cdot}m^2$, $I_2 = 803.10^{-7}\,kg{\cdot}m^2$,
$I_3 = 144.10^{-6}\,kg{\cdot}m^2$, $c_u = 0.0069\,N{\cdot}m/V$,
$c_v = 0.000099\,N{\cdot}m/s$, $U_0 = 19\,V$.
In open-loop the poles (eigenvalues) of linear system (14) are
$\lambda_1 = 1.3858$, $\lambda_{2,3} = -0.0003 \pm 1.0115i$,
$\lambda_4 = -0.2136$, $\lambda_5 = -1.3858$.
Our system contains only one unstable mode. Using control (20), we obtain the largest region of attraction (18). With inequality (18), the maximum initial tilt of link 1, which can be handled for linear model (14) under control (20) is $\varphi_1(0) = 12.38°$ ($\varphi_3(0) = \varphi_1(0) + \pi = 192.38°, \dot{\varphi}_{1,2,3}(0) = 0$).
However, eigenvalues $\lambda_{2,3}$ are too close to the imaginary axis and, with control (20), the transient process is too long. We hereafter prescribe the eigenvalues to avoid a long transient process.
Linear feedback control law (21) is designed to obtain in closed loop the following poles: $\lambda_{1,\ldots,5} = -1.0$.
In simulation, control law (21) with the prescribed poles and under saturation, i.e. control law (22) is applied to the nonlinear model (11)-(13). Considering the initial position of link 1, the maximum tilt which can be handled is $\varphi_1(0) = 10°$ ($\varphi_3(0)) = \varphi_1(0) + \pi = 190°$). The graphs of variables $\varphi_{1,2,3}(t)$ as functions of time are shown in Figure 2. At the end, the pendulum is steered

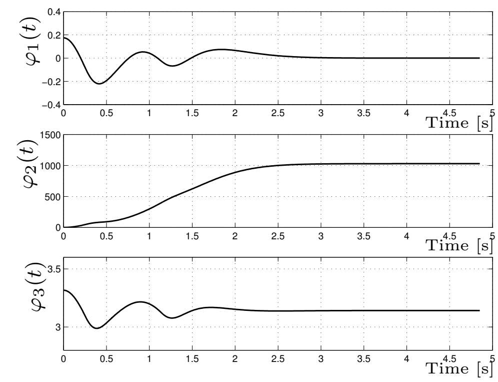

Fig. 2. Stabilization of Equilibrium $\varphi_1 = 0$, $\varphi_3 = \pi$: Angles $\varphi_{1,2,3}(t)$ (in radians).

to the desired equilibrium. When the stabilization of the pendulum is reached, the rotation of the flywheel is stopped. The maximum control voltage is applied at initial time, as shown in Figure 3.

5.3 Numerical results with link 1 inverted

In open-loop the poles of linear system (14) are
$\lambda_1 = 1.0113$, $\lambda_{2,3} = -7.0\,10^{-6} \pm 1.3858i$,

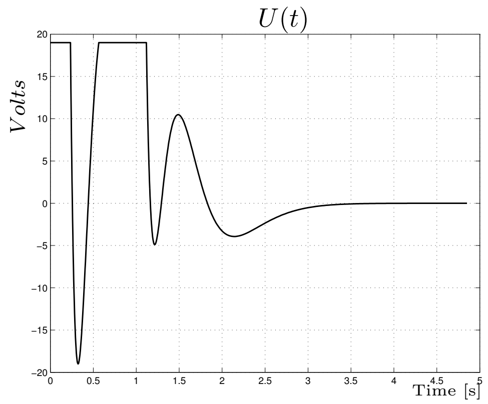

Fig. 3. Stabilization of the Equilibrium $\varphi_1 = 0$, $\varphi_3 = \pi$: Voltage supplied to the motor.

$\lambda_4 = -0.2136, \quad \lambda_5 = -1.0119.$
In this case, our system contains also one unstable mode only. This mode can be suppressed using feedback control (20) with saturation. Under this control we obtain for system (14), (2) the largest region of attraction (18). Using inequality (18), the maximum initial tilt of link 1, which can be handled for linear model (14) is $\varphi_1(0) = 186.1°$ ($\varphi_3(0) = \varphi_1(0) - \pi = 6.1°, \dot{\varphi}_{1,2,3}(0) = 0$). However, poles $\lambda_{2,3}$ are too close to the imaginary axis and we prescribe new poles to avoid a long transient process. To obtain large basin of attraction for nonlinear system we have tested different poles. We have obtained a "good" result with the following poles in closed loop:
$\lambda_1 = -1.0113, \quad \lambda_{2,3} = -1.04 \pm 1.092i,$
$\lambda_4 = -1.0, \quad \lambda_5 = -1.0113.$
In simulation, the corresponding control law (22), is applied to nonlinear model (11)-(13). Considering the initial position of link 1, the maximum tilt which can be handled is $\varphi_1(0) = 185.9°$ ($\varphi_3(0) = \varphi_1(0) - \pi = 5.9°$). Considering the graphs of variables $\varphi_{1,2,3}(t)$, we can see that at the end of process, the pendulum is steered to the equilibrium and the flywheel is stopped. The maximum voltage is supplied to the motor at initial time.

6. STABILIZATION OF THE EQUILIBRIUM WITH BOTH LINKS INVERTED

6.1 Control design for $\varphi_1 = \pi$, $\varphi_3 = \pi$

Matrix A has two real positive eigenvalues λ_1, λ_2 and three eigenvalues in the left half of the complex plane. Let us consider the first two scalar differential equations of the system (15), (16) corresponding to eigenvalues λ_1 and λ_2:

$$y_1' = \lambda_1 y_1 + d_1 u, \quad y_2' = \lambda_2 y_2 + d_2 u \tag{23}$$

System (14) is Kalman controllable. Then, (23) is controllable too (Kalman *et al.*, 1969). The controllability region Q of (23), and consequently of (15), is an open bounded set with the following boundaries (Boltyanskii, 1969)

$$y_i(t) = \pm \frac{d_i u_0}{\lambda_i}\left(2e^{-\lambda_i \tau} - 1\right), \quad (0 \le \tau < \infty) \quad \text{i=1,2} \tag{24}$$

We can "suppress" the instability of coordinates y_1 and y_2 by a linear feedback control,

$$u = k_1 y_1 + k_2 y_2 \tag{25}$$

With,

$$k_1 = -\frac{(\lambda_1 + \mu_0)^2}{d_1(\lambda_1 - \lambda_2)}, \quad k_2 = \frac{(\lambda_2 + \mu_0)^2}{d_2(\lambda_1 - \lambda_2)}, \tag{26}$$

the characteristic equation of system (23), (25) is:

$$(\mu + \mu_0)^2 = 0 \tag{27}$$

If we take into account constraints (2), we obtain a linear feedback control with saturation,

$$u(x) = \begin{cases} u_0, \; if \; k_1 y_1 + k_2 y_2 \ge u_0 \\ k_1 y_1 + k_2 y_2, \; if \; |k_1 y_1 + k_2 y_2| \le u_0 \\ -u_0, \; if \; k_1 y_1 + k_2 y_2 \le -u_0 \end{cases} \tag{28}$$

With $\mu_0 > 0$, solution $(y_1(t),\ y_2(t))$ of system (23), (28) tends to 0 as $t \to \infty$ for initial $y_1(0)$, $y_2(0)$ belonging to the basin of attraction of this system. But if $y_1(t) \to 0$ and $y_2(t) \to 0$, then according to expression (28) $u(t) \to 0$. Therefore, the solutions $y_i(t)$ $(i = 3, 4, 5)$ of the third, fourth and fifth equations of system (15) with any initial conditions $y_i(0)$ $(i = 3, 4, 5)$ converge to zero as $t \to \infty$ because $Re\lambda_i < 0$ for $i = 3, 4, 5$. Thus, under control (28) and with inequality $\mu_0 > 0$, the domain of attraction of system (15), (28) is described by the same relations, which describe the domain of attraction of (23), (28).

6.2 Numerical results

In open-loop the poles of linear system (14) are
$\lambda_1 = 1.4471$, $\lambda_2 = 0.9684$, $\lambda_3 = -0.2136$,
$\lambda_4 = -0.9689$, $\lambda_5 = -1.4472$.

Using (24), we have designed the controllability region Q for (23) that is shown in Figure 4. The region of attraction V for (23) under control (28) is also shown for $\mu_0 = 0.7$. The periodical motion (cycle) is the boundary of V. This cycle is computed using the backward motion of system (23), (28) from a state close to the origin. We have computed the domains of attraction for different values μ_0. For $\mu_0 = 0.7$, V is close to Q.
In simulation, control law (28) with $\mu_0 = 0.7$ is applied to nonlinear model (11)-(13). The initial position with maximum tilt of the straight

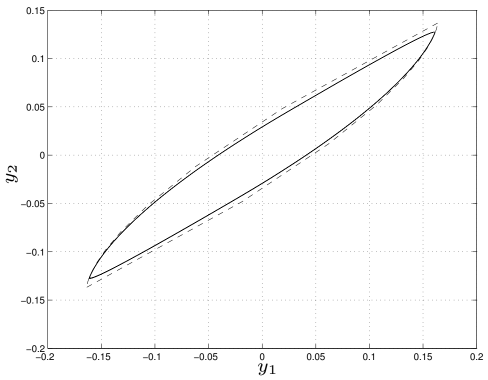

Fig. 4. Controllability region Q (dashed line) for system (23) and region of attraction V (solid line) with $\mu_0 = 0.7$.

($\varphi_1(0) = \varphi_3(0)$) double-link pendulum, which can be handled is $\varphi_{1,3}(0) = 181.757°$; the initial angular velocities are equal to zero. The graphs of variables $\varphi_{1,2,3}(t)$ are shown in Figure 5. At the

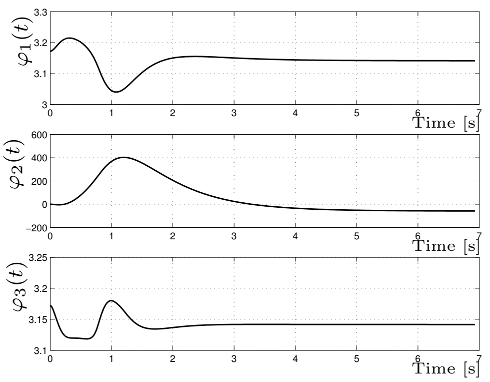

Fig. 5. Stabilization of Equilibrium $\varphi_1 = \pi$, $\varphi_3 = \pi$: Angles $\varphi_{1,2,3}(t)$ (in radians).

end, the pendulum is steered to the equilibrium and the flywheel is stopped. The transient process is long because λ_3 is close to the imaginary axis.

7. CONCLUSION

Different strategies are defined to stabilize a two-link pendulum with a flywheel around three unstable equilibriums. The Jordan form of the linear model of the double-link mechanism and a pole placement are used to extract the unstable modes and to suppress them by the feedback control. When the linear system in open loop has a unique positive pole, an optimal (largest) domain of attraction for the system can be obtained, *i.e.*, it coincides with the controllability domain. We try to increase the domain of attraction in the case with two positive poles as possible. All these realistic numerical results lead to test these control laws experimentally. The results of paper for the attraction domain allow to construct in future the nonlinear and global control to transfert the pendulum to the upright position.

REFERENCES

Akensson, J. (2000). Safe Manual Control of Unstable Systems. PhD thesis. Master thesis Lund Institute of Technology, Lund, Sweden.

Aoustin, Y. and A. M. Formal'sky (2005). On the stabilization of biped vertical posture in single support using internal torques. *Robotica,* **23**(1), 65–74.

Aström, K. J. and K. Furuta (2000). Swinging up a pendulum by energy control. *Automatica,* **36**(2), 287–296.

Aström, K. J. and S. Brufani (1997). Manual control of an unstable system with a saturating actuator. In: *Proc. IEEE Conf. Decision and Control.* pp. 964–965.

Beznos, A. V., A. A. Grishin, A. V. Lenskii, D. E. Okhotsimsky and A. M. Formal'skii (2003). A pendulum controlled by a flywheel. *Doklady Mathematical Sciences.*

Boltyanskii, V. G. (1969). *Mathematical Methods of Optimal Control.* Nauka, Moscow [In Russian], 408 p.

Chung, C. C. and J. Hauser (1995). Nonlinear control of a swinging pendulum. *Automatica,* **31**(6), 851–862.

Fantoni, I., R. Lozano and M. W. Spong (2001). Stabilization of the reaction wheel pendulum using an energy approach. In: *Proc. European Control Conf. ECC2001.* pp. 2552–2557.

Formal'sky, A. M. (1974). *Controllability and stability of systems with limited ressources.* Nauka, Moscow [In Russian], 368 p.

Gilbert, E., I. Kolmanovsky and T. Tan (1994). Nonlinear control of discret-time linear systems with state and control constraints: A reference governor with global convergences properties. In: *Proc. IEEE Conf. Decision and Control.* pp. 144–149.

Gorinevsky, D. M., A. M. Formal'sky and A. Yu. Schneider (1997). *Force Control of Robotics Systems.* CRC Press, Boca Raton, 350 p.

Grishin, A. A., A. V. Lenskii, D. E. Okhotsimsky, D. A. Panin and A. M. Formal'sky (2002). A control synthesis for an unstable object. An inverted pendulum. *Computer and System Sciences International,* **41**(5), 685–694.

Kalman, R. E., P. L. Falb and M. A. Arbib (1969). *Topics in mathematical system theory.* Mc Grow-Hill Book Compagny, 358 p.

Martynenko, Y. G., A. V. Lensky and A. I. Kobrin (2004). The problem of controlling a mobile single-wheel robot with an unperturbed gyrostabilized platform. In: *Proc. ISR 04.*

Slotine, J. J. E. and W. Li (1991). *Applied nonlinear control.* Prentice Hall International, Inc., 461 p.