REDUCING SECOND ORDER SYSTEMS BY AN INTEGRATED STATE SPACE AND BACK CONVERSION PROCEDURE

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Abstract: The Arnoldi algorithm is modified to match the first Markov parameter and some of the first moments in order reduction of large scale systems while preserving the properties of the standard Arnoldi algorithm: an upper Hessenberg matrix as a coefficient of $\dot{\mathbf{x}}_r$, the identity matrix as a coefficient of $\mathbf{x}_r$ and a multiple of the first unit vector as the input vector. These properties are then used for the reduction of large scale second order models by first reducing in state space and then converting into second order form by introducing a numerical algorithm.

Keywords: Model reduction, Second-order systems, Large scale systems, Markov parameters, Reduced-order models.

1. INTRODUCTION

Many mechanical and electrical systems are modelled as a set of second order differential equations. Such a modelling (for instance by finite element methods) often leads to high order systems. To reduce the computational effort for analysis and optimization, it is then advisable to construct a reduced order model while preserving the second-order structure of the system.

One option in reducing a second order system is to convert it into a state-space model (i.e. a set of first order differential equations) and then apply known techniques like balancing and truncation or Krylov-subspace methods (Villemagne and Skelton 1987, Feldmann and Freund 1995, Freund 2000*a*); see (Antoulas *et al.* 2001) for a survey of reduction methods in state space. However, in doing so, the reduced-order system will be of the first-order type as well, making a physical interpretation difficult.

In (Chahlaoui *et al.* 2002, Meyer and Srinivasan 1996), the method of balanced truncation is generalized to reduce medium size second order systems. In (Su and Craig Jr. 1989, Bastian and Haase 2003), moment matching approach using Krylov subspaces was used for reduced order modelling of large scale second order models which was generalized by introducing a Second Order Krylov Subspace, in (Salimbahrami and Lohmann 2004*a*, Salimbahrami and Lohmann 2005). These methods use a projection directly applied to the second order system, thereby preserving its structure however the number of matching moments is less than by doing the reduction in state space.

To increase the number of matching parameters, in (Salimbahrami and Lohmann 2005, Salimbahrami and Lohmann 2004*b*), it is proposed to do the reduction by first reducing the equivalent state space representation

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using specific Krylov subspaces, preserving the second order character inside, and then, in a second step, by back converting into a second order representation. This approach matches an increased number of moments up to quadrupled compared to (Su and Craig Jr. 1989, Bastian and Haase 2003) and doubled compared to second order Krylov methods.

The reduced state space system based on moment matching by Arnoldi (or Lanczos) algorithm has a special structure; coefficient of $\dot{\mathbf{x}}_r$ is an upper Hessenberg (or a tridiagonal) matrix directly calculated in the algorithm. However, in the approach based on back conversion, because of matching the first Markov parameter, the reduced state space model can only be calculated by applying a projection and the special structure of the reduced system is destroyed. Furthermore, there was a lack of efficient numerical algorithms for back conversion into second order form.

In this paper, the Arnoldi algorithm is modified such that the first Markov parameter matches and the structure in the reduced state space matrices is preserved. The proposed algorithm directly calculates the matrices of the reduced state equation. Then, using this structure in the reduced system, a numerical procedure is proposed to find a transformation matrix which transforms the state space system into a second order form. This procedure not only suggests a numerically reliable procedure to compute the transformation matrix, but also extracts sufficient conditions for the possibility of the back conversion into second order form.

2. KRYLOV SUBSPACE METHODS

Consider the state space model,

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{x}(t). \end{cases} \tag{1}$$

The *moments* of system (1) are defined as,

$$m_i = \mathbf{c}^T(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{b} \quad , \quad i = 0, 1, \cdots , \tag{2}$$

which are the negative coefficients of the Taylor series expansion (about zero) of the system transfer function (Villemagne and Skelton 1987).

The idea of order reduction is to find a reduced model with a certain number of moments matching exactly with the ones of the original model. This can be done using the *Krylov subspaces* defined as,

$$\mathcal{K}_q(\mathbf{A}_1, \mathbf{b}_1) = span\{\mathbf{b}_1, \mathbf{A}_1\mathbf{b}_1, \cdots , \mathbf{A}_1^{q-1}\mathbf{b}_1\}, \tag{3}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$, and $\mathbf{b}_1 \in \mathbb{R}^n$ is called the starting vector.

One of the numerically stable and commonly used algorithm in moment matching is Arnoldi algorithm (Freund 2000*b*). It finds matrix $\mathbf{V}_q$ whose columns form a basis for the subspace $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and an upper Hessenberg matrix $\mathbf{H}_q$ with the following properties,

$$\mathbf{V}_q^T\mathbf{V}_q = \mathbf{I}, \tag{4}$$

$$\mathbf{v}_1 = \frac{1}{\|\mathbf{A}^{-1}\mathbf{b}\|_2}\mathbf{A}^{-1}\mathbf{b} \tag{5}$$

$$\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_q = \mathbf{V}_q\mathbf{H} + h_{q+1,q}\mathbf{v}_{m+1}\mathbf{e}_m^T, \tag{6}$$

$$\mathbf{V}_q^T\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_q = \mathbf{H}_q, \tag{7}$$

where,

$$\mathbf{V}_q = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_q \end{bmatrix},$$

$$\mathbf{H}_q = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1,q-1} & h_{1q} \\ h_{21} & h_{22} & \cdots & h_{2,q-1} & h_{2q} \\ 0 & h_{32} & \cdots & h_{3,q-1} & h_{3q} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{q,q-1} & h_{qq} \end{bmatrix},$$

the columns of $\mathbf{V}_q$ and $\mathbf{v}_{q+1}$ form a basis for the Krylov subspace $\mathcal{K}_{q+1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathbf{e}_q$ is the q-th unit vector.

Now, consider the Arnoldi algorithm is applied to reduce the order of the system (1). Then, the first q moments of the reduced q-th order model,

$$\begin{cases} \mathbf{H}_q\dot{\mathbf{x}}_r = \mathbf{x}_r + \underbrace{\|\mathbf{A}^{-1}\mathbf{b}\|_2\mathbf{e}_1}_{\mathbf{b}_r} u, \\ y = \underbrace{\mathbf{c}^T\mathbf{V}_q}_{\mathbf{c}_r^T}\mathbf{x}_r, \end{cases} \tag{8}$$

match with the system (1). The reduced system (8) can also be calculated by multiplying the state equation of the system (1) with $\mathbf{A}^{-1}$ and applying the projection $\mathbf{x}_r = \mathbf{V}_q\mathbf{x}$ to it.

By considering two Krylov subspaces $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathcal{K}_q(\mathbf{E}^T\mathbf{A}^{-T}, \mathbf{c})$ and applying the Lanczos algorithm (Freund 2000*b*), it is possible to match a double number of moments compared to the one achieved by Arnoldi algorithm.

Lanczos algorithm finds two matrices $\mathbf{V}_q$ and $\mathbf{W}_q$ whose columns form bases for two given subspaces with the following properties,

$$\mathbf{W}_q^T\mathbf{V}_q = \mathbf{I},\ \mathbf{W}_q^T\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_q = \mathbf{T}_q. \tag{9}$$

Compared to Arnoldi algorithm, Lanczos finds two sets of bi-orthogonal vectors and the matrix $\mathbf{T}_q$ is tridiagonal. The reduced order model found by Lanczos algorithm is as follows,

$$\begin{cases} \mathbf{T}_q\dot{\mathbf{x}}_r = \mathbf{x}_r + \underbrace{sign(\mathbf{c}^T\mathbf{A}^{-1}\mathbf{b})\sqrt{|\mathbf{c}^T\mathbf{A}^{-1}\mathbf{b}|}\mathbf{e}_1}_{\mathbf{b}_r} u, \\ y = \underbrace{-sign(\mathbf{c}^T\mathbf{A}^{-1}\mathbf{b})\sqrt{|\mathbf{c}^T\mathbf{A}^{-1}\mathbf{b}|}\mathbf{e}_1^T}_{\mathbf{c}_r^T}\mathbf{x}_r, \end{cases} \tag{10}$$

matching $2q$ moments with the original system (1).

3. NUMERICAL ALGORITHMS

By using Krylov subspace methods, it is not only possible to match the moments, but also the Markov parameter can be matched, simultaneously (Villemagne and Skelton 1987). The Markov parameters of the system (1) are defined as,

$$M_i = \mathbf{c}^T(\mathbf{E}^{-1}\mathbf{A})^i\mathbf{E}^{-1}\mathbf{b},\ i = 0, 1, 2, \cdots .$$

The Markov parameters are related to the coefficients of the Taylor series of the transfer function about infinity.

In the following, it is described how to modify the related numerical algorithms in order to match the first Markov parameter together with some of the moments, leading to the reduced order models given in (8) and (10) with similar properties of the state space matrices.

3.1 One-sided Krylov subspace methods

Consider that the Krylov subspace $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{E}^{-1}\mathbf{b})$ is used for the reduction of system (1). The Arnoldi algorithm finds the orthonormal matrix $\mathbf{V}_q$ and by applying the projection $\mathbf{x}_r = \mathbf{V}_q\mathbf{x}$ a reduced order model is calculated matching the first Markov parameter and $q-1$ moments. However, because of changing the starting vector of the Krylov subspace, the reduced order model can not be written in the form (8) using the Hessenberg matrix $\mathbf{H}_q$ calculated directly from the Arnoldi algorithm and $\mathbf{b}_r$ is not a multiple of the first unit vector $\mathbf{e}_1$. To achieve the reduced system (8), the standard Arnoldi algorithm is changed into algorithm 1.

In this algorithm, because $\mathbf{v}_1 = \frac{1}{\|\mathbf{A}^{-1}\mathbf{b}\|_2}\mathbf{A}^{-1}\mathbf{b}$ and $\mathbf{V}_q$ is orthonormal, we have,

$$\mathbf{V}_q^T\mathbf{A}^{-1}\mathbf{b} = \mathbf{V}_q^T\mathbf{v}_1\|\mathbf{A}^{-1}\mathbf{b}\|_2 = \|\mathbf{A}^{-1}\mathbf{b}\|_2\mathbf{e}_1.$$

From the other side, because the Arnoldi algorithm is applied to calculate the first $q-1$ vectors, we have,

$$\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_{q-2} = \mathbf{V}_{q-2}\mathbf{H}_{q-2} + h_{q-1,q-2}\mathbf{v}_{q-1}e_{q-2}^T.$$

By calculating $\mathbf{v}_q$ and $\mathbf{v}_{q+1}$ in steps 2 and 3, we have,

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_q &= \alpha_0\mathbf{A}^{-1}\mathbf{E}\mathbf{E}^{-1}\mathbf{b} + \mathbf{A}^{-1}\mathbf{E}\mathbf{V}_{q-2}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_{q-2}\end{bmatrix}\\ &\quad + \alpha_{q-1}\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{q-1}\\ &= \alpha_0\mathbf{A}^{-1}\mathbf{b} + \mathbf{V}_{q-2}(\mathbf{H}_{q-2} + h_{q-1,q-2}\mathbf{v}_{q-1}e_{q-2}^T)\times\\ &\quad\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_{q-2}\end{bmatrix} + \alpha_{q-1}\sum_{i=1}^{q+1} h_{i,q-1}\mathbf{v}_i. \qquad (12)\end{aligned}$$

By knowing that $\mathbf{A}^{-1}\mathbf{b} = \mathbf{v}_1\|\mathbf{A}^{-1}\mathbf{b}\|_2$, equation (12) can be rewritten as,

Algorithm 1. Modified Arnoldi algorithm

(1) Apply the standard Arnoldi algorithm to the Krylov subspace $\mathcal{K}_{q-1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ to produce $\mathbf{V}_{q-1}$, $\mathbf{H}_{q-2}$ and $h_{q-1,q-2}$.
(2) Find the normalized vector $\mathbf{v}_q = \alpha_0\mathbf{E}^{-1}\mathbf{b} + \sum_{i=1}^{q-1}\alpha_i\mathbf{v}_i$ using modified Gram-Schmidt procedure (Golub and Van Loan 1996) such that $\mathbf{v}_q^T\mathbf{V}_{q-1} = \mathbf{0}$.
(3) Find the normalized vector $\mathbf{v}_{q+1}$ using modified Gram-Schmidt procedure such that $\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{q-1} = \sum_{i=1}^{q+1} h_{i,q-1}\mathbf{v}_i$.
(4) Calculate the q-th column of the matrix $\mathbf{H}_q$ as follows,

$$\begin{aligned}\begin{bmatrix}h_{1,q}\\ h_{2,q}\\ \vdots\\ h_{q-2,q}\\ h_{q-1,q}\\ h_{q,q}\end{bmatrix} &= \begin{bmatrix}\alpha_0\|\mathbf{A}^{-1}\mathbf{b}\|_2 + \alpha_{q-1}h_{1,q-1}\\ \alpha_{q-1}h_{2,q-1}\\ \vdots\\ \alpha_{q-1}h_{q-2,q-1}\\ \alpha_{q-1}h_{q-1,q-1}\\ \alpha_{q-1}h_{q,q-1}\end{bmatrix}\\ &+ \begin{bmatrix} & \mathbf{H}_{q-2} & \\ 0 \cdots 0 & h_{q-1,q-2}\\ 0 \cdots 0 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{q-2}\end{bmatrix}. \qquad (11)\end{aligned}$$

$$\begin{aligned}\mathbf{A}^{-1}\mathbf{E}\mathbf{v}_q =& \mathbf{V}_q\left(\begin{bmatrix}\alpha_0\|\mathbf{A}^{-1}\mathbf{b}\|_2 + \alpha_{q-1}h_{1,q-1}\\ \alpha_{q-1}h_{2,q-1}\\ \vdots\\ \alpha_{q-1}h_{q,q-1}\end{bmatrix}\right.\\ &\left. + \begin{bmatrix} & \mathbf{H}_{q-2} & \\ 0 \cdots 0 & h_{q-1,q-2}\\ 0 \cdots 0 & 0\end{bmatrix}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_{q-2}\end{bmatrix}\right)\\ &+ \alpha_{q-1}h_{q,q-1}\mathbf{v}_{q+1}\mathbf{e}_q^T. \qquad (13)\end{aligned}$$

The value inside the parentheses in (13) is the q-th column of $\mathbf{H}_q$ calculated in (11).

Now, if we combine the results of steps 1 and 3 of the algorithm 1 with equation (13) we conclude,

$$\begin{aligned}&\mathbf{A}^{-1}\mathbf{E}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_q\end{bmatrix} =\\ &\begin{bmatrix}\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_{q-2} & \mathbf{A}^{-1}\mathbf{E}\mathbf{v}_{q-1} & \mathbf{A}^{-1}\mathbf{E}\mathbf{v}_q\end{bmatrix} =\\ &\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_{q+1}\end{bmatrix}\times\\ &\underbrace{\begin{bmatrix} & \mathbf{H}_{q-2} & & h_{1,q-1} & h_{1,q}\\ & & & \vdots & \vdots\\ 0 \cdots 0 & h_{q-1,q-2} & h_{q-1,q-1} & h_{q-1,q}\\ 0 \cdots 0 & 0 & h_{q,q-1} & h_{q,q}\\ 0 \cdots 0 & 0 & h_{q+1,q-1} & 0\end{bmatrix}}_{\bar{\mathbf{H}}}.\end{aligned}$$

The matrix $\mathbf{H}_q = \mathbf{V}_q^T\mathbf{A}^{-1}\mathbf{E}\mathbf{V}_q$ is calculated by deleting the last row of the matrix $\bar{\mathbf{H}}$. All entries of the matrix $\mathbf{H}_q$ can directly be calculated from algorithm 1.

After calculating the projection matrix $\mathbf{V}$ by applying the algorithm 1, the reduced order model is of the form (8) which matches the first Markov parameter and $q-1$ first moments with the original system.

3.2 Two-sided Krylov subspace methods

In two-side methods, by applying the Lanczos algorithm to the Krylov subspaces $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathcal{K}_q(\mathbf{E}^T\mathbf{A}^{-T}, \mathbf{E}^{-T}\mathbf{c})$, it is possible to match the first Markov parameter and $2q-1$ first moments. The reduced order model is of the form,

$$\begin{cases} \mathbf{T}_q\dot{\mathbf{x}}_r = \mathbf{x}_r + sign(\mathbf{c}^T\mathbf{E}^{-1}\mathbf{b})\sqrt{|\mathbf{c}^T\mathbf{E}^{-1}\mathbf{b}|}\mathbf{e}_1 u \\ y = \mathbf{c}^T\mathbf{V}_q\mathbf{x}_r, \end{cases} \tag{14}$$

where the matrix $\mathbf{T}_q$ is tridiagonal. The difference to the reduced system (10) is that $\mathbf{c}^T\mathbf{V}_q$ is not a multiple of the first unit vector. However, it does not have any effect on the upcoming facts.

4. REDUCTION OF SECOND ORDER SYSTEMS

We consider the second order system,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{g}u(t), \\ y(t) = \mathbf{l}^T\mathbf{z}(t). \end{cases} \tag{15}$$

System (15) can be transformed into state space as,

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M} & \mathbf{D} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \ddot{\mathbf{z}} \\ \dot{\mathbf{z}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}} \\ \mathbf{z} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{g} \end{bmatrix}}_{\mathbf{b}} u, \\ y = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{l}^T \end{bmatrix}}_{\mathbf{c}^T} \begin{bmatrix} \dot{\mathbf{z}} \\ \mathbf{z} \end{bmatrix}. \end{cases} \tag{16}$$

We also consider that the matrices $\mathbf{K}$ and $\mathbf{M}$ are nonsingular. By this assumption, the matrices $\mathbf{E}$ and $\mathbf{A}$ become nonsingular and therefore, a method to match the moments and Markov parameters can be applied to reduce system (16).

A characteristic property of the second order type system (16) is that the first Markov parameter is zero,

$$\begin{aligned} M_0 &= \begin{bmatrix} \mathbf{0} & \mathbf{l}^T \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{g} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{l}^T \end{bmatrix} \begin{bmatrix} \mathbf{M}^{-1}\mathbf{g} \\ \mathbf{0} \end{bmatrix} = 0. \end{aligned}$$

For the reduction of second order systems, first we reduce the equivalent state space equation (16) using algorithm 1 while the first Markov parameter of the reduced system is zero by matching with the one of the original system. Then, the reduced order system (8) (or its special case in (14)) is transformed into the second order form (16) by applying a similarity transformation as explained in the following.

To calculate the transformation matrix, consider that the first nonzero entry of $\mathbf{r} = \mathbf{c}_r^T\mathbf{H}_q^{-1}$ is r_k. Because $\mathbf{c}_r^T\mathbf{H}_q^{-1}\mathbf{b}_r = 0$ and $\mathbf{b}_r$ is a multiple of the first unit vector, $k > 1$. The transformation matrix is constructed for two different cases. First consider k is an even number. Then, the matrix $\mathbf{S} \in \mathbb{R}^{q\times\frac{q}{2}}$ is constructed as,

$$\mathbf{S} = \begin{bmatrix} \mathbf{H}_q^{-T}\mathbf{c}_r & \mathbf{e}_2 & \mathbf{e}_4 & \cdots & \mathbf{e}_{k-2} & \mathbf{e}_{k+2} & \cdots & \mathbf{e}_q \end{bmatrix}.$$

Because $\mathbf{c}_r^T\mathbf{H}_q^{-1}\mathbf{b} = 0$, the first row of $\mathbf{S}$ is zero and therefore $\mathbf{S}^T\mathbf{b}_r = \mathbf{0}$. The transformation matrix $\mathbf{T}$ is constructed as,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{S}^T\mathbf{H}_q \end{bmatrix}. \tag{17}$$

If we interchange the rows of $\mathbf{T}^{-1}$ then,

$$\bar{\mathbf{T}} = \begin{bmatrix} \mathbf{e}_2^T\mathbf{H}_q \\ \mathbf{e}_2^T \\ \mathbf{e}_4^T\mathbf{H}_q \\ \mathbf{e}_4^T \\ \vdots \\ \mathbf{e}_{k-2}^T\mathbf{H}_q \\ \mathbf{e}_{k-2}^T \\ \mathbf{c}_r^T\mathbf{H}_q^{-1}\mathbf{H}_q \\ \mathbf{c}_r^T\mathbf{H}_q^{-1} \\ \mathbf{e}_{k+2}^T\mathbf{H}_q \\ \mathbf{e}_{k+2}^T \\ \vdots \\ \mathbf{e}_q^T\mathbf{H}_q \\ \mathbf{e}_q^T \end{bmatrix} = \begin{bmatrix} \mathbf{h}_2^T \\ \mathbf{e}_2^T \\ \mathbf{h}_4^T \\ \mathbf{e}_4^T \\ \vdots \\ \mathbf{h}_{k-2}^T \\ \mathbf{e}_{k-2}^T \\ \mathbf{c}_r^T \\ \mathbf{c}_r^T\mathbf{H}_q^{-1} \\ \mathbf{h}_{k+2}^T \\ \mathbf{e}_{k+2}^T \\ \vdots \\ \mathbf{h}_q^T \\ \mathbf{e}_q^T \end{bmatrix},$$

where $\mathbf{h}_j^T$ is the j-th row of the matrix $\mathbf{H}_q$. Because the matrix $\mathbf{H}_q$ is upper Hessenberg, the first $j-2$ entries of $\mathbf{h}_j$ are zero and the matrix $\bar{\mathbf{T}}$ is upper triangular whose diagonal entries are one or the sub-diagonal entries of even rows of $\mathbf{H}_q$, except for the $k-1$-st row with $r_k h_{k,k-1}$ and k-th row with r_k as,

$$\bar{\mathbf{T}} = \begin{bmatrix} h_{2,1} & * & \cdots & * & * & * & * & \cdots & * \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{k,k-1}r_k & * & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & r_k & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & h_{k+2,k+1} & * & \cdots & * \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Therefore, $\mathbf{T}^{-1}$ is full rank if all sub-diagonal entries of even rows of $\mathbf{H}_q$ are nonzero.

If k is odd, then the matrix $\mathbf{S}$ is constructed as

$$\mathbf{S} = \begin{bmatrix} \mathbf{c}_r & \mathbf{e}_2 & \mathbf{e}_4 & \cdots & \mathbf{e}_{k-1} + \beta\mathbf{e}_{k+1} & \mathbf{e}_{k+3} & \cdots & \mathbf{e}_q \end{bmatrix},$$

where β is a parameter. Again $\mathbf{S}^T\mathbf{b}_r = \mathbf{0}$ and the transformation matrix $\mathbf{T}$ is constructed using (17). To show that $\mathbf{T}^{-1}$ is full rank, we interchange the rows as follows,

$$\bar{\mathbf{T}} = \begin{bmatrix} \mathbf{e}_2^T\mathbf{H}_q \\ \mathbf{e}_2^T \\ \mathbf{e}_4^T\mathbf{H}_q \\ \mathbf{e}_4^T \\ \vdots \\ (\mathbf{e}_{k-1}+r\mathbf{e}_{k+1})^T\mathbf{H}_q \\ \mathbf{e}_{k-1}^T+\beta\mathbf{e}_{k+1}^T \\ \mathbf{c}_r^T\mathbf{H}_q^{-1}\mathbf{H}_q \\ \mathbf{c}_r^T\mathbf{H}_q \\ \mathbf{e}_{k+3}^T\mathbf{H}_q \\ \mathbf{e}_{k+3}^T \\ \vdots \\ \mathbf{e}_q^T\mathbf{H}_q \\ \mathbf{e}_q^T \end{bmatrix} = \begin{bmatrix} \mathbf{h}_2^T \\ \mathbf{e}_2^T \\ \mathbf{h}_4^T \\ \mathbf{e}_4^T \\ \vdots \\ \mathbf{h}_{k-1}^T+r\mathbf{h}_{k+1}^T \\ \mathbf{e}_{k-1}^T+\beta\mathbf{e}_{k+1}^T \\ \mathbf{c}_r^T \\ \mathbf{c}_r^T\mathbf{H}_q \\ \mathbf{h}_{k+2}^T \\ \mathbf{e}_{k+2}^T \\ \vdots \\ \mathbf{h}_q^T \\ \mathbf{e}_q^T \end{bmatrix}$$

$$= \begin{bmatrix} h_{2,1} & * & \cdots & * & * & * & * & * & \cdots \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h_{k,k-1} & * & * & * & * & \cdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \beta & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \alpha_3 & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & r_k & r_{k+1} & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & h_{k+2,k+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\alpha_1 = r_k h_{k,k-1}$, $\alpha_2 = r_k h_{k,k} + r_{k+1}h_{k+1,k}$, $\alpha_3 = r_k h_{k,k+1} + r_{k+1}h_{k+1,k+1} + r_{k+2}h_{k+2,k+2}$. If $\beta \neq \frac{1}{\alpha_1}\left(\alpha_3 - \frac{r_{k+1}}{r_k}\alpha_2\right)$ is chosen and the sub-diagonal entries of $\mathbf{H}_q$ at even rows are nonzero then the matrix $\bar{\mathbf{T}}$ is full rank, knowing that $\alpha_1 \neq 0$ because $h_{k,k-1}, r_k$ are assumed to be nonzero.

Now, consider the similarity transformation $\mathbf{x}_r = \mathbf{T}\mathbf{x}_t$ is applied to the system (8),

$$\begin{cases} \mathbf{T}^{-1}\mathbf{H}_q\mathbf{T}\dot{\mathbf{x}}_t = \mathbf{x}_t + \mathbf{T}^{-1}\mathbf{b}_r u, \\ y = \mathbf{c}_r^T\mathbf{T}\mathbf{x}_t. \end{cases} \tag{18}$$

Considering the facts that,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{S}^T \\ \mathbf{S}^T\mathbf{H}_q \end{bmatrix}, \mathbf{S}^T\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{19}$$

$$\mathbf{S}^T\mathbf{H}_q\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{S}^T\mathbf{b} = \mathbf{0}, \tag{20}$$

the system (18) can be rewritten as follows,

$$\begin{cases} \begin{bmatrix} \mathbf{S}^T\mathbf{H}_q\mathbf{T} \\ \mathbf{S}^T\mathbf{H}_q^2\mathbf{T} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{S}^T\mathbf{b}_r \\ \mathbf{S}^T\mathbf{H}_q\mathbf{b}_r \end{bmatrix} u, \\ y = \mathbf{c}_r^T\mathbf{T}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{cases}$$

which is equivalent to,

$$\begin{cases} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M}_r & \mathbf{D}_r \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K}_r \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\ \qquad + \begin{bmatrix} \mathbf{0} \\ \mathbf{g}_r \end{bmatrix} u, \\ y = \begin{bmatrix} \mathbf{0} & \mathbf{l}_r^T \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{cases} \tag{21}$$

where,

$$\mathbf{K}_r = -\mathbf{I}, \tag{22}$$

$$\mathbf{g}_r = \mathbf{S}^T\mathbf{H}_q\mathbf{b}_r, \tag{23}$$

$$\begin{bmatrix} \mathbf{M}_r & \mathbf{D}_r \end{bmatrix} = \mathbf{S}^T\mathbf{H}_q^2\mathbf{T}. \tag{24}$$

The output equation in (18), $y = \mathbf{c}_r^T\mathbf{T}\mathbf{x}_t$, is simplified to $y = \mathbf{e}_{\frac{q}{2}+1}\mathbf{x}_t$, because $\mathbf{c}_r^T$ is the first line of $\mathbf{S}^T\mathbf{H}_q$ in $\mathbf{T}^{-1}$. Thereby, it is concluded

$$\mathbf{l}_r^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{25}$$

Because, in system (21), $\dot{\mathbf{x}}_2 = \mathbf{x}_1$ by defining $\mathbf{z}_r = \mathbf{x}_1$ then the state space equation (21) is equivalent to second order system,

$$\begin{cases} \mathbf{M}_r\ddot{\mathbf{z}}_r + \mathbf{D}_r\dot{\mathbf{z}}_r + \mathbf{K}_r\mathbf{z}_r = \mathbf{p}_r u(t), \\ y = \mathbf{l}_r^T\mathbf{z}_r. \end{cases} \tag{26}$$

We conclude the results of the back conversion procedure as the following theorem:

Theorem 1. The sufficient conditions for a state space model of the form (8) to be converted to a second order type model are: the first Markov parameter is zero, the order of the system is even, $\mathbf{H}_q$ is full rank and its sub-diagonal entries at even rows are nonzero.

Lemma 2. Consider the Hessenberg matrix $\mathbf{H}_q$ in system (8) is full rank. The system (8) is controllable if and only if all sub-diagonal entries of $\mathbf{H}_q$ are nonzero.

PROOF. The Kalman controllability matrix of system (8) is

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{b}_r, & \mathbf{H}_q^{-1}\mathbf{b}_r, & \cdots, & \mathbf{H}_q^{-q+1}\mathbf{b}_r \end{bmatrix} \\ &= \mathbf{H}_q^{-q+1} \underbrace{\begin{bmatrix} \mathbf{H}_q^{q-1}\mathbf{b}_r, & \mathbf{H}_q^{q-2}\mathbf{b}_r, & \cdots, & \mathbf{b}_r \end{bmatrix}}_{\mathcal{C}_H}, \end{aligned}$$

where $\mathbf{b}_r$ is a multiple of the first unit vector. Because $\mathbf{H}_q$ is nonsingular, the system (8) is controllable if and only if the matrix $\mathcal{C}_H$ is full rank. If we change the sequence of the columns of $\mathcal{C}_H$ to

$$\mathcal{C}_t = \begin{bmatrix} \mathbf{b}_r, & \mathbf{H}_q\mathbf{b}_r, & \cdots, & \mathbf{H}_q^{q-1}\mathbf{b}_r \end{bmatrix},$$

then using the structure of $\mathbf{H}_q$ and $\mathbf{b}_r$, the matrix $\mathcal{C}_t$ is an upper triangular matrix whose diagonal entries are the sub-diagonal entries of the matrix $\mathbf{H}_q$. Therefore, $\mathcal{C}_H$ is full rank (or the system is controllable) if all sub-diagonal entries of $\mathbf{H}_q$ are nonzero and vice versa.

By using lemma 2, if a system is controllable then the third condition of theorem 1 is fulfilled. If a two-sided method using Lanczos is applied as explained in section 3.2, then the matrix $\mathbf{H}_q$ becomes tridiagonal which is a special case of Hessenberg form and all steps of back conversion to second order form are quite similar.

So, the steps of reducing second order type models are:

(1) Apply algorithm 1 (or Lanczos as explained in section 3.2) to find a reduced system of the form (8) (using Lanczos $\mathbf{H}_q$ is substituted by $\mathbf{T}_q$).
(2) Calculate the matrix $\mathbf{S}$ as explained in this section and $\mathbf{T}$ using (17).
(3) calculate the state space matrices of the reduced second order system (26) using equations (22), (23), (24) and (25).

In this way, the number of matching moments of the reduced second order type model is the maximum that can be achieved. More precisely: if a model of order n is reduced to order q, this method matches at most $2q - 1$ moments.

4.1 Matching the moments about s_0

To match the moments about $s_0 \neq 0$, it is sufficient to substitute $\mathbf{A}$ with $\mathbf{A} - s_0\mathbf{E}$ in the the corresponding Krylov subspaces in the algorithm 1 or in Lanczos algorithm. Then the moments of the reduced system (8) or (10) about zero are matched with the moments of the original system about s_0. Therefore, after transforming such a reduced order model into second order form (26) the reduced matrices should be modified as follows,

$$\begin{aligned}\mathbf{M}_{s_0} &= \mathbf{M}_r, \\ \mathbf{D}_{s_0} &= \mathbf{D}_r - 2s_0\mathbf{M}_r, \\ \mathbf{K}_{s_0} &= \mathbf{K}_r - s_0\mathbf{D}_r - s_0^2\mathbf{M}_r,\end{aligned}$$

5. CONCLUSION

The reduced order modelling of large scale second order systems can be done by first reduction in state space and then back conversion into second order form. In this paper, a modified Arnoldi algorithm has been proposed to reduce a state space equation matching the first Markov parameter which is zero for second order model and some of the first moments. The structure of the reduced state space matrices found by applying the modified Arnoldi algorithm is the same as the one by moment matching in standard Arnoldi algorithm.

The structure in the reduced state space matrices found by the proposed algorithm was used to calculate a similarity transformation to transform the reduced system into a second order form.

It was also explained how to use the method to match the moments about other points and how to increase the number of matching parameters by applying the Lanczos algorithm.

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