TRACKING OF NON-SQUARE NONLINEAR SYSTEMS WITH MODEL PREDICTIVE CONTROL

L. Magni, [*,1] **R. Scattolini** [**]

[*] *Dipartimento di Informatica e Sistemistica*
Università degli Studi di Pavia
via Ferrata 1, 27100 Pavia, Italy, e-mail: lalo.magni@unipv.it
[**] *Dipartimento di Elettronica e Informazione, Politecnico di Milano, p.za Leonardo da Vinci, 32, 20133 Milano, Italy*
e-mail: riccardo.scattolini@elet.polimi.it

Abstract: This paper presents a Model Predictive Control (MPC) algorithm for nonlinear systems. It allows the tracking of reference signals constant after a certain time by imposing integral action on the error variables, rather than on the control moves as in standard MPC methods. The plant under control, the state and control constraints and the performance index to be minimized are described in continuous time, while the manipulated variables are allowed to change at fixed and uniformly distributed sampling times. A simulation example is reported and discussed in detail.

Keywords: Nonlinear Model Predictive Control, Tracking, Stability

1. INTRODUCTION

This paper presents an MPC method for nonlinear, multivariable systems. It has been specifically designed for the tracking of reference signals constant after a certain time. In standard (linear and nonlinear) MPC, asymptotic zero error regulation for constant reference signals is usually obtained by forcing integral action on the manipulated variables. This solution has the significant drawback that a state observer is required even when the state itself is measurable, otherwise any plant-model mismatch could lead to closed loop instability, see (Magni, 2002). On the contrary, in the control scheme proposed in this paper, integral action is directly imposed on the error variables. In so doing, when the plant state is available the use of an observer is not required to compensate for uncertainties. Moreover, it is easy to deal with nonsquare systems, where the number of manipulated variables is greater than the one of controlled variables.

The proposed algorithm is derived by extending the results reported in (Magni and Scattolini, 2004), where a pure regulation problem was considered. Specifically, the plant under control, the state and control constraints and the performance index to be minimized are described in continuous time, while the manipulated variables are allowed to change at fixed and uniformly distributed sampling times. In so doing, one has to deal with the optimization with respect to sequences, as in discrete time nonlinear MPC, while taking into account the continuous time evolution of the system.

A simulation example is reported and discussed in depth to illustrate the characteristics of the method. The proofs are not reported for space limitations.

[1] The authors acknowledge the partial financial support by $MURST$ Project "New techniques for the identification and adaptive control of industrial systems"

2. PROBLEM STATEMENT AND PRELIMINARY RESULTS

In the paper, for any vector $x \in R^n$, $\|x\|$ denotes the Euclidean norm in R^n, $\|x\|_\Pi^2 := x'\Pi x$, where $\Pi > 0$ is an arbitrary Hermitian matrix, denotes the weighted norm. For any Hermitian matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest and the smallest real part of the eigenvalues of the matrix A, respectively and $\|A\|$ stands for the induced 2-norm of A. $B_r^\Pi(\bar{x})$ denotes the closed ball of radius r and center $\bar{x}$ defined with the weighted norm Π, i.e. $B_r^\Pi(\bar{x}) = \{x \in R^n : \|x - \bar{x}\|_\Pi \leq r\}, \Pi > 0$. To properly formulate the MPC problem, the plant and a number of state augmentations must be defined.

Plant: The plant P is described by the nonlinear continuous-time dynamic system

$$\dot{x}_p(t) = f_p(x_p(t), u(t)),\ t \geq 0,\ x_p(0) = x_{p0} \quad (1)$$
$$y(t) = h_p(x_p(t))$$

where $x_p \in R^n$ is the state, $u \in R^m$ is the input, $y \in R^p$ is the output. In (1) it is assumed that: (i) $m \geq p$; (ii) $f_p(\cdot,\cdot)$ is a C^1 function of its arguments; (iii) $h_p(\cdot)$ is a Lipschitz function with Lipschitz constant L_h; (iv) given a suitable sampling period T_s, and letting $t_k = kT_s$, k nonnegative integer, be the sampling instants, the control variable u is restricted to be constant in $[t_k, t_{k+1})$; (v) the following constants must be fulfilled

$$x_p(t) \in X_p, \quad u(t) \in U, \quad t \geq 0 \quad (2)$$

where X_p and U are closed and compact subsets of R^n and R^m respectively. The movement of (1) from the initial time $\bar{t}$ and initial state $x_p(\bar{t})$ for a control signal $u(\cdot)$ is denoted by $\varphi_p(t, \bar{t}, x_p(\bar{t}), u(\cdot))$.

The tracking problem here considered consists in finding a "sampled" feedback control law guaranteeing $\lim_{t\longrightarrow\infty} e(t) = 0$ with $e(t) = y^0(t) - y(t)$, for reference signals y^0 such that

$$y^0(t) = \bar{y}^0, \quad \forall t \geq t_{k+N_r}$$

where N_r is a nonnegative integer.

Letting $\bar{x}_p(\bar{y}^0)$ and $\bar{u}(\bar{y}^0)$ such that $f_p(\bar{x}_p(\bar{y}^0), \bar{u}(\bar{y}^0)) = 0$ and $h_p(\bar{x}_p(\bar{y}^0)) = \bar{y}^0$, the following preliminary assumptions are required.

Assumption 1. For a given $\bar{y}^0$, there exists at least one equilibrium $\bar{x}_p(\bar{y}^0) \in X_p$, $\bar{u}(\bar{y}^0) \in U$ of system (1) such that, letting $A_p(\bar{y}^0) = \partial f_p/\partial x_p|_{\bar{x}_p(\bar{y}^0),\bar{u}(\bar{y}^0)}$, $B_p(\bar{y}^0) = \partial f_p/\partial u|_{\bar{x}_p(\bar{y}^0),\bar{u}(\bar{y}^0)}$,

$C_p(\bar{y}^0) = \partial h_p/\partial x_p|_{\bar{x}_p(\bar{y}^0)}$, (i) the pair ($A_p$, B_p) is stabilizable; (ii) there are no transmission zeros of (A_p, B_p, C_p) equal to zero.

Assumption 2. For any equilibrium $\bar{x}_p(\bar{y}^0)$, $\bar{u}(\bar{y}^0)$, if $lim_{t\to\infty} u(t) = \bar{u}(\bar{y}^0)$ and $lim_{t\to\infty} y(t) = \bar{y}^0$, then $lim_{t\to\infty} x_p(t) = \bar{x}_p(\bar{y}^0)$.

Plant and integrators: According to the Internal Model Principle, see (Davison, 1976), the tracking problem is solved by passing the error variable through a set of p integral actions described by

$$\dot{z}(t) = e(t), \quad z(0) = z_0, \quad t \geq 0 \quad (3)$$

Then, the regulator to be determined must stabilize the system formed by the cascade connection of (1), (3), described by

$$\dot{x}_a(t) = f_a(x_a(t), u(t), y^0(t)), \quad t \geq 0 \quad (4)$$
$$e(t) = h_a(x_a(t), y^0(t))$$

where $x_a = [x_p'\ z']' \in R^{n+p}$, $x_a(0) = x_{a0} = [x_{p0}'\ z_0']'$, while $f_a(\cdot,\cdot,\cdot)$, and $h_a(\cdot,\cdot)$ are derived from (1) and (3).

Plant, integrators and past control variable (enlarged plant): According to the MPC approach, at any sampling instant t_k a performance index penalizing the future error and control variations $\delta u(t_{k+i}) := u(t_{k+i}) - u(t_{k+i-1})$, $i \geq 0$ is minimized with respect to the future control moves. To formulate the optimization problem, it is convenient to enlarge the state vector x_a of (4) with the previous value of the control variable, so obtaining the new augmented system:

$$\dot{x}(t) = \begin{bmatrix} f_a(x_a(t), u(t), y^0(t)) \\ 0_{m,1} \end{bmatrix}, \quad (5)$$
$$t \in [t_k, t_{k+1}),\ x(t_k) = \begin{bmatrix} x_a(t_k^-) \\ u(t_k^-) \end{bmatrix}$$
$$e(t) = h(x(t), y^0(t)) \quad (6)$$

where $x = [x_a'\ x_{uv}']' \in R^{n+p+m}$, $x(0) = x_0 = [x_{a0}'\ u(t_{-1})']'$ and $h(\cdot,\cdot)$ is easily derived from (4). In this way $u(t_{k-1}) = \Xi x(t_k)$, $\Xi = [0_{m,n+p}\ I_m]$ where $0_{n,m}$ and I_m are a $n \times m$ zero matrix and the identity matrix of dimension m, respectively. For any control signal $u(\cdot)$, the movement of (5) from the initial time $\bar{t}$ and initial state $x(\bar{t})$ is denoted by $\varphi(t, \bar{t}, x(\bar{t}), u(\cdot))$.

Enlarged plant and sampled control law: Given a generic sampled feedback control law

$$u(t) \equiv \kappa(x(t_k), y^0(\cdot)) \quad , t \in [t_k, t_{k+1}), \quad (7)$$

the description of the hold mechanism implicit in (7) calls for a further state augmentation. Letting $x_c := [x'\ x_u']' \in R^{n+p+2m}$, the closed loop system (5)-(6)-(7) is

$$\dot{x}_c(t) = \begin{bmatrix} f_a(x_a(t), x_u(t), y^0(t)) \\ 0_{2m,1} \end{bmatrix}, \quad (8)$$
$$t \in [t_k, t_{k+1}),\ x_c(t_k) = \begin{bmatrix} x_a(t_k^-) \\ x_u(t_k^-) \\ \kappa(x(t_k^-), y^0(\cdot)) \end{bmatrix}$$

and its movement from the initial time $\bar{t}$ and initial state $x_c(\bar{t})$ is denoted by

$$\varphi_c(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) = \begin{bmatrix} \varphi_c^{x_p}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_c^{I}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_c^{uv}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_c^{u}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \end{bmatrix} = \begin{bmatrix} \varphi_c^{x}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_c^{u}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \end{bmatrix}$$

$$\varphi_c^{x_p} \in R^n, \varphi_c^I \in R^p, \varphi_c^{uv} \in R^m, \\ \varphi_c^x \in R^{n+p+m}, \varphi_c^u \in R^m$$

With reference to the closed-loop system (8) it is possible to define the following sets.

Definition 1. A *sampled output admissible set* associated to (8) is a set $\Gamma_s^c(\kappa, y^0(\cdot)) \in R^{n+p+m}$ such that for all $x \in \Gamma_s^c(\kappa, y^0(\cdot))$, $\varphi_c^x(t_{k+1}, t_k, [x' \ \kappa(x, y^0(\cdot))']', y^0(\cdot)) \in \Gamma_s^c(\kappa, y^0(\cdot))$, $\varphi_c^{x_p}(t, t_k, [x' \ \kappa(x, y^0(\cdot))']', y^0(\cdot)) \in X_p, t \in [t_k, t_{k+1}), \kappa(x, y^0(\cdot)) \in U$, $\lim_{t\to\infty} ||h_p(\varphi_c^{x_p}(t,\bar{t},x_c(\bar{t}),y^0(\cdot))) - \bar{y}^0|| = 0$, $\lim_{i\to\infty} ||\kappa(\varphi_c^x(t_{k+i}, t_k, [x'\kappa(x,y^0(\cdot))']', \bar{y}^0) - \kappa(\varphi_c^x(t_{k+i-1}, t_k, [x'\kappa(x,y^0(\cdot))']', \bar{y}^0)|| = 0$. In other words, $\Gamma_s^c(\kappa, y^0(\cdot))$ is a state invariant set, associated to the closed loop system (8), defined at the sampling instants t_k and such that (i) the state and control constraints (2) are satisfied in all the future continuous-time instants, (ii) the tracking problem is asymptotically solved. The (unique) maximal sampled output admissible set $X_s^c(\kappa, y^0(\cdot))$ is defined as the union of all sampled output admissible sets.

Definition 2. An *output admissible set* associated to (8) is a set $\Gamma^c(t, \kappa, y^0(\cdot)) \in R^{n+p+2m}$ such that for all $x_c \in \Gamma^c(t, \kappa, y^0(\cdot))$, $\varphi_c^x(t_k, t, x_c) \in X_s^c(\kappa, y^0(\cdot))$, where t_k is the closest sampling time in the future, $\varphi_c^{x_p}(\tau, t, x_c, y^0(\cdot)) \in X_p, \tau \in [t, t_k), \varphi_c^u(t, t, x_c, y^0(\cdot)) \in U$. The (unique) maximal output admissible set $X^c(t, y^0(\cdot))$ is defined as the union of all output admissible sets. ■

The tracking problem can now be formally stated as the problem of finding a feasible sampled control law (7) with the largest output admissible set X^c and which optimizes a given performance index.

Assume now to know a feasible control law (7) satisfying the following assumption.

Assumption 3. The feasible control law (7) is a C^1 function with Lipschitz constant L_κ. ■

Let $(\overline{x}_c(\bar{y}^0, \kappa), \bar{y}^0)$ with $\overline{x}_c(\bar{y}^0, \kappa) := [\overline{x}_p(\bar{y}^0, \kappa)' \ \bar{z}(\bar{y}^0, \kappa)' \ \bar{u}(\bar{y}^0, \kappa)' \ \bar{u}(\bar{y}^0, \kappa)']'$ be an equilibrium associated to the closed-loop system (8) such that $\bar{e} = h_a(\bar{x}_a(\bar{y}^0, \kappa), \bar{y}^0) = 0$, with $\bar{x}_a(\bar{y}^0, \kappa) := [\overline{x}_p(\bar{y}^0, \kappa)' \ \bar{z}(\bar{y}^0, \kappa)']'$. If Assumption 1 is satisfied, in view of the Implicit Function Theorem, $(\overline{x}_c(\bar{y}^0, \kappa), \bar{y}^0)$ is an isolated equilibrium for system (8) (for sake of simplicity the dependence of the equilibrium point on κ will be omitted whenever possible).

For this control law, an associated sampled output admissible set can be computed as follows. Define the linearization of system (5) around the equilibrium as

$$\delta\dot{x}(t) = \begin{bmatrix} A_a(\bar{y}^0)\delta x_a(t) + B_a(\bar{y}^0)\delta u(t) \\ 0_{m,n+p+m} \end{bmatrix}, \quad (9)$$
$$\delta x(t_k) = \begin{bmatrix} \delta x_a(t_k^-) \\ \delta u(t_k^-) \end{bmatrix}$$

where $\delta x = x - \overline{x}(\bar{y}^0)$, $\delta u = u - \bar{u}(\bar{y}^0)$, $\delta x_a = x_a - \bar{x}_a(\bar{y}^0)$, $\overline{x}(\bar{y}^0) = [\overline{x}_p(\bar{y}^0, \kappa)' \ \bar{z}(\bar{y}^0, \kappa)' \ \bar{u}(\bar{y}^0, \kappa)']'$and

$$A_a(\bar{y}^0) = \partial f_a/\partial x_a|_{\bar{x}_a(\bar{y}^0), \bar{u}(\bar{y}^0), \bar{y}^0}, \\ B_a(\bar{y}^0) = \partial f_a/\partial u|_{\bar{x}_a(\bar{y}^0), \bar{u}(\bar{y}^0), \bar{y}^0}$$

Then introduce the discretization of (9) given by

$$\delta x(t_{k+1}) = A_D(\bar{y}^0)\delta x(t_k) + B_D(\bar{y}^0)\delta u(t_k) \quad (10)$$

with

$$A_D(\bar{y}^0) := \begin{bmatrix} e^{A_a(\bar{y}^0)T_s} & 0_{n+p,m} \\ 0_{m,n+p} & 0_{m,m} \end{bmatrix},$$
$$B_D(\bar{y}^0) := \begin{bmatrix} \int_0^{T_s} e^{A_a(\bar{y}^0)\eta} B_a(\bar{y}^0) d\eta \\ I_m \end{bmatrix}$$

Finally let

$$K(\bar{y}^0) = \left. \frac{\partial \kappa(x, y^0)}{\partial x} \right|_{\bar{x}(\bar{y}^0), \bar{y}^0}$$

In view of the feasibility of (7), it is then easy to show that the closed-loop matrix $A_D^{cl}(\bar{y}^0) := A_D(\bar{y}^0) + B_D(\bar{y}^0)K(\bar{y}^0)$ of the linearized discrete-time system (10) is Hurwitz and the following result holds.

Lemma 4. Let $\kappa(x, y^0)$ be a feasible control law. Consider an equilibrium $(\bar{x}_p(\bar{y}^0), \bar{u}(\bar{y}^0))$ of system (1) satisfying Assumptions 1 and 3, a positive definite matrix $\tilde{Q}$ and two real positive scalars γ and γ_2 such that $\gamma < \lambda_{\min}(\tilde{Q})$. Define by Π the unique symmetric positive definite solution of the following Lyapunov equation:

$$A_D^{cl}(\bar{y}^0)'\Pi A_D^{cl}(\bar{y}^0) - \Pi + \bar{Q} = 0 \quad (11)$$

where

$$\bar{Q} = \int_0^{T_s} A_c^{ZOH}(\eta)'\tilde{Q}A_c^{ZOH}(\eta)d\eta + \gamma_2 I_{n+p+m}$$

and

$$A_c^{ZOH}(t) := \begin{cases} \begin{bmatrix} \pi_1 & \pi_2 \\ 0_{m,n+p} & 0_{m,m} \end{bmatrix} & t \in [0, T_s) \\ A_D^{cl}(\bar{y}^0) & t = T_s \end{cases}$$

with $\pi_1 = e^{A_a(\bar{y}^0)t} + \left(\int_0^t e^{A_a(t-\tau)}d\tau\right) B_a K_1$, $\pi_2 = \left(\int_0^t e^{A_a(t-\tau)}d\tau\right) B_a K_2$ and $K := [K_1 K_2]$, $K_1 \in R^{m,n+p}$.

Then, there exists two constants $T_s \in (0,\infty)$ and $\sigma \in (0,\infty)$ specifying a neighborhood $\Omega_\sigma(\bar{x}(\bar{y}^0),\kappa,T_s)$ of $\bar{x}(\bar{y}^0)$ of the form

$$\begin{aligned}&\Omega_\sigma(\bar{x}(\bar{y}^0),\kappa,T_s) \\ &= \left\{ x \in \Re^{n+p+m} \mid \left\| x - \bar{x}(\bar{y}^0)\right\|_\Pi^2 \le \sigma \right\}\end{aligned} \tag{12}$$

such that $\forall x \in \Omega_\sigma(\bar{x}(\bar{y}^0),\kappa,T_s)$:

(a) $\varphi_c^{x_p}(t,t_k,[x' \ \kappa(x,\bar{y}^0)']',\bar{y}^0) \in X_p$, $t \in [t_k,t_{k+1})$, $\kappa(x,\bar{y}^0) \in U$;

(b)

$$\begin{aligned}&\left\|\varphi_c^x(t_{k+1},t_k,[x'\kappa(x,\bar{y}^0)']',\bar{y}^0) - \bar{x}(\bar{y}^0)\right\|_\Pi^2 \\ &- \left\|x - \bar{x}(\bar{y}^0)\right\|_\Pi^2 \\ &\le -\gamma \int_{t_k}^{t_{k+1}} \left\|\varphi_c^x(\eta,t_k,[x'\kappa(x,\bar{y}^0)']',\bar{y}^0) - \bar{x}(\bar{y}^0)\right\|^2 d\eta \\ &-\gamma_2 \left\|x - \bar{x}(\bar{y}^0)\right\|^2\end{aligned} \tag{13}$$

2.1 The sampled MPC control law

Let $u = \kappa(x,y^0)$ be a feasible auxiliary control law, assumed to be known together with its sampled output admissible set and Lyapunov function given by Lemma 4. Moreover, given a control sequence

$$\bar{u}_{1,N_c}(t_k) := [u_1(t_k), u_2(t_k), ..., u_{N_c}(t_k)]$$

with $N_c \ge 1$, define the *Finite Horizon* piece-wise constant control signal

$$u_{t_k}^{FH}(t) := \begin{cases} u_j(t_k) \\ t \in [t_{k+j-1},t_{k+j}), j = 1,...,N_c \\ \bar{\varphi}_c^u(t) \\ t \in [t_{k+N_c},t_{k+N_p}) \end{cases} \tag{14}$$

where $\bar{\varphi}_c^u(t) := \varphi_c^u(t,t_{k+N_c},[\bar{x}_{N_c} \quad \kappa(\bar{x}_{N_c},y^0(\cdot))]$ $,y^0(\cdot))$ with $\bar{x}_{N_c} = \varphi(t_{k+N_c},t_k,x(t_k),u_{t_k}^{FH}(\cdot),y^0(\cdot))$ and $N_p \ge N_c$ and denote by $\bar{u}_{t_k}^{FH}(t_{fin},t_{in})$ the signal $u_{t_k}^{FH}(t)$ in the interval $t \in [t_{in},t_{fin})$.

For system (5), (6) the MPC technique is applied to enlarge the output admissible set of $\kappa(\cdot,\cdot)$ and to improve the control performance by solving the following

Finite Horizon Optimal Control Problem (FHOCP). Given the sampling time T_s, the control horizon N_c, the prediction horizon N_p, $N_c \le N_p$, two positive definite matrices Q and R, the reference signal duration $N_r \le N_p$, a feasible auxiliary control law $\kappa(x,\bar{y}^0)$, the matrix Π and the region $\Omega_\sigma(\bar{x}(\bar{y}^0),\kappa,T_s)$ given in Lemma 4 with $\gamma > c_{\max} := (\lambda_{\max}(Q)L_h + \lambda_{\max}(\Xi' R\Xi))$, $\gamma_2 > T_s\lambda_{\max}(R)L_\kappa$, at every sampling time instant t_k, minimize, with respect to $\bar{u}_{1,N_c}(t_k)$,

$$\begin{aligned}&J_{FH}(x_{t_k},\bar{u}_{1,N_c}(t_k),N_c,N_p,y^0(\cdot)) \\ &= \int_{t_k}^{t_{k+N_p}} \left\{\|e(\tau)\|_Q^2 + \|u(\tau) - \Xi x(\tau)\|_R^2\right\} d\tau \\ &+V_f(\varphi(t_{k+N_p},t_k,x(t_k),\bar{u}_{t_k}^{FH}(t_{k+N_p},t_k),\bar{y}^0))\end{aligned} \tag{15}$$

where the terminal penalty V_f is selected as

$$V_f(x) = \left\|x - \bar{x}(\bar{y}^0)\right\|_\Pi^2$$

The minimization of (15) must be performed under the following constraints:

(i) the state dynamics (5)-(6) with $x(t_k) = x_{t_k}$;
(ii) the constraints (2), $t \in [t_k,t_{k+N_p})$ with u given by (14);
(iii) the terminal state constraint

$$x(t_{k+N_p}) \in \Omega_\sigma(\bar{x}(\bar{y}^0),\kappa,T_s)$$

■

The state-feedback MPC control law

$$u(t) = \kappa^{RH}(x(t_k),y^0(\cdot)) \quad , \quad t \in [t_k,t_{k+1}) \tag{16}$$

is then derived by solving $FHOCP$ at every sampling time instant t_k, and applying the constant control signal $u(t) = u_1^o(x(t_k))$, $t \in [t_k,t_{k+1})$ where $u_1^o(x(t_k))$ is the first column of the optimal sequence $\bar{u}_{1,N_c}^o(x(t_k))$.

Let

$$\varphi^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) = \begin{bmatrix}\varphi_{x_p}^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_I^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_{uv}^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_u^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot))\end{bmatrix} = \begin{bmatrix}\varphi_x^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot)) \\ \varphi_u^{RH}(t,\bar{t},x_c(\bar{t}),y^0(\cdot))\end{bmatrix}$$

$\varphi_{x_p}^{RH} \in R^n$, $\varphi_I^{RH} \in R^p$, $\varphi_{uv}^{RH} \in R^m$, $\varphi_x^{RH} \in R^{n+p+m}$, $\varphi_u^{RH} \in R^m$ be the movement of (8) with $\kappa(\cdot,\cdot) = \kappa^{RH}(\cdot,\cdot)$ and define the following sets.

Definition 3. Let $X_s^0(N_c,N_p,y^0) \in R^{n+m+p}$ be the set of states x_{t_k} of system (5), (6) at the sampling times t_k such that there exists a feasible control sequence $\bar{u}_{1,N_c}(t_k)$ for $FHOCP$.

Definition 4. Let $X^0(t,N_c,N_p,y^0) \in R^{n+p+2m}$ be the set of states x_c such that for all $x_c(t) \in X^0(t,N_c,N_p,y^0)$, $\varphi_x^{RH}(t_k,t,x_c,y^0) \in X_s^0(N_c,N_p,y^0)$, $\varphi_{x_p}^{RH}(\tau,t,x_c,y^0(\cdot)) \in X_p$, $\tau \in [t,t_k)$, $\varphi_u^{RH}(t,t,x_c,$

$y^0(\cdot)) \in U$ where t_k is the closest sampling time in the future. ■

The main stability results of the proposed MPC algorithm can now be stated.

Theorem 5. Under Assumptions 1, 2 and 3,

(i) $([\bar{x}(\bar{y}^0, \kappa)' \; \kappa^{RH}(\bar{x}(\bar{y}^0, \kappa), \bar{y}^0)' \;]', \bar{y}^0)$ is an exponentially stable equilibrium point for the closed-loop system formed by (5), (6) and (16) with output admissible set $X^0(t, N_c, N_p, y^0)$;
(ii) $X_s^0(N_c, N_p{+}1, y^0) \supseteq X_s^0(N_c, N_p, y^0), \forall N_c, N_p$;
(iii) $X_s^0(N_c, N_p, y^0) \supseteq \Omega_\sigma(\bar{x}(\bar{y}^0), \kappa, T_s), \forall N_c, N_p$;
(iv) there exist a finite $\bar{N}_p$ such that $X_k^0(N_c, \bar{N}_p, y^0) \supseteq X_s^c(\bar{x}(\bar{y}^0), \kappa, T_s), \forall N_c$.

Remark 1. An usual way to solve the tracking problem is to compute the reference trajectories of the state and control variables corresponding to the reference signal and to resort to a proper coordinate transformation. However, this procedure does not guarantee that asymptotic zero error regulation is preserved for modelling errors or plant parameters variations. On the contrary, the proposed method is such that the error asymptotically vanishes even in the presence of a plant-model mismatch provided that stability is preserved, although the computation of the allowed uncertainty is usually difficult. This property is due to the integrators directly applied on the error signal. In fact, if the feasibility of the $FHOCP$ and the asymptotic stability are preserved, then the input to the integrators must go asymptotically to zero. ■

In the $FHOCP$ optimization problem continuous time state constraints are considered. It can appear that this approach is only conceptual, because any numerical implementation needs a time discretization and the constraints satisfaction can be checked only in the integration time instants. However this is not a significant limitation; in fact, following Theorem 3 in (Magni and Scattolini, 2004) one can choose the maximum integration step δ and a more conservative discrete-time state constraint so as to guarantee continuous-time state constraint satisfaction.

3. SIMULATION EXAMPLE

In this section, the MPC control law is applied to a continuous fermenter. The volume of the fermenter is assumed constant, its contents well-mixed, and the feed sterile. The manipulated inputs are the dilution rate D and the feed substrate concentration S_f. The state variables are the effluent cell-mass or biomass concentration X_b, the substrate concentration S and the product concentration P. In the sequel, X_b, S and P are assumed to be measurable.

Assuming that the fermenter culture consists of a single, homogeneously growing organism, a simple and widely used model is (Henson and Seborg, 1997)

$$\dot{X}_b = -DX_b + \mu X_b \tag{17}$$

$$\dot{S} = D(S_f - S) - \frac{1}{Y_{X_b/S}}\mu X_b \tag{18}$$

$$\dot{P} = -DP + (\alpha\mu + \beta)X_b \tag{19}$$

where

$$\mu = \frac{\mu_m(1 - \frac{P}{P_m})S}{K_m + S + \frac{S^2}{K_i}} \tag{20}$$

is the specific growth rate, $Y_{X_b/S}$ is the cell-mass yield, and α and β are yield parameters for the product. In (20) the maximum specific growth rate μ_m, the product saturation constant P_m, the substrate saturation constant K_m, and the substrate inhibition constant K_i must be chosen to fit experimental data. The nominal operating conditions and model parameters are $Y_{X_b/S} = 0.4\ g/g$, $\beta = 0.2\ h^{-1}$, $P_m = 50\ g/L$, $K_i = 22\ g/L$, $D = 0.202\ h^{-1}$, $S = 5.0\ g/L$, $\alpha = 2.2 g/g$, $\mu_m = 0.48\ h^{-1}$, $K_m = 1.2\ g/L$, $S_f = 20 g/L$, $X_b = 6.0\ g/L$, $P = 19.14\ g/L$.

The control objective is to move the biomass concentration X_b along a pre-defined trajectory within a large range. The responses to step changes of the dilution rate are not symmetrical. The model exhibits more severe nonlinear behavior for changes in the feed substrate concentration. In particular the gain from S_f to X_b can ever change sign. Different single-input/single-output (SISO) control strategies have been proposed in order to control this class of systems (Henson and Seborg, 1997). However, the open loop behavior shows that with a single input control strategy is not possible to move the biomass concentration X to a value greater than $7.4\ g/L$. This motivates the interest for a multi-input/single-output control law. In the following the MPC algorithm proposed in this paper will be used to control the plant along a pre-prescribed reference bringing the biomass concentration X_b from the initial equilibrium value to the final steady state value of $7.5\ g/L$. The following input and state constraints are considered: $0.05h^{-1} \leq D \leq 0.3h^{-1}$, $16 g/L \leq S_f \leq 25 g/L$, $3 g/L \leq X_b \leq 10 g/L$, $1 g/L \leq S \leq 10 g/L$, $10 g/L \leq P \leq 35 g/L$. The nonlinear continuous-time state space model (1) of system (17)-(20) is obtained by defining the normalized state vector $x_p = [\frac{X_b - 6}{6}, \frac{S-5}{5}, \frac{P-19.14}{19.14}]'$, the manipulated input $u = [\frac{D-0.202}{0.202}, \frac{S_f-20}{20}]'$ and the output $y = \frac{X_b-6}{6}$. The initial equilibrium point is defined by $\bar{x} = 0$, $\bar{u} = 0$ and $\bar{y} = 0$, while the linear auxiliary stabilizing control law is given by

$$u(t) = K_x x(t_k) + K_{y^0}\Delta Y^0(t_k) \tag{21}$$

where K_x and K_{y^0} are obtained with an MPC control law synthesized on the linearization of (5)-(6) around $(x,\ u,\ y^0) = (0,0,0)$ discretized with a sampling period $T_s = 1h$. Finally, $\Delta Y^0(t_k) := [y^0(t_k),\ y^0(t_{k+1}), \ldots, y^0(t_{k+N_p^L})]'$ and $N_p^L = 30$ is the prediction and control horizon of the linear MPC. The cost function minimized to synthesize the linear MPC has the same stage cost of (15) with an additional terminal equality constraint and a penalty on the state variable with matrix Q_x to guarantee the stability of the linearized closed-loop system. Letting $Q = 1$, $R = diag(1,1)$, $Q_x = diag(1,1,1,3,1,1) * 10^{-3}$, $\gamma = 2$, $\gamma_2 = 0.99$, $\tilde{Q} = 2.02 * I_{n+p+m}$ and $\bar{y}^0 = 0.25$ (which corresponds to $X_b = 7.5\ g/L$) a region $\Omega_\sigma(\bar{x}(\bar{y}^0), \kappa, T_s)$ satisfying Lemma 4 is computed with the Π solution of (11) and $\sigma = 0.027$. In order to guarantee continuous time state-constraints satisfaction, following Theorem 3 in (Magni and Scattolini, 2004), the constraints $\|x_p - \bar{x}_p\|_\nu \leq \bar{g}$ with $\bar{x}_p = [0.0833, 0.1000, 0.1755]$, ν = diag(2.9388,1.2346,2.3446), $\bar{g} = 0.9$ are introduced so that, with a constant integration step $\delta = 0.05h$, $x_p(t) \in \left\{x_p : \|x_p - \bar{x}_p\|_\nu \leq 1\right\} \subseteq X_p, t > 0$. The sampled MPC control law described in Section 2.1 has been synthesized letting $Q = 1$, $R = diag(1,1)$, $N_c = 6$ and $N_p = 200$.

The results obtained by comparing the auxiliary linear MPC and the nonlinear MPC algorithm here proposed are reported in Fig. 1. Fig. 1.a shows that both the methods allow to reach the required steady-state value $X_b = 7.5\ g/L$. However, the nonlinear MPC law achieves a significant performance improvement. In fact the infinite horizon cost obtained with the linear MPC law is 0.2447 while with the nonlinear one is 0.1390. Note also that as stated in Theorem 5, the state and control variables with nonlinear MPC converge to the same steady-state values of the auxiliary linear control law.

A second simulation experiment has been performed in order to emphasize the robustness property of the control scheme with respect to model uncertainty. To this end, the parameter μ_m has been changed from the nominal value $0.48h^{-1}$ to the perturbed value $0.45h^{-1}$ at time $10h$, while the set point is constant at $6g/L$. The results obtained with the control algorithm based on the solution of the regulation problem through a standard change of coordinates (see Remark 1) and with the tracking control algorithm based on the solution of the $FHOCP$ are reported in Fig. 2. It is apparent that the introduction of the integral action guarantees robust asymptotic zero error regulation.

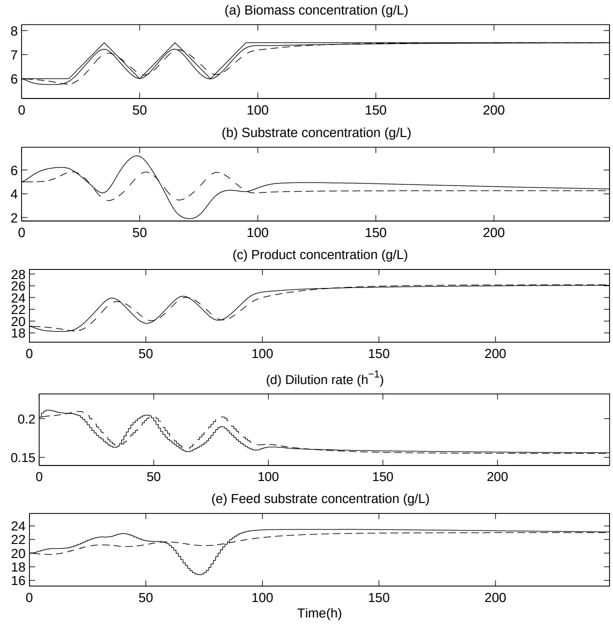

Fig. 1. Responses of closed-loop system with linear MPC (dashed line) and nonlinear MPC (solid line)

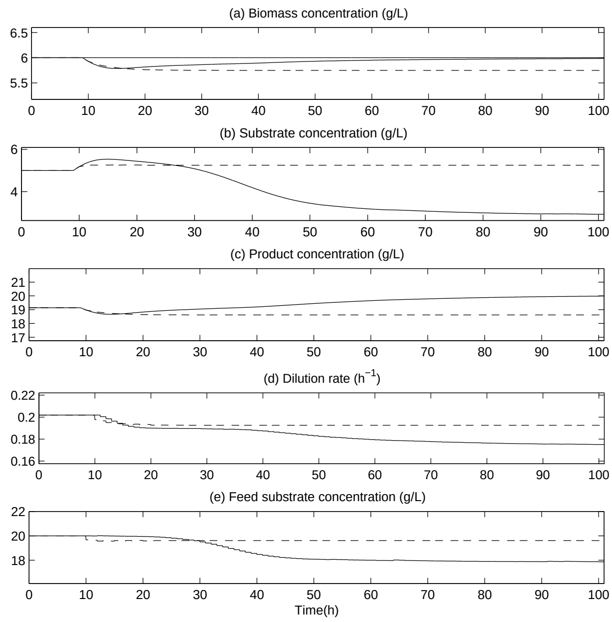

Fig. 2. Closed-loop system responses with nonlinear MPC based on a pure regulation algorithm (dashed line) and on the tracking algorithm (solid liner) when a change of μ_m occurs

REFERENCES

Davison, E. J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Transactions on Automatic Control* **21**, 25–34.

Henson, M. A. and D. E. Seborg (1997). Feedback linearizing control. In: *Nonlinear Process Control* (M. A. Henson and D. E. Seborg, Eds.). pp. 149–231. Prentice Hall.

Magni, L. (2002). On robust tracking with nonlinear model predictive control. *International Journal of Control* **75**, 399–407.

Magni, L. and R. Scattolini (2004). Model predictive control of continuous-time nonlinear systems with piecewise constant control. *IEEE Transactions on Automatic Control* **49**, 900–906.