PID CONTROLLER DESIGN FOR MULTIVARIABLE SYSTEMS USING GERSHGORIN BANDS

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Abstract: A method to design decentralized PID controllers for MIMO systems is presented in this paper. Each loop is designed separately, but the Gershgorin bands are considered to take interactions into account. The method uses different design parameters: The infinity norm of the complementary sensitivity function as well as the crossover frequency are considered to represent the closed-loop system performances. A third design parameter, defined as the minimal distance from the critical point to the Gershgorin band is used to provide the desired stability robustness to the MIMO closed-loop system.

Keywords: Multiloop control, PID controllers, robust control

1. INTRODUCTION

PID controllers are considerably used in industrial processes, because their structure, consisting of only three parameters is very simple to implement and many different techniques are nowadays available for their tuning for SISO systems.

Many systems encountered in practice consist, however, of several interconnected loops. Classical MIMO techniques solve usually the controller design problem successfully. Their drawback consists mainly in the fact that the results are state-space high-order controllers. Moreover, systems containing non negligible time-delays cannot be handle by such procedures.

On the other hand, considerable attention has been given to the use of SISO procedures for the tuning of decentralized PID controllers for MIMO systems. Motivations comes from the fact that many systems can be made diagonally dominant (i.e. interactions between loops are not predominant) by designing appropriated decoupling compensators. Furthermore the stability of MIMO systems in closed-loop can be directly taken into account by SISO approaches thanks to the Gershgorin bands. W.K. Ho *et al.* (1997) proposed analytical formulas for the design of multiloop PID controllers by specifying the gain and phase margins for the Gershgorin bands. But this approach is restricted to a particular model structure. In D. Chen and D. E. Seborg (2002) the ultimate gain and frequency are defined for MIMO systems based on Gershgorin bands and a design method is derived from the modified Ziegler-Nichols rules. The approach suffers from the need of a full model knowledge to compute the ultimate point, and it finally uses only this information to design the controller. Furthermore the stability of the closed-loop system is not guaranteed.

The proposed approach, derived from the procedure presented in Garcia *et al.* (2005) to adjust robust PID controller for SISO systems, uses the following design parameters: The infinity norm of the complementary sensitivity function and the crossover frequency, which are specified for each loop independently. These represent the closed-loop performances. The minimal distance from the critical point to the Gershgorin bands is also

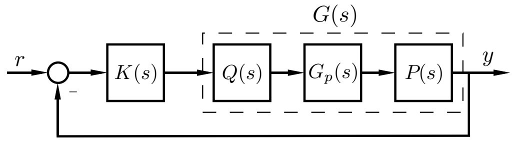

Fig. 1. Classical multi-loop control system configuration

specified. This guarantees the desired stability robustness of the MIMO system. The problem which consists of finding the controller parameters to satisfy the specifications is then solved by minimizing iteratively a frequency criterion (Karimi *et al.*, 2003). This one is defined as the weighted sum of squared errors between the achieved and specified values of the design parameters.

The paper is organized as follows: The configuration of decentralized feedback control system with decoupling compensators is presented in Section 2. Stability analysis considerations for diagonally dominant systems are recalled in Section 3. In Section 4 the controller design procedure is exposed and some examples are provided in Section 5. Finally some concluding remarks are offered in Section 6.

2. SYSTEM CONFIGURATION

Notation: In this paper, the element in the row i and column j of a transfer matrix $L(s)$ is indicated by $l_{ij}(s)$.

The classical configuration of a multi-loop (decentralized) feedback control system is shown in Fig. 1. $K(s)$, $Q(s)$, $G_p(s)$ and $P(s)$ are $m \times m$ transfer function matrices. $G_p(s)$ describes the process transfer matrix, $K(s) = \text{diag}\{k_1(s), \ldots, k_m(s)\}$ is a diagonal matrix of controller transfer functions, $Q(s)$ and $P(s)$ stand for the transfer matrices of the precompensator and postcompensator, respectively. Compensators are used in order to decouple the loops, so that the overall control can still be obtained by independent SISO design of diagonal loops. If $G(s) = P(s)G_p(s)Q(s)$ is diagonal, the system will consist of a number of independent SISO diagonal control loops, each of them can be designed independently by classical techniques. However, aside from conditions on the existence, stability and causality, the decoupling compensators tend to be of the same order of complexity of the plant itself. Moreover, exact decoupling (which implies the knowledge of an exact plant model) means that the compensator is used to cancel dynamics of $G_p(s)$. These cancelled modes will still exist in the presence of disturbances and could be uncontrollable. In view of these difficulties, compensators are designed only in order to limit interactions between loops and to obtain diagonal dominancy. This represents an interesting property under which interactions are reduced sufficiently and allows to design the controller by considering each loop independently. Diagonal dominancy can usually be achieved by matrices Q and P consisting of constant elements (Van de Vegte, 1994). Numerous techniques are nowadays available for their design.

3. STABILITY ANALYSIS OF DIAGONALLY DOMINANT SYSTEMS

Nyquist array analysis, which has been investigated by Rosenbrock (1970), provides useful theoretical basis for stability analysis and controller design for diagonally dominant systems. This considerations are based on the stability theorem for MIMO systems in the frequency domain, which is repeated hereafter for convenience.

Consider the closed-loop system of Fig. 1, define $L(s)$ as the loop transfer matrix, and $D(s)$ as the return difference transfer matrix:

$$L(s) = G(s)K(s) \tag{1}$$

$$D(s) = I + G(s)K(s) \tag{2}$$

It can be shown (Van de Vegte, 1994) that, in the similar way as for SISO systems, the numerator of the determinant of $D(s)$ is the closed-loop characteristic polynomial while its denominator constitutes the open-loop characteristic polynomial. Assume that p_0 is the number of roots of the open-loop characteristic polynomial inside the Nyquist contour C. The latter consists of the imaginary axis and a right semicircle of radius $R \to \infty$ and, in effect, encloses the entire right half-plane. The basic stability theorem follows from the principle of the argument:

Theorem 1. If a plot of $\det(D(s))$ as s travels once clockwise around the Nyquist contour C encircles the origin N_d times clockwise, the system is stable if and only if $N_d = -p_0$.

This stability theorem is however difficult to apply for the design of multivariable systems. On the other hand, in the special case of diagonally dominant matrices, the stability condition can be expressed in a much more convenient way for controller tuning using Nyquist array technique.

Definition 1. An $m \times m$ transfer matrix $Z(s)$ is column diagonally dominant on the Nyquist contour C if $\forall s \in C$ and $\forall i = 1, \ldots, m$:

$$|z_{ii}(s)| > r_i(s) = \sum_{j=1, j \neq i}^{m} |z_{ji}(s)| \tag{3}$$

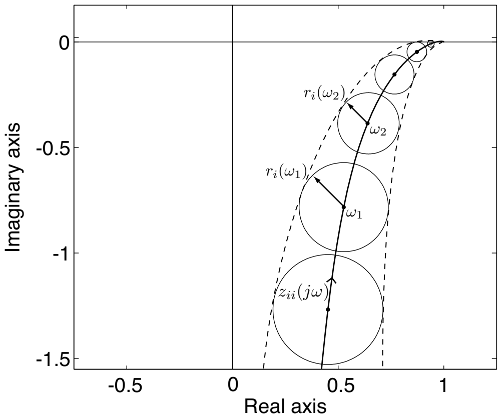

Fig. 2. Nyquist plot of a diagonal element of $Z(s)$ with the Gershgorin bands

A graphical interpretation of this condition is based on the Gershgorin bands: $Z(s)$ is column diagonally dominant if the Nyquist plots of the diagonal elements $z_{ii}(s)$, with the band generated by the circles of radii $r_i(s)$ and centered of $z_{ii}(s)$ at the corresponding frequencies exclude the origin. Fig. 2 illustrates this interpretation.

Stability analysis for diagonally dominant system depends directly of the following theorem:

Theorem 2. For a diagonally dominant matrix $Z(s)$, the origin encirclements N_z of $\det(Z(s))$ as s travels once clockwise around the Nyquist contour equal the sum of the encirclements N_{zi} of the diagonal elements z_{ii}

$$N_z = \sum_{i=1}^{m} N_{zi} \tag{4}$$

The Nyquist stability theorem (Rosenbrock, 1970), that follows directly from the preceding theorem can now be expressed for the considered closed-loop system of Fig. 1:

Theorem 3. If the Gershgorin bands centered on the $d_{ii}(s)$ (diagonal elements of the return difference transfer matrix) exclude the origin (i.e. $D(s)$ is column diagonally dominant), the system is stable if and only if:

$$\sum_i (\text{clockwise encirclements of } d_{ii}(s) \text{ about the origin}) = -p_0$$

Since I is a diagonal matrix, the Gershgorin circle radii of $D(s) = I + G(s)K(s)$ are the same as those of $L(s) = G(s)K(s)$. It follows, that the preceding theorem can be formulated in a form that resembles the classical Nyquist criterion for SISO system: **If the Gershgorin bands centered on the $l_{ii}(s)$ exclude the critical point -1 (i.e. D(s) is diagonally dominant), the system is stable if and only if :**

$$\sum_i (\textbf{clockwise encirclements of } l_{ii}(s) \textbf{ about the critical point } -1) = -p_0$$

The radii of the circle that generate the bands are:

$$r_i(s) = \sum_{j=1, j\neq i}^{m} |l_{ji}(s)|\,, \quad \forall i = 1, \dots, m \tag{5}$$

Remark 1: This Nyquist array analysis represents only sufficient, but not necessary stability conditions. If the bands overlap the critical point (i.e. $D(s)$ is not diagonally dominant), conclusions about the stability or instability of the closed-loop system cannot be made.

Remark 2: Note that the time-delays of the functions $g_{ij}(s)$ $(i \neq j)$ are not involved in the stability analysis: If the preceding theorem is satisfied for a given system, changing the time delays values in any transfer functions $g_{ij}(s)$ $(i \neq j)$, will not affect the closed loop stability.

Since most industrial processes are open-loop stable, the controller design procedure will be restricted to those systems. Assuming open-loop stability, the Gershgorin bands must not encircle nor include the the critical point -1 to ensure closed-loop stability.

4. CONTROLLER DESIGN PROCEDURE

Consider the closed-loop system of Fig. 1 and assume that the transfer matrix $G(s)$ is diagonally dominant. Thus the interactions between loops are sufficiently reduced so that the controller can still be obtained by independent SISO design of the diagonal loops (Van de Vegte, 1994). That is what the proposed controller design method does. But for stability considerations, it also takes into account the interactions with the Gershgorin bands, because these bands provide about the same type of stability information for MIMO systems as the Nyquist diagram does for SISO systems. The procedure is derived from the method proposed in Garcia *et al.* (2005), where appropriate design parameters are chosen for PID controller design and bands are also considered in the complex plane for the stability robustness against the model uncertainties. Moreover the method does not require any parametric plant models. Only the knowledge of a non-parametric transfer matrix $G(j\omega)$ in a frequency range is necessary for the design.

4.1 Design Parameters

The design parameters used by the method are the following :

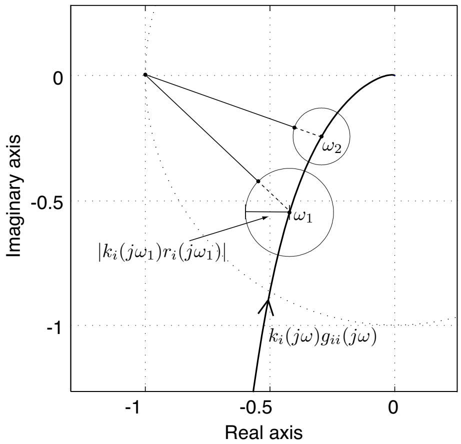

Fig. 3. Nyquist plot of a diagonal element of $K(s)G(s)$ with circles that generate the Gershgorin bands

- Modulus margin M_m: For the i-th loop which is characterized by the loop transfer function $l_{ii}(j\omega) = k_i(j\omega)g_{ii}(j\omega)$, the modulus margin M_{m_i} is defined as the minimal distance from the critical point -1 to the Gershgorin band of the loop: Since the band is generated by circles, an analytical expression of the band can be given by:

$$k_i(j\omega)g_{ii}(j\omega) + |k_i(j\omega)r_i(j\omega)|e^{j\theta}, \quad (6)$$

with $\theta \in [0, 2\pi)$ and $\omega \in [0, \infty)$. M_{m_i} can thus be formulated as:

$$M_{m_i} = \inf_{\omega} \Big(\inf_{\theta} |1 + k_i(j\omega)g_{ii}(j\omega) + |k_i(j\omega)r_i(j\omega)|\, e^{j\theta}| \Big) \quad (7)$$

It can easily be seen on Fig. 3 that for a given frequency ω_1, the minimal distance from the critical point to the corresponding circle is equal to the distance from the critical point to $k_i(j\omega_1)g_{ii}(j\omega_1)$ minus the radius $|k_i(j\omega_1)r_i(j\omega_1)|$ of the circle. Hence:

$$M_{m_i} = \inf_{\omega} \Big(|1 + k_i(j\omega)g_{ii}(j\omega)| - |k_i(j\omega)r_i(j\omega)| \Big) \quad (8)$$

This term can easily be computed numerically. Satisfying a specified modulus margin for each loop ensures the desired robust stability of the MIMO closed-loop system. Moreover it gives an upper bound for the magnitude of the sensitivity functions of each loop.

- Complementary modulus margin M_c: Let the complementary sensitivity function of the i-th loop be $T_i(s) = \frac{l_{ii}(s)}{1+l_{ii}(s)}$, which represents the transfer function from setpoint to process output of the SISO system. The second design parameter, called the complementary modulus margin, is defined as being the inverse of the infinity-norm of $T_i(s)$:

$$M_{c_i} = ||T_i(s)||_{\infty}^{-1} \quad (9)$$

Its value is directly related to the maximum peak overshoot to a setpoint change of the closed-loop system and thus constitutes an important performance indicator. Moreover this specification can be directly interpreted in the complex plane, since the loci for constant complementary modulus margin are circles (Garcia *et al.*, 2005).

- Crossover frequency ω_c: The proposed method also allows the crossover frequency to be considered as a design parameter. For the i-th loop, ω_{c_i} is defined as the frequency at which the loop amplitude is one ($|l_{ii}(j\omega_{c_i})| = 1$). A specified value for the crossover frequency is however not a priori known and depends especially on the plant dynamics. If either the closed-loop bandwidth or the desired rise time to setpoint changes are approximatively known a specification can however be formulated.

4.2 Frequency Criterion

The problem which consists of finding the controller parameters in order to satisfy the specifications on the design parameters can now be formulated as an optimization: For each loop independently, find the controller parameters that minimizes a frequency criterion. The frequency criterion for the i-th loop J_i is defined as the weighted sum of squared errors between the specified and computed values of the design parameters:

$$J_i(\rho_i) = \frac{1}{2}\Big(\lambda_{1_i}(M_{m_i}(\rho) - M^*_{m_i})^2 + \lambda_{2_i} \times (M_{c_i}(\rho) - M^*_{c_i})^2 + \lambda_{3_i}(\omega_{c_i}(\rho) - \omega^*_{c_i})^2\Big) \quad (10)$$

where ρ_i is the vector of the controller parameters, λ_{1_i}, λ_{2_i} and λ_{3_i} are weighting factors, M_{m_i} and $M^*_{m_i}$ are respectively the achieved and specified values of the modulus margin. M_{c_i} and $M^*_{c_i}$ are the achieved and desired complementary modulus margin and ω_{c_i} and $\omega^*_{c_i}$ the achieved and desired crossover frequency. The weightings factors are usually chosen as:

$$\lambda_{1_i} = 1/M^{*2}_{m_i},\ \lambda_{2_i} = 1/M^{*2}_{c_i},\ \lambda_{3_i} = 1/\omega^{*2}_{c_i} \quad (11)$$

in order to normalize the terms in the criterion. It is assumed that the values of M_{m_i}, M_{c_i} and ω_{c_i} can be computed numerically using the plant model and the current controller transfer function. The controller parameters of each loop, minimizing the corresponding criterion can be obtained using the iterative Gauss-Newton algorithm. Details of the minimization procedure can be found in Garcia *et al.* (2005). The minimization is done numerically and does not requires a parametric model of the transfer matrix $G(s)$.

5. SIMULATION EXAMPLES

Two simulation examples are now considered to demonstrate the closed-loop performances of decentralized PID designed with the proposed method.

5.1 Example 1

Consider a MIMO process described by the following process transfer matrix G_{p1}:

$$G_{p1}(s) = \begin{pmatrix} \frac{8e^{-0.05s}}{4s^2+3s+2} & \frac{0.5e^{-s}}{(s+1)(2s+1)} \\ \frac{1}{(s+1)(s+2)} & \frac{-0.2s+1}{(s+1)^2} \end{pmatrix} \quad (12)$$

Because this process model has the property of column diagonal dominance, no decoupling compensators are required. It should be noted that the transfer functions on the diagonal of $G_{p1}(s)$ correspond to an oscillatory as well as a non-minimum phase system. this kind of systems often represents a problem when classical methods based on first-order plus dead time are used to tune the controller, because this model is not representative of the plant behavior.

From initial controllers obtained with the Kappa-Tau tuning rules (K. J. Aström and T. Hägglund, 1995) (by considering only the transfer functions on the diagonal of $G_{p_1}(s)$), it is now desired to adjust its parameters with the proposed method by taking into account the Gershgorin bands to ensure the stability of the MIMO system. The same specifications on design parameters are chosen for each loop: $M^*_{m_{1,2}} = 0.4$ is chosen for the minimal distances to the critical point and 1.05 for the maximal value of the complementary functions. This value corresponds to $M^*_{c_i} = 0.952$. No specification is however given on the crossover frequencies ($\lambda_{3_{1,2}} = 0$).

The following controller structure is used:

$$k_i(s) = K_{p_i}\left(1 + \frac{1}{T_{i_i}s} + \frac{T_{d_i}}{\frac{T_{d_i}}{20}s + 1}\right) \quad (13)$$

since it is usual to include a filter in the derivative term. Controllers having only 2 parameters can be sufficient to minimize the criteria. Thus, the number of controller parameters are set to two by choosing the constant ratio $T_{i_i} = 4T_{d_i}$ between integral and derivative times. It is pointed out in K. J. Aström and T. Hägglund (1995) that this ratio is appropriated for many industrial processes.

Nyquist plots of the two designed open-loop transfer function with the Gershgorin bands and prohibited disks defined by the specifications are shown in Fig. 4. It can be seen that the designed systems fulfill the design specifications. Closed-loop responses of the MIMO system with the proposed controllers are simulated and compared with those resulting from the method proposed in D. Chen and D. E. Seborg (2002). The simulation consists of unit set-point changes for the first output at $t = 0$ and for the second one at $t = 15$ s. Fig. 5 shows the responses of the first plant output y_1, while Fig. 6 represents the responses of the second one. It can be seen that the proposed controllers perform well. In particular settling-time and overshoot of the step responses are considerably reduced. Concerning the interactions between loops, the proposed controllers provide a better performances for the time of rejection. Overshoots due to the interactions are reduced in the first output but amplified in the second one. Details of the controllers settings are presented in Table 1.

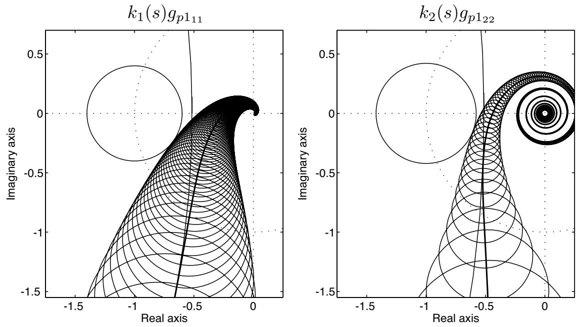

Fig. 4. Nyquist plots of the designed loops with Gershgorin bands

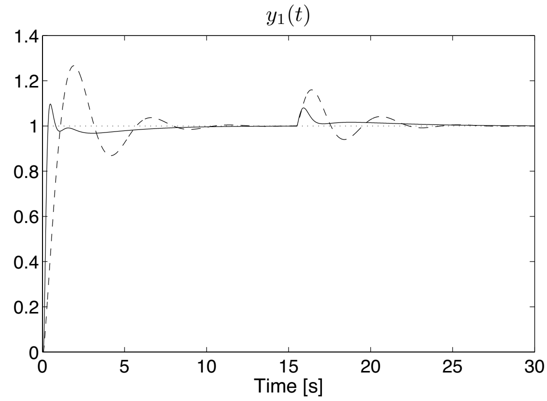

Fig. 5. Step response of the first output (dashed line: Chen, solid line: Proposed)

Controller	Method	K_p	T_i	T_d
$k_1(s)$	Chen	1.08	1.57	0.39
	Proposed	2.92	3.55	0.89
$k_2(s)$	Chen	2.24	1.65	0.41
	Proposed	2.62	2.06	0.51

Table 1. Controller parameters (ex. 1)

5.2 Example 2

The following third by third process model is now considered:

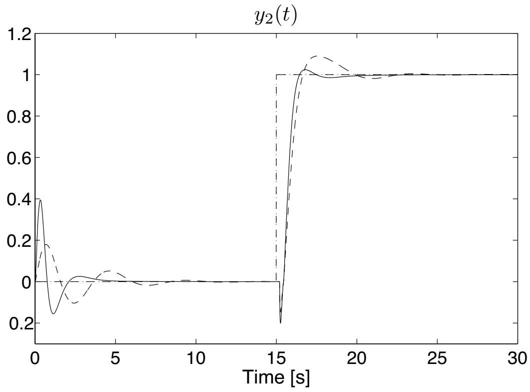

Fig. 6. Step response of the second output (dashed line: Chen, solid line: Proposed)

Controller	Method	K_p	T_i	T_d
$k_i(s)$	Chen	5.25	1.08	0.271
	Proposed	9.33	1.97	0.493

Table 2. Controller parameters (ex. 2)

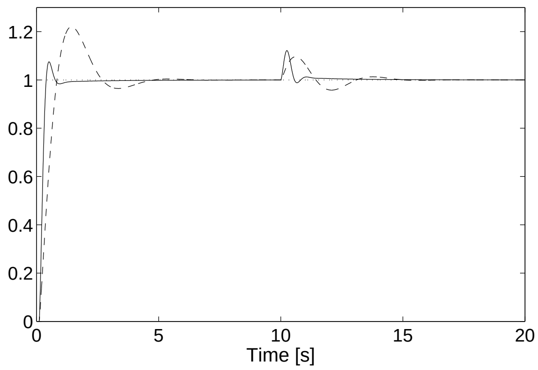

Fig. 7. Step response of the first output (dashed line: Chen, solid line: Proposed)

$$G_{p_2}(s) = \begin{pmatrix} \frac{e^{-0.1s}}{(s+1)^2} & \frac{0.3}{(s+1)(2s+1)} & \frac{0.3}{(s+1)(2s+1)} \\ \frac{0.3}{(s+1)(2s+1)} & \frac{e^{-0.1s}}{(s+1)^2} & \frac{0.3}{(s+1)(2s+1)} \\ \frac{0.3}{(s+1)(2s+1)} & \frac{0.3}{(s+1)(2s+1)} & \frac{e^{-0.1s}}{(s+1)^2} \end{pmatrix}$$

This transfer matrix is diagonally dominant and thus no decoupling compensators are required. Due to the symmetric structure of the system, the same controller will be used for each loop. The same values of design specifications are used as previously. Again, an initial controller is designed using the Kappa-Tau tuning rule and then the proposed method is used to adjust its parameters in order to satisfy the design specifications.

The resulting controller (Table 2) is compared with that obtained by the method of D. Chen and D. E. Seborg (2002): Fig. 7 shows the behavior of the first system output $y_1(t)$ for a unit setpoint change of the first output at $t = 0$ and a unit setpoint change of the second output at $t = 10$ s. Since the process is symmetric, other system outputs will be identical. Again the proposed controller perform well, since it reduces drastically step response overshoot as well as settling time. The disturbance rejection overshoot is, on the other hand slightly increased. Finally it should be remarked that both methods require the same information about the plant model (i.e a non-parametric model of the system).

6. CONCLUSION

A controller design method has been proposed for decentralized PID control systems. The approach is restricted to diagonally dominant MIMO systems or systems that can be made diagonally dominant by using decoupling compensators. But since no other assumptions have been made, it is not restricted to any particular models nor controller structures. The design procedure considers each loop separately for the closed-loop performances but also takes the Gershgorin bands into account to ensure a stability robustness of the closed-loop MIMO system. Simulation examples illustrate the effectiveness of the method for controller design of moderately interacting systems.

ACKNOWLEDGEMENTS

This research work is financially supported by the Swiss National Science Foundation under grant No. 2100-064931.01

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