BAYESIAN APPROACH TO MODELLING OF QUASI-PERIODIC INTERMITTENT DEMAND

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Abstract: The paper focuses on the stochastic modelling of the quasi-periodic intermittent demand patterns, which arise in the inventory management of the "slow moving items" such as service parts or high-priced capital goods. It is proposed a new stochastic model, which describes the demand patterns with essentially non-exponential distribution of the inter-arrival times. The model is based on generalized beta-binomial distribution and the Bayesian inference using the historical data array describing the demand repeatability within the time periods. For this model, there were derived explicit expressions for the forecast distributions, its moments and relevant Bayesian risk. The efficiency of the proposed approach is confirmed by computer simulation and an application example. *Copyright © 2005 IFAC*

Keywords: inventory control, demand modelling, statistical inference, parameter estimation, prediction methods.

1. INTRODUCTION

Accurate demand forecasting is a principal component of supply chain management allowing companies to balance service levels against investment over large assortment of stock-keeping units (SKU) (Harris, 1997; Silver *et al*., 1998). As follows from previous research (Snyder, 2002), large portion of SKUs is usually made up by "slow moving items" with irregular (intermittent) demand patterns, i.e. random sequences with a large proportion of zeros and great variability among the remaining nonzero integer values (Willemain *et al*., 2004). This paper addresses a special case of the intermittent demand possessing quasi-periodic regularity typical for many supply planning systems, from retail sales to manufacturing of high-priced capital goods.

The first heuristic technique for the intermittent demand forecasting was developed by Croston (1972) who applied separate exponential smoothing to both the non-zero demand sizes and inter-arrival times between successive demands. Later, the problem has been addressed by a number of authors (Johnston and

Boylan, 1996, Syntetos and Boylan, 2001) who concentrated on improving the forecast accuracy and proposed several modifications of the Croston's method, including EWMA-smoothing and log-transformation of the data (both the demand values and inter-arrival times) in order to avoid the positive-space constraint. However, the corresponding model variables were still deterministic and defined on a continuous space. Consequently, all of these methods do not produce forecast distributions and associated prediction intervals (Shenstone and Hyndman, 2003).

The first stochastic demand models were based on the Poison or NBD (negative-binomial) distribution, they originated from research on stochastic inter-purchase times and proved to be very accurate in fitting of the aggregated data describing frequently purchased goods (Dunn et al., 1983; Wagner and Taudes, 1987; Gupta, 1991; Agrawal and Smith, 1996, Grange 1998). However, their basic assumption on exponential (or gamma-exponential) distribution of the inter-arrival times does not allow to take into account quasi-periodicity in purchases/visits, which becomes a vital issue for personalization of marketing decisions. To overcome this problem, Telang *et al.* (2004) proposed recently a hierarchical probabilistic model of user's repeat visits that incorporates Weibull, Laplace and double-exponential distribution mixture to account for users' schedule. This model was successfully applied to forecasting of visits to massively popular Internet web sites, but it seems to be hardly applied to intermittent demand modelling, which possesses the above mentioned specificities.

In this paper, the problem of quasi-periodicity is solved within Bayesian framework, using the generalized beta-binomial demand distribution (GBBD) developed in our previous paper (Dolgui *et al.*, 2004). The demand repeatability is described by a historical data array corresponding to a typical time period, which also allows generating personalised forecasts using non-aggregated demand data.

1. QUASI- PERIODIC DEMAND MODEL

Let us assume that the intermittent demand patterns posses natural regularity over the time period T (24 hours, a week, etc.), the examined time interval $[0, kT]$ includes k periods, and the period T is uniformly divided in m time segments (1 hour, a day, etc.). Following our previous paper (Dolgui *et al.*, 2004), let us also assume that, within these segments, the demand data are arranged in an integer matrix $\{s_{ij},\ i=1\ldots k,\ i=1\ldots m\}$, where each observed value s_{ij} is presented as a sum of n binary random variables $s_{ij}=\sum_{l=1}^{n} b_{ijl}$ obtained via the Bernoulli trials with the random probabilities p_j following the generalized beta distribution

$$f(p)=\frac{1}{B(\alpha_j,\beta_j)\,(\pi_1-\pi_0)}\cdot\left(\frac{p-\pi_0}{\pi_1-\pi_0}\right)^{\alpha_j-1}\left(\frac{\pi_1-p}{\pi_1-\pi_0}\right)^{\beta_j-1}\qquad(1)$$

where $B(\alpha,\beta)$ is the complete beta function; α,β and π_0,π_1 are the shape and range distribution parameters respectively. It should be noted that the above formulation accounts the quasi-periodicity via the parameter arrays $\{\alpha_j, \beta_j, j=1{:}m\}$ defined and varying on the time period $[0, T]$, however below we also apply the Bayesian framework based on the historical data averaged for the similar time periods.

Under such assumptions, the columns of the demand matrix $\{s_{ij}\}$ obey the generalized beta-binomial demand distribution, which is computed using the above proposition, where $P_r^{(j)}$ is the probability that the demand for the j-th segment is equal to r.

Proposition 1. *For the general-beta prior, the probability distribution of the intermittent demand can be represented as the weighted sum of the shifted beta-binomial pdfs*

$$P_r^{(j)}=\sum_{l=0}^{n} w_{rl}\cdot\binom{n}{l}\frac{B(\alpha_j+l,\beta_j+n-l)}{B(\alpha_j,\beta_j)}\qquad(2)$$

where $P_r^{(j)}\in[\pi_0,\pi_1]$ *is the probability that the demand value for the j-th segment of the period is equal to r, and the weights are computed as*

$$w_{rl}=\sum_{s=\max(r,l)}^{\min(n,r+l)}\binom{l}{s-r}\cdot\binom{n-l}{s-l}\cdot\pi_0^{s-l}(1-\pi_0)^{n-s}\pi_1^{l+r}(1-\pi_1)^{s-r}$$

Proof of Proposition 1 is based on application of the hypergeometric expansion and properties of the complete beta function. First, using the linear transformation $p=\pi_0+\Delta\cdot\theta$ with the standard-beta random variable $\theta\in[0,1]$ and the width parameter $\Delta=\pi_1-\pi_0$, the expression for the marginal distribution of r is rewritten as

$$P_r=\frac{1}{B(\alpha,\beta)}\cdot\binom{n}{r}\cdot\int_0^1(\pi_0+\Delta\cdot\theta)^r(\rho_0-\Delta\cdot\theta)^{n-r}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta$$

where $\rho_0=1-\pi_0$ and the subscripts "j" are omitted for brevity. Then, the product $(\pi_0+\Delta\cdot\theta)^r(\rho_0+\Delta\cdot\theta)^{n-r}$ is expanded into the sum $\sum_{l=0}^{n} w_{rl}\,\theta^l\,(1-\theta)^{n-l}$, where the coefficients w_{rl} are using the Binomial theorem. Finally, to derive Eq.(2), the obtained sum of the integrals is expressed via the beta-functions. For computing convenience, the ratios of the beta-functions $B(\alpha_j+l,\beta_j+n-l)/B(\alpha_j,\beta_j)$ can be further simplified and Eq.(2) can be rewritten as

$$P_r^{(j)} = \sum_{l=1}^{n} w_{rl} \cdot \binom{n}{l} \frac{\prod_{s=0}^{l-1}\left(\alpha_j + s\right)\prod_{s=0}^{l-1}\left(\beta_j + n - s\right)}{\prod_{s=0}^{n-1}\left(\alpha_j + \beta_j + s\right)} \quad (3)$$

Using the same approach, the demand mean $E(r)$ can be expressed via the conditional expectation as $E_\theta[E(r|\theta)] = n\pi_0 + \Delta \cdot E(n\theta)$, that after substitution of $E(\theta)$, yields the following average demand value

$$E(r) = \pi_0 \cdot n\beta_j/(\alpha_j + \beta_j) + \pi_1 \cdot n\alpha_j/(\alpha_j + \beta_j). \quad (4)$$

where $n\alpha/(\alpha+\beta)$ and $n\beta/(\alpha+\beta)$ can be interpreted as the average success/failure number for the standard BBD-model. Similarly, the demand variance $V(r)$ be expressed via the conditional mean and variance as $V(r) = V_\theta(np|\theta) + E_\theta(np(1-p)|\theta)$ that after computing of the conditional components and substitution of $E(\theta)$ and $V(\theta)$ yields

$$V(r) = V_0 + \pi_0(1-\pi_0) \cdot \frac{n\beta_j}{\alpha_j+\beta_j} + \pi_1(1-\pi_1) \cdot \frac{n\alpha_j}{\alpha_j+\beta_j}, \quad (5)$$

where the first term

$$V_0 = \Delta^2 \cdot n\alpha_j\beta_j(\alpha_j+\beta_j+n_j)/(\alpha_j+\beta_j)^2(\alpha_j+\beta_j+1)$$

can be treated as the weighted variance of the standard beta-binomial distribution with the scale factor equal to the square range of p, and the remainder ones are the weighted variances of the binomial distribution with parameters π_0, π_1.

3. ESTIMATION OF MODEL PARAMETERS

The proposed demand model includes two types of parameters, the shape parameters α_j, β_j varying over the period T and the range parameters π_0, π_1, which are assumed to be similar for all the time segments. To estimate them, let us apply the MM technique combined with the minimisation of the Pirson's χ^2-statistics, sequentially considering cases of known and unknown π_0, π_1 (such approach simplifies the general identification procedure, which adjusts the range parameters in the outer loop while the inner loop tunes the shape parameters).

If the probability range $[\pi_0, \pi_1]$ is assumed to be known, the remaining model parameters α, β should ensure equality of the mean π^* and variance δ^* for the model and normalised demand data $\{s_{ij}/n\}$, i.e. $\pi^* = E(r)/n$; $\delta^* = V(r)/n$, where the explicit expressions for $E(r)$ and $V(r)$ are given in Section 2 and the subscript "j" is omitted. Then, from the equation for the first moment, the fractions $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$ may be expressed respectively as $(\pi^* - \pi_0)/\Delta$ and $(\pi_1 - \pi^*)/\Delta$. Then, after substitution, the equation for the second moment may be solved for $(\alpha+\beta)$. So, the parameters of interest are computed as

$$\alpha = \eta \cdot (\pi^* - \pi_0)/\Delta; \quad \beta = \eta \cdot (\pi_1 - \pi^*)/\Delta \quad (6)$$

where

$$\eta = (n-1)\frac{\delta^* - \pi^*(1-\pi^*)}{(\pi^* - \pi_0)(\pi_1 - \pi^*)} - 1$$

Then, let us release the assumption concerning the known probability range and estimate the parameters π_0, π_1 minimizing the Pirson's χ^2-statistics that describes goodness-of-fit for the empirical distribution $\{p_r^*\}$. The corresponding optimisation problem may be written as

$$F = \sum_{r=0}^{n} (p_r^*)^2 / P_r(\alpha, \beta, \pi_0, \pi_1) \to \min_{\alpha,\beta,\pi_0,\pi_1} \quad (7)$$

subject to

$$m(\alpha, \beta, p_0, p_1) = np^*; \quad d(\alpha, \beta, p_0, p_1) = n\delta^*. \quad (8)$$

where $\{p_k^*,\ k = \overline{1,n}\}$ are the empirical frequencies and $m(.), d(.)$ denote respectively the mean and dispersion of the considered probability distribution model. It should be noted that the objective function $F(.)$ is directly related to the standard χ^2-statistics. Since the desired parameters π_0, π_1 must belong to the intervals $[0, \pi^*)$ and $(\pi^*, 1]$, this optimisation problem has been solved numerically, by sequentially applying a one-dimensional search for π_0, π_1 and re-computing α, β from the expressions (6).

As follows from our research, the algorithms is rather sensitive to initial estimates. Thus, to ensure good convergence, the developed numerical routines includes the extremum localization step (using grid-based search) and exponential smoothing of the updates for the range parameters.

4. FORECASTING DEMAND DISTRIBUTION

Let assume now that, in addition to the parameter arrays $\{\alpha_j, \beta_j, j=1{:}m\}$, the quasi-periodicity is also described by the historical array $\{v_j, j=1{:}m\}$ obtained by the demand averaging on the similar time segments. Then it can be proved that the Bayesian approach leads to the following posterior distribution for the probabilities of the non-zero demand binary components b_{jl}

$$f(p\,|\,v_j) = \frac{p^{v_j}(p-\pi_0)^{\alpha_j-1}(1-p)^{n-v_j}(\pi_1-p)^{\beta_j-1}}{\int_{\pi_0}^{\pi_1} p^{v_j}(p-\pi_0)^{\alpha_j-1}(1-p)^{n-v_j}(\pi_1-p)^{\beta_j-1}dp} \quad (9)$$

that is computed using the following proposition.

Proposition 2. *For the adopted demand model, the Bayesian forecasting distribution can be represented as the weighted sum of the shifted beta pdfs*

$$f(p\,|\,v) = \sum_{l=0}^{n} \omega_{vl} \cdot \frac{p^{\alpha_j+l-1} \cdot (1-p)^{\beta_j+n-l-1}}{B(\alpha_j+l, \beta_j+n-l)} \qquad (10)$$

and the mean square optimal predictor function is

$$\hat{p}(v) = \sum_{l=0}^{n} \omega_{vl} \cdot (\alpha_j + l)/(\alpha_j + \beta_j + n), \qquad (11)$$

where the weights are expressed as

$$\omega_{vl} = \frac{\binom{n}{l} w_{vi} B(\alpha + l, \beta + n - l)}{\sum_{j=0}^{n} \binom{n}{j} w_{vj} B(\alpha + j, \beta + n - j)}.$$

Proof of Proposition 2 uses the same technique as Proposition 1 (variable expression via θ, hyper-geometric expansion, and relevant transformation). It has been also proved that the developed predictor ensures the following mean square forecast error

$$\varepsilon^2(p) = \frac{\alpha^{[2+]}}{(\alpha+\beta)^{[2+]}} - \sum_{v=0}^{n} \left(\left(\sum_{l=0}^{n} \omega_{vl} \frac{\alpha + l}{\alpha+\beta+n} \right)^2 \cdot \sum_{j=0}^{n} w_{vl} \binom{n}{l} \frac{\alpha^{[l+]} \beta^{[(n-l)+]}}{(\alpha+\beta)^{[n+]}} \right) \qquad (12)$$

where $\alpha^{[l+]}$ denotes the ascending factorial defined as the product $\alpha(\alpha+1)\cdot \ldots \cdot(\alpha+l-1)$.

5. SIMULATION STUDY

To demonstrate the applicability of the developed model to describing the demand quasi-periodicity, a simulation study was performed. Following research of Telang *et al.* (2004), it was considered a demand model with the time period 24 hours and hourly data aggregation, with the smallest time unit within the segment of 3 minutes that corresponds to a single binary demand fraction, i.e. m=24 and n=20. It was assumed that the prior probability of the non-zero demand within each time unit follows the beta distribution with "flat" parameters $\alpha = 0.02$, $\beta = 4.98$, while the repeatability is described by the historical array v = {0 0 0 0 0 1 2 3 1 1 0 0 0 0 0 0 1 2 5 2 1 0 0 0} with two evident peaks during the 24-hour period. It should be stressed that both the prior model and the historical data yield the same average demand value, 0.08 units/hour, which is low enough to describe orders of the "slow-moving" items considered in this paper.

Simulation results are presented in Figs. 1, 2 and Table 1. As expected, the demand patterns (Fig. 1) exhibit the quasi-periodicity having strong concentration of the non-zero values close to the historical data peaks; at the same time, they posses typical intermittent properties (large proportion of zeros and great variability of the non-zeros, see Table 1). Also, the corresponding inter-arrival-time distribution (Fig. 2) possesses a bi-modal shape, which can not be properly described by the exponential function adopted in the classical negative-binomial model (Agrawal and Smith, 1996; Telang *et al.* 2004).

Hence, the simulation study confirms appropriateness of the proposed approach in modelling of the quasi-periodic demand patterns. Besides, the proposed model incorporates less statistical hierarchy levels in comparison with the known one (Telang *et al.*, 2004) and relies on robust identification routines, which can compensate the reporting errors (Dolgui *et al.*, 2004).

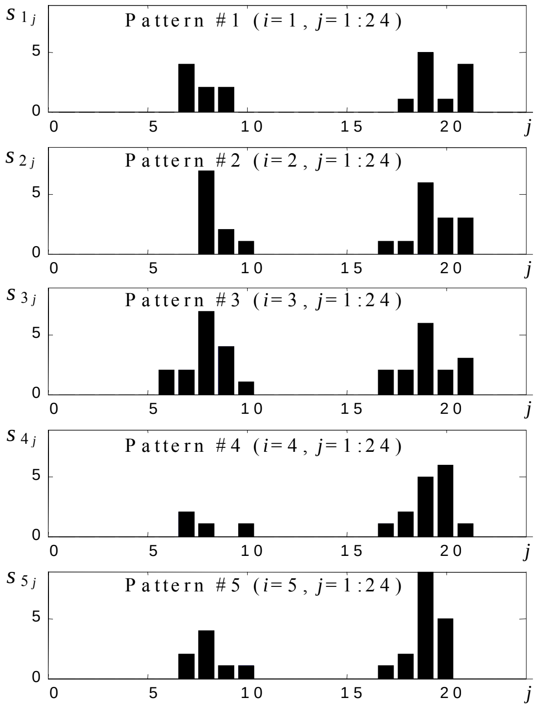

Fig. 1. Example patterns of quasi-periodic demand

Table 1. Distribution of the demand values

r	P_0	P_1	P_2	P_3	P_4	P_5	P_{6+}
P_r	0.967	0.016	0.007	0.004	0.002	0.001	0.004

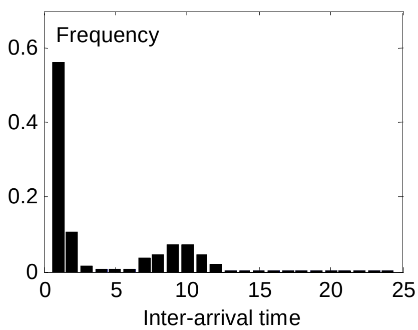

Fig. 2. Distribution of the time intervals

Table 2. Demand data for three time periods

Segment	1	2	3	4	5	6	7	8	9	10	11	12
Period #1	3	0	2	0	0	0	0	1	0	0	1	2
Period #2	0	1	0	0	1	1	2	1	0	2	0	0
Period #3	0	1	1	2	2	2	1	0	0	2	0	0

Table 3. Fitting of the demand data

Data		Model distributions			
r	p_r*	$n = 3$	$n = 4$	$n = 5$	$n = 6$
0	0.541	0.533	0.523	0.518	0.516
1	0.250	0.272	0.301	0.310	0.315
2	0.167	0.142	0.130	0.126	0.124
3	0.042	0.050	0.040	0.037	0.036
α		0.729	1.263	1.701	2.068
β		2.050	5.157	9.108	13.70
χ^2		0.154	0.510	0.663	0.741

Table 4. Forecast based on the previous period

Segment	Forecast			
	$n = 3$	$n = 4$	$n = 5$	$n = 6$
1	0.408	0.510	0.559	0.589
2	0.957	0.906	0.883	0.870
3	0.408	0.510	0.559	0.589
4	0.408	0.510	0.559	0.589
5	0.957	0.906	0.883	0.870
6	0.957	0.906	0.883	0.870
7	1.478	1.293	1.202	1.148
8	0.958	0.906	0.883	0.870
9	0.408	0.510	0.559	0.589
10	1.478	1.292	1.202	1.148
11	0.408	0.510	0.559	0.589
12	0.408	0.510	0.559	0.589
ε	0.791	0.801	0.812	0.819

Table 5. Forecast based on the weighted sum of the previous period and previous segment

Segment	Forecast			
	$n = 3$	$n = 4$	$n = 5$	$n = 6$
1	0.408	0.509	0.559	0.5887
2	1.340	1.268	1.236	1.2177
3	1.531	1.449	1.412	1.3916
4	1.531	1.449	1.412	1.3916
5	2.364	2.068	1.923	1.8370
6	2.364	2.068	1.923	1.8370
7	1.478	1.292	1.202	1.1481
8	0.957	0.906	0.883	0.8698
9	0.408	0.509	0.559	0.5887
10	2.660	2.326	2.1634	2.0666
11	1.149	1.087	1.059	1.0437
12	0.408	0.509	0.559	0.5887
ε	0.390	0.509	0.450	0.474

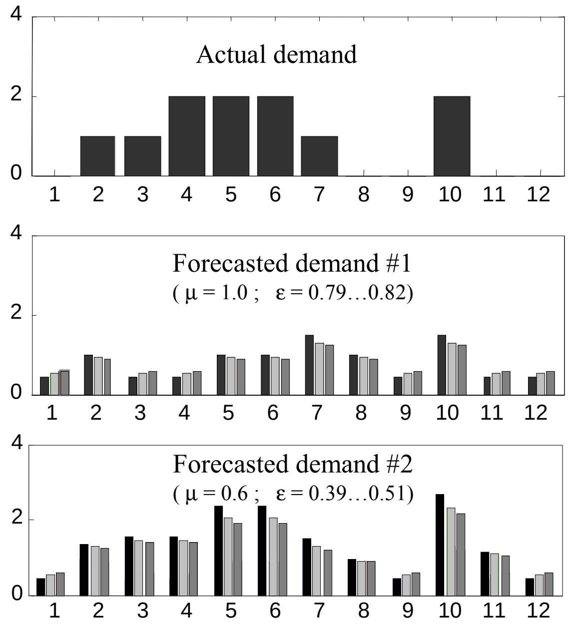

Fig. 3. Actual demand versus forecasts for different values of n and μ (- n=3 ; - n=4 ; - n=5)

6. APPLICATION EXAMPLE

To validate the efficiency of the proposed technique, there were explored a data set for a car spare part sold in Australia (Snyder, 2002), which demonstrates typical intermittent properties and possesses an obvious quasi-periodicity within the 12-months periods (Table 2). The aggregation time-segment was assumed to be equal to one month, and the demand values for 36 months were available. For comparison purposes, there were generated two forecasts, which differ in methods of constructing the historical arrays employed in the Bayesian expressions.

The demand model parameters $\{\alpha, \beta, \pi_0, \pi_1, n\}$ incorporated in the prior distribution (1) were estimated using the numerical routines described in Section 3, using the statistical data for the months 1-24. There were investigated several cases that differ by parameter n (number of binary items in the demand representation). The minimum value of n was set as the upper demand level for these segments. The estimated results are presented in Table 3, which confirms that such selection of n ensures the best fitting of the experimental data, but for the practical reasons this value should be slightly increased to allow some larger demands, that were not observed in a particular experimental data set.

Using the demand statistical model extracted from the months 1-24, there were generated two Bayesian forecasts for the months 25-36. For the first forecast (Table 4), the historical array was created by straightforward coping the demands for the months 13-24, which is the most natural way of accounting the seasonal repeatability (since the demand of the month 25 should be in certain degree similar to the demand of the month 13, etc.). Such approach yielded the forecast error about 0.8 items/months, which slightly increases while increasing *n*.

The second forecast (Table 5) relies on another historical array, which is created from both the previous-year and the previous-month demand values (with certain weights μ and $1-\mu$). The reason behind this is that, in addition to the seasonality, it should be also strong correlation between the demands of the successive months. As follows from our research, for this particular example, the best forecast is obtained for $\mu \approx 0.6$, which corresponds to the forecast error 0.39…0.47 items/month. It is almost twice lower, than in the previous case.

Hence, this application example confirms the applicability of the developed method to the modelling of real-life intermittent demand patterns and efficiency of the relevant Bayesian forecast.

7. CONCLUSION

Forecasting the intermittent demand for service parts and high-priced capital goods is a challenging problem of the inventory management. Current practice in such demand modelling favours the exponential smoothing of the demand values or applying exponential smoothing separately to the intervals between nonzero demands and their sizes (Croston's method). An alternative recommended in inventory control literature is based on the stochastic demand modelling using the negative binominal distribution that assumes exponentially distributed inter-arrival times.

In this paper, it is proposed a new stochastic model, which describes quasi-periodic intermittent demand patterns with essentially non-exponential (poly-modal) distribution of the inter-arrival times. It is based on generalized beta-binomial distribution and the Bayesian inference using the historical data array describing the demand repeatability within the time periods. For this model, there were derived explicit expressions for the forecast distributions, its moments and relevant Bayesian risk. The efficiency of the proposed approach is confirmed by computer simulation and is illustrated by an application example for modelling of the demand patterns for car spare parts.

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