PARAMETER DEPENDENT H_∞ CONTROLLER DESIGN BY FINITE DIMENSIONAL LMI OPTIMIZATION: APPLICATION TO TRADE-OFF DEPENDENT CONTROL

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Abstract: The design of a parameter dependent H_∞ controller for a parameter dependent plant is considered. A solution can be proposed as a parameter dependent LMI optimization problem, that is, an infinite dimensional problem. In the case of rational dependence, an approach is proposed involving an optimization problem over parameter independent LMI constraints (which is finite dimensional). The obtained conditions are less conservative than previous ones. An application to the trade-off dependent controller design with the control of a DC motor is developed to emphasize the interest of our approach.

Keywords: Parameter dependent H_∞ control, Linear Matrix Inequality (LMI) optimization, trade-off dependent control.

1. INTRODUCTION

In this paper, we focus on the design of a parameter dependent H_∞ controller for a parameter dependent plant. This problem can be applied to numerous controller design problems: gain scheduling control (Stilwell and Rugh, 1999), saturated system control (Megretski, 1996), spatial system control (de Castro and Paganini, 2002), trade-off dependent control (Dinh *et al.*, 2003), to cite a few. It can be naturally cast into optimization problems involving parameter dependent Linear Matrix Inequalities, that is, an infinite dimensional optimization problem whose solution cannot be efficiently computed. The major difficulty is to replace it by a parameter independent LMI optimization problem while avoiding the introduction of conservatism. Remember that parameter independent LMI optimization problems are finite dimensional ones which can be efficiently solved. Such an approach was considered in numerous problems: robust analysis, LPV control.. The reader is referred to (Dinh *et al.*, accepted in 2005) for the bibliography on these approaches. For sake of shortness, it is not developed here. In these approaches, the choice of function sets for the decision variables of the infinite dimensional optimization problem is the crucial point, as pointed out in (Dinh *et al.*, 2003). In this last paper, we investigated the choice of rational functions with fixed denominator, which encompasses all previous ones.

In this paper, we consider LMI which rationally depend on a scalar parameter θ. We prove that when the set of decision variables is the set of rational functions of a given order N (whose denominator is free), a considered parameter dependent LMI constraint can be equivalently replaced by parameter independent LMI constraints. This point is a dramatic improvement with respect to previous approaches, *e.g.* (Dinh *et al.*, 2003): we improve this approach by deriving conditions of similar complexity. The proposed approach can be applied to *several* parameters but by introducing conservatism. It is based on an extension of the Kalman Yakubovich Popov lemma. The obtained result allows to derive a parameter independent LMI formulation for the parameter dependent control problem in the next section. In Section 3, this solution is applied to the trade-off dependent control with the control of a DC motor as a numerical example. A trade-off dependent controller is a controller $K(s,\theta)$ such that a set of specification trade-offs parameterized by a scalar $\theta \in [0,1]$ is satisfied for a given LTI plant (Dinh *et al.*, 2003). A trade-off is, here, formulated as an optimization problem involving a weighted H_∞ norm. A motivating application is controller (re)tuning.

Notations I_n (respectively $0_{m\times p}$) denotes the $n\times n$ identity matrix (resp. the $m\times p$ zero matrix). The subscribes will be dropped when they are clear from the context. $P>0$ denotes that P is positive definite. The Redheffer star product (Skogestad and Postlethwaite, 1996) is denoted by $\star$. Let us also introduce

$$\mathcal{R}_\mathcal{H} = \left[\mathcal{H}_N \ \cdots \ \mathcal{H}_0\right],\ \mathcal{R}_{d,p} = \left[d_N I_p \ \cdots \ d_1 I_p \ I_p\right],$$

$$\Lambda_p = \left\{\mathcal{R}_\mathcal{H} \mid \mathcal{H}_i = \mathcal{H}_i^T \in \mathbb{R}^{p\times p},\ i=0,\ldots,N\right\},$$

$$J_p = \left[\begin{array}{ccccc|c} 0 & I_p & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & I_p & 0 \\ -c_N I_p & \cdots & \cdots & \cdots & -c_1 I_p & I_p \\ \hline & & I_{N\times p} & & & 0 \\ \hline -c_N I_p & \cdots & \cdots & \cdots & -c_1 I_p & I_p \end{array}\right]$$

where c_i, $i=1,\ldots,N$, are given real scalars such that for any $\theta \in [0,1]$ $1+\sum_{i=1}^N \theta^i c_i \neq 0$,

$$\mathcal{L}(A,B,C,D,M,\mathcal{S},\mathcal{G}) = \begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \ldots$$
$$\cdots + \begin{bmatrix} A^T(\mathcal{S}-\mathcal{G}) + (\mathcal{S}+\mathcal{G})A - 2\mathcal{S} & (\mathcal{S}+\mathcal{G})B \\ B^T(\mathcal{S}-\mathcal{G}) & 0 \end{bmatrix}.$$

2. PARAMETER DEPENDENT CONTROLLER DESIGN

2.1 Problem formulation

We consider the following generalized plant $P(s,\theta)$:

$$\begin{cases} \dot{x}(t) = A(\theta)x(t) + B_w(\theta)w(t) + B_u(\theta)u(t) \\ z(t) = C_z(\theta)x(t) + D_{zw}(\theta)w(t) + D_{zu}(\theta)u(t) \\ y(t) = C_y(\theta)x(t) + D_{yw}(\theta)w(t) \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $z(t) \in \mathbb{R}^{n_z}$, $w(t) \in \mathbb{R}^{n_w}$ and where θ is a constant parameter (conventionally $\theta \in [0,1]$). The state space matrices of $P(s,\theta)$ are assumed to be rational functions of θ, well-posed on $[0,1]$.

PROBLEM: Given $P(s,\theta)$ as in (1) and $\gamma > 0$, compute, if there exists, a parameter dependent controller

$$K(s,\theta) = \frac{1}{s} I_n \star \left[\begin{array}{c|c} A_K(\theta) & B_K(\theta) \\ \hline C_K(\theta) & D_K(\theta) \end{array}\right] \quad (2)$$

where $A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ are rational functions of θ, well-posed on $[0,1]$, ensuring for any $\theta \in [0,1]$:

- the asymptotic stability of $P(s,\theta) \star K(s,\theta)$;
- $\|P(s,\theta) \star K(s,\theta)\|_\infty < \gamma$.

$A_K(\theta)$, $B_K(\theta)$, $C_K(\theta)$ and $D_K(\theta)$ are required to be rational in θ in order to obtain a controller implementation of reasonable complexity. The proposed approach can be readily applied to other criteria (such as H_2 norm, multiobjective..).

2.2 Proposed approach

Note that PROBLEM is in fact an extension of the H_∞ control problem as both the controller and the generalized plant are, here, dependent on a parameter θ. Thus a solution to PROBLEM can be straightforwardly obtained by extending the LMI solution of (Scherer *et al.*, 1997) (conditions (41) and (42)) to the H_∞ control problem. We obtain the optimization problem over parameter dependent LMI constraints presented in (Dinh *et al.*, 2003). This is an infinite dimensional optimization problem along two aspects:

- as functions of θ, the decision variables are in an infinite dimensional space;
- as parameterized by θ, there is an infinite number of constraints.

This infinite nature prevents an efficient computation of a solution to the optimization problem.

The proposed approach is an interesting way to compute a solution to this infinite dimensional optimization problem via a finite dimensional one. Following (Rossignol *et al.*, 2003), it is obtained along two steps.

- For the first step, we introduce for a decision variable, say $\Upsilon(\theta)$, the following finite parameterization
$$\Upsilon(\theta) = \frac{\sum_{i=0}^N \theta^i \Upsilon_i}{1+\sum_{i=1}^N \theta^i d_i}, \quad (3)$$
parameterized by the $N+1$ matrices Υ_i and the N scalars d_i.
- In order to obtain a finite number of constraints, the second step is based on the following lemma, which is an extension of the Kalman Yakubovich Popov lemma.

Lemma 2.1. (Rossignol *et al.*, 2003) Let M be a symmetric matrix and let $\Phi(\theta)$ be a rational matrix function of θ, well-posed on $[0,1]$, and define one of its LFT realization by

$$\Phi(\theta) = \theta I \star \left[\begin{array}{c|c} A_\Phi & B_\Phi \\ \hline C_\Phi & D_\Phi \end{array}\right].$$

Then the following condition holds

$$\forall \theta \in [0,1], \quad \Phi(\theta)^T M \Phi(\theta) < 0$$

if and only if there exist $\mathcal{S} = \mathcal{S}^T > 0$ and $\mathcal{G} = -\mathcal{G}^T$ such that

$$\mathcal{L}(A_\Phi, B_\Phi, C_\Phi, D_\Phi, M, \mathcal{S}, \mathcal{G}) < 0.$$

We can then state the following lemma.

Lemma 2.2. Let $H_1(\theta)$ and $H_2(\theta)$ be two matrices of rational functions of θ, well-posed on $[0,1]$. Let C be a matrix and N be a positive integer.
There exists a (possibly structured) matrix $\Upsilon(\theta)$ as defined in (3), well-posed on $[0,1]$, such that

$$\begin{aligned} \forall \theta \in [0,1], \; & H_1(\theta)(C + \Upsilon(\theta))H_2(\theta) + \cdots \\ & \cdots + (H_1(\theta)(C + \Upsilon(\theta))H_2(\theta))^T < 0 \end{aligned} \tag{4}$$

if and only if there exist $N+1$ matrices Υ_i, $i = 0, \ldots, N$ and N scalars d_i, $i = 1, \cdots, N$, such that the two following conditions hold:
(i) there exist $\mathcal{S}_d = \mathcal{S}_d^T > 0$ and $\mathcal{G}_d = -\mathcal{G}_d^T$ such that

$$\mathcal{L}\left(A_d, B_d, C_d, D_d, \begin{bmatrix} 0 & -\mathcal{R}_{d,1} \\ -\mathcal{R}_{d,1}^T & 0 \end{bmatrix}, \mathcal{S}_d, \mathcal{G}_d\right) < 0 \tag{5}$$

where

$$\theta I \star \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right] = \left[\begin{array}{c} 1 \\ \hline \theta \star J_1 \end{array}\right]$$

(ii) there exist $\mathcal{S} = \mathcal{S}^T > 0$ and $\mathcal{G} = -\mathcal{G}^T$ such that

$$\mathcal{L}\left(A_H, B_H, C_H, D_H, \begin{bmatrix} 0 & \mathcal{U}(\Upsilon_i, d_i) \\ \mathcal{U}(\Upsilon_i, d_i)^T & 0 \end{bmatrix}, \mathcal{S}, \mathcal{G}\right) < 0 \tag{6}$$

where

$$\theta I \star \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] \triangleq \left[\begin{array}{c} H_1(\theta)^T \\ \bar{H}(\theta) H_2(\theta) \end{array}\right]$$

and where $\mathcal{U}(\Upsilon_i, d_i)$ is an affine function of Υ_i and of d_i such that

$$\mathcal{U}(\Upsilon_i, d_i)\bar{H}(\theta) = \frac{(\Upsilon_0 + C) + \sum_{i=1}^N \theta^i (\Upsilon_i + d_i C)}{1 + \sum_{i=1}^N \theta^i c_i}. \tag{7}$$

Note that the infinite dimensional optimization problem defined by (4) is equivalently replaced by the finite dimensional optimization problem over the LMI constraints (5) and (6). Note also that the factorization (7) is always possible although not unique.

Proof of Lemma 2.2 The well-posedness of $\Upsilon(\theta)$ on $[0,1]$ is equivalent to $\forall \theta \in [0,1]$, $1 + \sum_{i=1}^N d_i \theta^i \neq 0$. As $1 + \sum_{i=1}^N d_i \theta^i$ is a real valued polynomial, it is sign (positive) definite. The well-posedness of $\Upsilon(\theta)$ on $[0,1]$ is then further equivalent to

$$\forall \theta \in [0,1], \quad \frac{1 + \sum_{i=1}^N d_i \theta^i}{1 + \sum_{i=1}^N c_i \theta^i} > 0 \tag{8}$$

for any polynomial $1 + \sum_{i=1}^N c_i \theta^i$ that does not vanish on $[0,1]$. Since $\dfrac{1 + \sum_{i=1}^N d_i \theta^i}{1 + \sum_{i=1}^N c_i \theta^i} = \mathcal{R}_{d,1} \times \theta \star J_1$, condition (8) is equivalent to condition (5) by applying Lemma 2.1.

Using (8), condition (4) can be rewritten as:

$$\begin{aligned} \forall \theta \in [0,1], \; & H_1(\theta)\mathcal{U}(\Upsilon_i, d_i)\bar{H}(\theta)H_2(\theta) + \cdots \\ & \cdots + (H_1(\theta)\mathcal{U}(\Upsilon_i, d_i)\bar{H}(\theta)H_2(\theta))^T < 0. \end{aligned} \tag{9}$$

with $\mathcal{U}(\Upsilon_i, d_i)$ and $\bar{H}(\theta)$ as defined in (7). Condition (9) is then equivalent to condition (6) by applying Lemma 2.1. □

2.3 Finite dimensional solution

From the previous result, we have the following theorem.

Theorem 2.1. Given N, there exists a controller (2) solving PROBLEM if there exist

- matrices $\mathcal{R}_\mathcal{X} \in \Lambda_n$, $\mathcal{R}_\mathcal{Y} \in \Lambda_n$, and a matrix $\mathcal{R}_\mathcal{V} \in \mathbb{R}^{(n+n_u)\times(N+1)(n+n_y)}$;
- scalars d_i, $i = 1, \ldots, N$

such that the two following conditions hold:
(i) there exist $\mathcal{S}_0 = \mathcal{S}_0^T > 0$ and $\mathcal{G}_0 = -\mathcal{G}_0^T$ such that

$$\mathcal{L}\left(A_{\Omega_0}, B_{\Omega_0}, C_{\Omega_0}, D_{\Omega_0}, \begin{bmatrix} 0 & \mathcal{W} \\ \mathcal{W}^T & 0 \end{bmatrix}, \mathcal{S}_0, \mathcal{G}_0\right) < 0 \tag{10}$$

with $\mathcal{W} = -\begin{bmatrix} \mathcal{R}_X & \mathcal{R}_{d,n} \\ 0 & \mathcal{R}_\mathcal{Y} \end{bmatrix}$ and with

$$\theta I \star \left[\begin{array}{c|c} A_{\Omega_0} & B_{\Omega_0} \\ \hline C_{\Omega_0} & D_{\Omega_0} \end{array}\right] = \left[\begin{array}{cc} \multicolumn{2}{c}{I_{2n}} \\ \hline \theta I \star J_n & 0 \\ 0 & \theta I \star J_n \end{array}\right];$$

(ii) there exist $\mathcal{S} = \mathcal{S}^T > 0$ and $\mathcal{G} = -\mathcal{G}^T$ such that

$$\mathcal{L}\left(A_\Omega, B_\Omega, C_\Omega, D_\Omega, \begin{bmatrix} 0 & \mathcal{Z} \\ \mathcal{Z}^T & 0 \end{bmatrix}, \mathcal{S}_0, \mathcal{G}_0\right) < 0 \tag{11}$$

with

$$\mathcal{Z} = \left[\begin{array}{ccccc} \mathcal{R}_\mathcal{V} & 0 & 0 & 0 & 0 \\ \hline 0 & \mathcal{R}_\mathcal{X} & \mathcal{R}_{d,n} & 0 & 0 \\ 0 & 0 & \mathcal{R}_\mathcal{Y} & 0 & 0 \\ \hline 0 & \mathcal{R}_{d,n} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{d,n_w} & 0 \\ 0 & 0 & 0 & \gamma\mathcal{R}_{d,n_w} & 0 \\ 0 & 0 & 0 & 0 & \gamma\mathcal{R}_{d,n_z} \end{array}\right]$$

and with

$$\theta I \star \left[\begin{array}{c|c} A_\Omega & B_\Omega \\ \hline C_\Omega & D_\Omega \end{array}\right] = \left[\begin{array}{c} F_1(\theta)^T \\ F_2(\theta)F_3(\theta) \end{array}\right]$$

where

$$F_1(\theta) = \left[\begin{array}{cc|cc|cccc} 0 & B_u(\theta) & A(\theta) & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & A(\theta)^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_w(\theta)^T & B_w(\theta)^T & 0 & -\frac{1}{2}I_{n_w} & 0 \\ 0 & D_{zu}(\theta) & C_z(\theta) & 0 & 0 & D_{zw}(\theta) & 0 & -\frac{1}{2}I_{n_z} \end{array}\right]$$

$$F_2(\theta) = \left[\begin{array}{c|cccc} \theta I \star J_{n+n_y} & 0 & 0 & 0 & 0 \\ 0 & \theta I \star J_n & 0 & 0 & 0 \\ 0 & 0 & \theta I \star J_n & 0 & 0 \\ 0 & 0 & 0 & \theta I \star J_{n_w} & 0 \\ 0 & 0 & 0 & 0 & \theta I \star J_{n_z} \end{array}\right]$$

$$F_3(\theta) = \left[\begin{array}{cccc} I_n & 0 & 0 & 0 \\ 0 & C_y(\theta) & D_{yw}(\theta) & 0 \\ \hline & I_{n+n+n_w+n_z} & & \end{array}\right].$$

If a controller exists, its state space matrices are then obtained with

$$\begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix} = \begin{bmatrix} L(\theta) & -M(\theta) & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix} \times \ldots \quad (12)$$

$$\ldots \times \left(\begin{bmatrix} I_n & 0 \\ 0 & B_u(\theta) \\ 0 & I_{n_u} \end{bmatrix} \mathcal{V}(\theta) \begin{bmatrix} \mathcal{X}(\theta)^{-1} & 0 \\ -C_y(\theta) & I_{n_y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A(\theta) & 0 \\ 0 & 0 \end{bmatrix} \right)$$

with $\begin{bmatrix} L(\theta) & -M(\theta) \end{bmatrix}$ given by

$$\left(\begin{bmatrix} I_n \\ I_n \end{bmatrix} \mathcal{X}(\theta) \begin{bmatrix} I_n & \mathcal{Y}(\theta) \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & -I_n & I_n \end{bmatrix} \right) \star I_n$$

and where

$$\mathcal{X}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{X}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \; \mathcal{Y}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{Y}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}, \quad (13)$$
$$\mathcal{V}(\theta) = \frac{\sum_{i=0}^{N} \theta^i \mathcal{V}_i}{1 + \sum_{i=1}^{N} \theta^i d_i}.$$

Proof of Theorem 2.1 It is proved by the application of Lemma 2.2 to the optimization problem over parameter dependent LMI constraints presented in (Dinh *et al.*, 2003), that is, the extension of conditions (41) and (42) in (Scherer *et al.*, 1997). □

Computation: for a given value of γ, the optimization problem defined by (10) and (11) is an LMI feasibility problem. Another interesting problem is to minimize γ over LMI constraints (10) and (11). As this minimization is a quasi convex optimization problem [1], the minimum value of γ can be found by performing a dichotomy on γ.

2.4 Discussion of our approach

$\mathcal{X}(\theta)$, $\mathcal{Y}(\theta)$ and $\mathcal{V}(\theta)$ are in fact the parameter dependent decision variables of the infinite dimensional optimization problem presented in (Dinh *et al.*, 2003). The dependence assumed in (13) is not restricting with respect to PROBLEM since, from equation (12), this dependence is enforced as rational matrices function of θ are wanted for the state spaces matrices of the controller (2).

N is the trade-off parameter: increasing N allows to decrease the performance level γ with the drawback of increasing the controller complexity, that is its state space matrices are rational functions of increasing degree. As N is the only trade-off parameter, a trade-off is easily obtained. The numerical example of Section 3.2 illustrates that good performance can be obtained with a small N.

For a given N, the approach of this paper gives a better performance level γ than the one of (Dinh *et al.*, 2003); or equivalently for a given performance level γ, a lower N is needed, that is, a controller of lower complexity is obtained. Furthermore, we consider here a more general problem.

3. APPLICATION: DESIGN OF A TRADE-OFF DEPENDENT CONTROLLER

3.1 Problem formulation

In the H_∞ control approach (see Figure 1), the

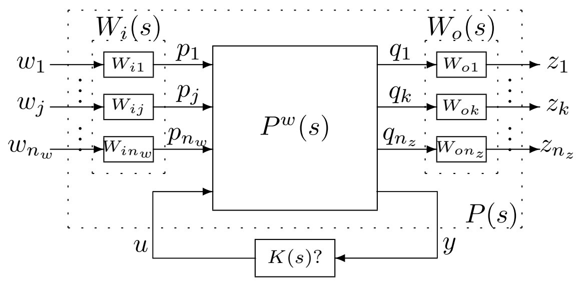

Fig. 1. General H_∞ problem

design of a controller $K(s)$ is recast as an optimization problem on weighted closed-loop transfer functions. The considered closed-loop transfer functions are defined by $P^w(s)$ (which depends on the plant):

$$\begin{cases} \dot{x}^w(t) = A^w x^w(t) + B_p^w p(t) + B_u^w u(t) \\ q(t) = C_q^w x^w(t) + D_{qp}^w p(t) + D_{zu}^w u(t) \\ y(t) = C_y^w x^w(t) + D_{yp}^w p(t) \end{cases}.$$

The desired performance specifications are introduced through the choice of the weighting functions $W_i(s)$ and $W_o(s)$.

A set of performance trade-offs parameterized by a scalar $\theta \in [0, 1]$ is then defined by weighting functions that are dependent on θ:

$$W_i(s,\theta) = \frac{1}{s} I \star \left[\begin{array}{c|c} A_{W_i}(\theta) & B_{W_i}(\theta) \\ \hline C_{W_i}(\theta) & D_{W_i}(\theta) \end{array}\right],$$

$$W_o(s,\theta) = \frac{1}{s} I \star \left[\begin{array}{c|c} A_{W_o}(\theta) & B_{W_o}(\theta) \\ \hline C_{W_o}(\theta) & D_{W_o}(\theta) \end{array}\right]$$

where the state space matrices are assumed rational functions of θ, well-posed on $[0, 1]$. The generalized plant is then defined by:

$$P(s,\theta) = \begin{bmatrix} W_o(s,\theta) & 0 \\ 0 & I \end{bmatrix} P^w(s) \begin{bmatrix} W_i(s,\theta) & 0 \\ 0 & I \end{bmatrix}.$$

[1] Quasi convexity can be proved by a simple adaptation of the proof of the (LMI) Generalized Eignevalue Problems, see (Boyd *et al.*, 1994).

Given $\gamma > 0$, the problem is to compute a controller $K(s,\theta)$ whose state space matrices are (explicit) rational functions of θ such that

$$\forall\theta \in [0,1],\ \|P(s,\theta) \star K(s,\theta)\|_\infty < \gamma \qquad (14)$$

The considered problem is thus a subcase of PROBLEM. Theorem 2.1 can then be applied.

3.2 Numerical example: DC motor control

A DC motor can be modeled by

$$G(s) = \frac{235}{s(\frac{s}{66}+1)} = \frac{1}{s}I \star \left[\begin{array}{cc|c} -66 & 0 & 32 \\ 32 & 0 & 0 \\ \hline 0 & 15 & 0 \end{array}\right].$$

The purpose is to design a one degree of freedom controller ensuring that the closed-loop system output is able to track, with a small error, step and low frequency sinusoidal reference signals with a specified transient time response (from 0.06 s for $\theta = 0$ down to 0.02 s for $\theta = 1$) and with the most limited possible control input energy. It should also be able to reject step and low frequency sinusoidal input disturbance signals.

Such a problem is addressed by the weighted H_∞ problem depicted in Figure 2 (Skogestad and Postlethwaite, 1996). A trade-off depends on the

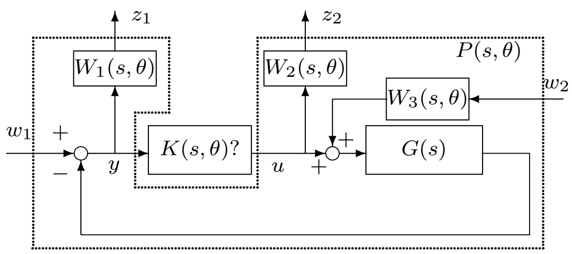

Fig. 2. Weighted sensitivity H_∞ problem

time response, which is related to the crossover frequency of $W_1(s,\theta)$ leading to choose it from 20 rad/s for $\theta = 0$ up to 80 rad/s for $\theta = 1$. θ is chosen as an affine function of the crossover frequency of $W_1(s,\theta)$ for a clear interpretation of θ. $W_3(s,\theta)$ is chosen in order to specify the input disturbance rejection. The crossover frequency of $W_2(s,\theta)$ is chosen to limit the most possible the control input energy leading to set it as follows: 23.33 rad/s for $\theta = 0$, 180 rad/s for $\theta = 0.5$ and 700 rad/s for $\theta = 1$. Using a least square technique, we obtain it as a function of θ. The weighting functions $W_i(s,\theta)$ can be written as:

$$\frac{1}{s} \star \left[\begin{array}{c|c} -\omega_{ci}(\theta)\sqrt{\dfrac{|G_{\infty i}^2 - 1|}{|G_{0i}^2 - 1|}} & \sqrt{\dfrac{|G_{\infty i}^2 - 1|}{|G_{0i}^2 - 1|}} \\ \hline \omega_{ci}(\theta)(G_{0i} - G_{\infty i}) & G_{\infty i} \end{array}\right]$$

where $G_{0i} = |W_i(0,\theta)|$, $G_{\infty i} = \lim_{\omega\to\infty} |W_i(j\omega,\theta)|$ and $\omega_{ci}(\theta)$ such that $|W_i(j\omega_{ci}(\theta),\theta)| = 1$, with:

- $20\log(G_{01}) = -40dB$, $20\log(G_{\infty 1}) = 6dB$, $\omega_{c1} = 20 + 60\theta$;
- $20\log(G_{02}) = 10dB$, $20\log(G_{\infty 2}) = -60dB$, $\omega_{c2}(\theta) = 23.33 + \dfrac{204\theta}{1-0.7\theta}$;
- for simplicity, $W_3(s,\theta)$ is set to 0.05.

$P(s,\theta)$ is then obtained where the parameter dependent matrices ($A(\theta)$ and $C_z(\theta)$) are rational functions of θ with the denominator $1-0.7\theta+0\theta^2$.

For comparison, several experiments are performed:

- $N = 2$ and a fixed denominator (the scalars d_i are *a priori* chosen). A natural choice is $1-0.7\theta$ as it is the denominator obtained for the state space matrices of $P(s,\theta)$;
- $N = 2$ to evaluate the benefit of a free denominator;
- $N = 3$ for the benefit of increasing N.

For a given N, the question of performance level γ loss arises with respect to more general function sets for the decision variables. A possible evaluation can be obtained by (i) finding the smallest γ, denoted γ_r, such that there exists a controller of the considered structure solving PROBLEM, (ii) comparing γ_r with the smallest γ, denoted γ_{best}, by considering $K(s,\theta)$ without any constraint on its state space matrices (except well-posedness). Even if the computation of γ_{best} is still open, a lower bound γ_l can be easily obtained: $\gamma_l = \max_{\theta_i} \gamma_{\theta_i}$ where γ_{θ_i} is the smallest γ such that there exists $K_{\theta_i}(s)$ with $\|P(s,\theta_i) \star K_{\theta_i}(s)\|_\infty < \gamma$. In the sequel, $K_{\theta_i}(s)$ is referred to as a *"pointwise"* controller. The following criterion is then evaluated: $\frac{\gamma_r - \gamma_l}{\gamma_l}$. For this example, $\gamma_l = 0.998$.

The optimization problems are solved using `Matlab 6.5` with the LMI control toolbox (Gahinet *et al.*, 1995). For an easier numerical resolution, we choose $1 + c_1\theta + c_2\theta^2 = 1 - 0.7\theta$ for $N = 2$ and $1+c_1\theta+c_2\theta^2+c_3\theta^3 = (1-0.7\theta)(1+3\theta)$ for $N = 3$. The term $1 - 0.7\theta$ allows to obtain a low order realization of $\Omega(\theta)$. The term $1 + 3\theta$ is arbitrary.

Table 1. Results: performance level γ

	$N=3$	$N=2$	$N=2$, fixed denominator
γ_r	1	1.06	1.105
$\frac{\gamma_r - \gamma_l}{\gamma_l}$	$< 1\%$	$\approx 6\%$	$\approx 11\%$

The results are presented in Table 1. As planned, the results are better when optimizing with $N = 3$ than with $N = 2$ and when optimizing with $N = 2$ than with $N = 2$ and a fixed denominator. Note that for $N = 3$, γ_r is very close to γ_{best}.

To illustrate the difficulty of choosing *a priori* the denominator, let us focus on the obtained one when optimizing on its coefficients. We obtain $1 - 1.12\theta + 3.37\theta^2$ for $N = 2$ with complex roots. It is difficult to *a priori* select such roots.

Now, let us focus on the case $N = 3$. Even for this low value of N, the results are "perfect" in the sense that the transient responses obtained with the parameter dependent controller and with the pointwise controllers are superposed (see Figure 3) [2] . The same statement holds for the magnitude

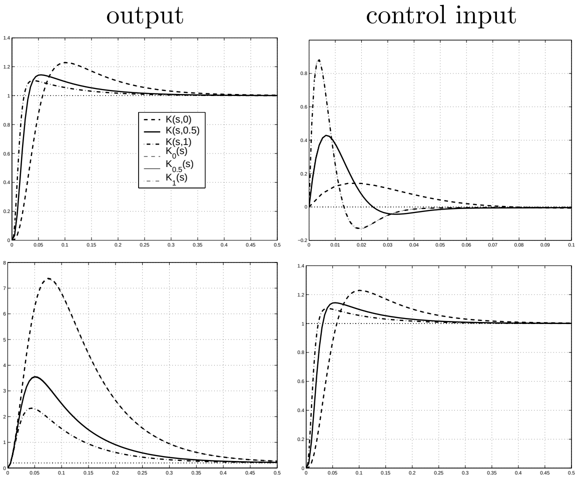

Fig. 3. Transient responses to a unit step reference signal (top) and to a unit step disturbance signal (bottom) with $N = 3$

of the closed-loop transfer functions. The figure is omitted due to space limitation. Due to the problem formulation, if it is natural that the magnitude of the closed-loop transfer functions and the transient responses can be recovered, it is interesting to see that the usual pointwise controllers are also recovered: a PI with a lead effect and a low pass filter with the variation of 37° for the lead effect recovered (see Figure 4).

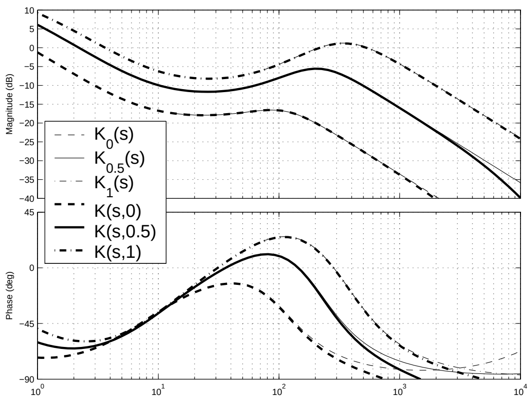

Fig. 4. Bode plots of controllers with $N = 3$

3.3 Interest of our approach

For $\theta = 0$, the lead effect is close to 0: the structure can be readily reduced to a PI with a low pass filter. Whereas for $\theta = 1$, the lead effect is important (25°) and cannot be neglected. Thus for this set of specifications the structure of the parameter dependent controller changes from $\theta = 0$ to $\theta = 1$ which corresponds to usual tuning know-how.

Moreover, using classical rules of automatic control, know-how... a qualitative link between the performance specifications and the controller gains can be established. Our approach allows to express the controller gains as an *explicit* function of the performance specifications. A quantitative link between performance specifications and the controller gains is thus obtained.

[2] In Figure 3, the figure at the top right does not have the same time scale than the others. The figures at the top left and at the bottom right are the same since the considered transfer functions are the same $(GK(I + GK)^{-1})$.

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