ON SPREADING DYNAMICS IN DISCRETE SMALL-WORLD NETWORKS

Xiang Li [*,1] **Guanrong Chen** [**] **Xiaofan Wang** [***]

[*] *Department of Automation*
Shanghai Jiao Tong University, Shanghai 200030, China
E-mail: xli@sjtu.edu.cn

[**] *Department of Electronic Engineering*
City University of Hong Kong, Hong Kong SAR, China
E-mail: gchen@ee.cityu.edu.hk

[***] *Department of Automation*
Shanghai Jiao Tong University, Shanghai 200030, China
E-mail: xfwang@sjtu.edu.cn

Abstract: The dynamical behaviors of some general spreading phenomena in discrete small-world networks are investigated. A new discrete small-world spreading model is proposed, where typical spreading dynamics, including period-doubling bifurcations and chaos, are analyzed on the small-world probability and the nonlinear interaction gain constant.

Keywords: Network topology, disease propagation, discrete-time network, dynamical behavior, bifurcation, chaos

1. INTRODUCTION

Can you imagine that among more than 6 billion people in the world, the length of the acquaintance chain between two randomly chosen persons are just about 6? But basically it is true. This well-known small-world phenomenon has been rediscovered in many practical networks, ranging from the Internet, the World Wide Web, human society, power grids, to economic and biological systems (Albert and Barabasi, 2002; Wang and Chen, 2003; Watts, 1999).

Recently, to study this important subject, Newman, Watts, and Strogatz have proposed the several small-world models (Newman and Watts, 1999; Watts, 1999) to mimic the large-cluster-and-short-path property, which universally exists in various large-scale networks.

On the other hand, it is now well known that network structures significantly affect the emergent network dynamics (Li and Chen, 2003; Li, *et al.*, 2004b, Lü, *et al.*, 2004; Wang, 2002). Especially, the spreading (propagation) of diseases, viruses, and disasters through a huge-scale small-world network, including such similar phenomena as power blackouts and financial crises, has

[1] partially supported by the City University of Hong Kong under the Strategic Research Grant 7001593, and the National Natural Science Foundation of P. R. China under Grant No. 90412004, 70431002, 70271072, and 70471082. Author X. Li also acknowledges the support of the Alexander von Humboldt Foundation and SRF for ROCS, SEM.

become one of the most concerned issues today. How do social networks and computer networks mediate the transmission of a human disease or a computer virus? How do cascading failures and crises propagate throughout a large power transmission grid, or a global financial network? All these point to the question of how does the small-world networking property affect the dynamical behaviors through the network.

Concerning network dynamics, bifurcation and chaos with stability analysis of some continuous spreading models have been investigated recently, particularly for small-world networks (Li and Chen, 2004; Li, *et al.*, 2004a; Li and Chen, 2005; Yang, 2001). As a continued effort, in this paper a new discrete spreading model is proposed within the framework of the Newman-Watts small-world networks. With the proposed model, the effects of the so-called small-world parameter and the practical nonlinear interaction gain constant will be further investigated, regarding with some typical discrete spreading dynamics such as period-doubling bifurcations and chaos.

2. A DISCRETE SPREADING MODEL OF THE N-W SMALL-WORLD NETWORKS

Newman and Watts proposed a model (Newman and Watts, 1999), called the N-W small-world network model hereafter, as a variant of the original Watts-Strogatz small-world model (Watts, 1999), which describes a transition between an ordered lattice and a random graph. A N-W small-world network is evolved as follows: (1) Start with a ring lattice of N nodes, in which every node is connected to its first K neighbors ($K/2$ on each side, and generally assume $1 < K \ll N$). (2) Add a new long-rang edge (short-cut) into the lattice with probability $0 < p \ll 1$ between a randomly chosen pair of nodes. It was shown that for sufficiently large N and small p, the N-W model is equivalent to the W-S model, both networks have short average path lengths and large clustering coefficients.

Afterwards, Newman and Watts (Newman and Watts, 1999) looked at the problem of disease spreading over small-world networks. In the N-W model, a disease spreads from neighbor to neighbor on the small-world network, where the disease can only spread within the connected cluster of susceptible individuals in which it first started. Then, Moukarzel (1999) reexamined and extended this basic spreading mechanism, followed by the studies of Yang, Li, and Chen et al. (Li and Chen, 2004; Li and Chen, 2005; Li, *et al.*, 2004a; Yang, 2001), using continuous differential-difference spreading equations for modelling.

In this paper, for simplicity of presentation, it is assumed that the N-W small-world network model is an one-dimensional lattice which started from a ring with $K = 2$ with constant shape factor $\Gamma_d = 1$ (Moukarzel, 1999; Newman and Watts, 1999; Yang, 2001).

Consider that a disease (virus, power failure alike) spreads with a constant radial velocity $v = 1$ from an original infected site (node) of the N-W small-world network, in which the total volume of infected individuals is $V(k)$ at time k.

According to the N-W evolving model, the incremental volume $\Delta V(k) = V(k+1) - V(k)$ at time step k to the next step $k+1$ includes two parts: $\Delta V_1(k)$ and $\Delta V_2(k)$, where $\Delta V_1(k)$ is the part of incremental volume that comes from the spreading between neighbors in the regular ring of the N-W model. With the assumption of constant spreading velocity $v = 1$, one has

$$\Delta V_1(k) = 1. \qquad (1)$$

Meanwhile, at every time step, there is a new long-rang connection (short-cut) being added into the regular ring with probability $0 < p \ll 1$, and the infected individuals $V(k)$ have probability $2p$ to be selected to connect with other individuals far from them. Therefore, this part of incremental volume is

$$\Delta V_2(k) = 2pV(k). \qquad (2)$$

Besides these two parts, the nonlinear interactions during the spreading process should not be neglected. The nonlinear interactions include, for example, (i) frictions due to congestion as in the case of Internet and transportation traffic jams, (ii) inability of re-firing due to the lack of sufficient oxygen (Yang, 2001), (iii) inability of self-recovery and immunity in epidemiology (Li, *et al.*, 2004a). Such a negative nonlinear interaction effect, denoted $\Delta V_3(k)$, is described by

$$\Delta V_3(k) = -\mu(1+2p)V^2(k), \qquad (3)$$

where $\mu > 0$ is the nonlinear interaction gain constant. Obviously, this negative term varies when p varies from 0 to 1 in the N-W model, because different p means different amount of short-cuts being added into the originally regular ring, which results in different nonlinear interactions affecting the volume $V(k)$ (Li, *et al.*, 2004a). The total incremental infected volume is

$$\Delta V(k) = 1 + 2pV(k) - \mu(1+2p)V^2(k) \quad (4)$$

and the discrete N-W spreading model is thus established as

$$V(k+1) = 1 + (1+2p)V(k)$$

$$-\mu(1+2p)V^2(k). \quad (5)$$

3. DYNAMICAL ANALYSIS OF THE DISCRETE N-W SPREADING MODEL

To analyze the effect of the small-world parameter p and the nonlinear interaction gain μ on the spreading dynamical behaviors, the discrete spreading model (5) is first cast into the form of the well-studied logistic map.

In doing so, denote

$$v(k+1) = A(V(k+1)+B) \quad (6)$$

$$v(k) = A(V(k)+B), \quad (7)$$

where

$$A = \frac{\mu(1+2p)}{1+\sqrt{4p^2+4\mu(1+2p)}}, \quad (8)$$

$$B = \frac{-2p+\sqrt{4p^2+4\mu(1+2p)}}{2\mu(1+2p)}. \quad (9)$$

Then, model (5) can be rewritten as

$$v(k+1) = \lambda v(k)(1-v(k)), \quad (10)$$

which has the familiar format of the logistic map, with

$$\lambda = 1+\sqrt{4p^2+4\mu(1+2p)}. \quad (11)$$

It is well know that the parameter λ determines period-doubling bifurcations and chaos in the logistic map: When $0 < \lambda \leq \lambda_0 \approx 3$, the logistic map trends to a stable fixed point; when $\lambda_0 < \lambda \leq \lambda_1 \approx 3.5699$, the logistic map is in a period doubling cascade; when $\lambda_1 < \lambda \leq \lambda_2 = 4$, the logistic map is in chaotic states.

The interest of this paper is to uncover how the parameters $0 \leq p \leq 1$ and $0 < \mu$ affect the dynamical spreading behaviors of the discrete spreading model (5). To do so, first, fix $0 < \lambda \leq 4$ and $0 \leq p \leq 1$, to obtain

$$\mu = \frac{(\lambda-1)^2-4p^2}{4(1+2p)}. \quad (12)$$

This means, with fixed $0 < \lambda \leq 4$ and $0 \leq p \leq 1$, when $0 < \mu \leq \mu_0$, model (5) approaches to a stable fixed point; when $\mu_0 < \mu \leq \mu_1$, the model has spreading behaviors in the fashion of period-doubling bifurcations; when $\mu_1 < \mu \leq \mu_2$, it has chaotic spreading behaviors. Here,

$$\mu_0 \approx \frac{1-p^2}{1+2p}, \quad (13)$$

$$\mu_1 \approx \frac{1.6511-p^2}{1+2p}, \quad (14)$$

$$\mu_2 = \frac{2.25-p^2}{1+2p}. \quad (15)$$

Figure 1 illustrates the dependence of μ on p in (13)-(15), where it can be observed that when p increases from 0 to 1, those critical values of μ that correspond to the cases of $\lambda = 3, 3.5699, 4$ all decrease. It should be noted that when $\lambda = 3$ and $p = 1$, the nonlinear interaction gain $\mu = 0$. It means that for any $\lambda < 3$, there does not exist any positive $\mu > 0$ for $0 \leq p \leq 1$ in model (5). Figure 1 shows the case of $\lambda = 2.5$, where it can be observed that when $p = 0.76$, $\mu = 0$.

Now, the bifurcating diagram of volume $V(k)$ vs μ in a small-world network is plotted. Set the small-world probability p to be the typical value of 0.1. By (13)-(15), it can be calculated to get $\mu_0(0.1) = 0.825$, $\mu_1(0.1) = 1.368$, and $\mu_2(0.1) = 1.867$. Therefore, in this N-W small-world network: when the gain $0 < \mu < 0.825$, the spreading volume $V(k)$ will finally reach a fixed point; when $0.825 < \mu < 1.368$, there is a period-doubling cascade of $V(k)$; when $1.368 < \mu < 1.867$, the volume $V(k)$ is in a chaotic state.

To guarantee that $V(k) > 0$ in the spreading process, the discrete spreading model (5) is slightly modified to have a cutoff, as

$$V(k+1) = \begin{cases} V'(k+1) & \text{if } 0 \leq V'(k+1) \\ 0 & \text{if } V'(k+1) < 0, \end{cases} \quad (16)$$

where

$$V'(k+1) = 1+(1+2p)V(k) -\mu(1+2p)V^2(k). \quad (17)$$

Figure 2 shows the bifurcating diagram of $V(k)$ vs μ, where blue dots represent the results of the original spreading model (5) without cutoff, and the red circles are the results of the modified spreading model (16) with cutoff.

It can be observed that when $\mu \leq \mu_1(0.1) = 1.368$, both of cases have the same spreading process of approaching a fixed point and then having period-doubling bifurcations. When $\mu > \mu_1(0.1) = 1.368$, the original model (5) enters the chaotic region, while the modified model (16), with increasing μ, further evolves from the chaotic region to period-doubling bifurcations, and finally arrives at a period-two state.

Denote μ_3 as the nonlinear interaction gain at which the period-two region occurs. Then,

$$\begin{aligned} V(k) &= 0 \\ V(k+1) &= 1+(1+2p)*0 \\ &\quad -\mu_3*(1+2p)*0 = 1, \end{aligned} \quad (18)$$

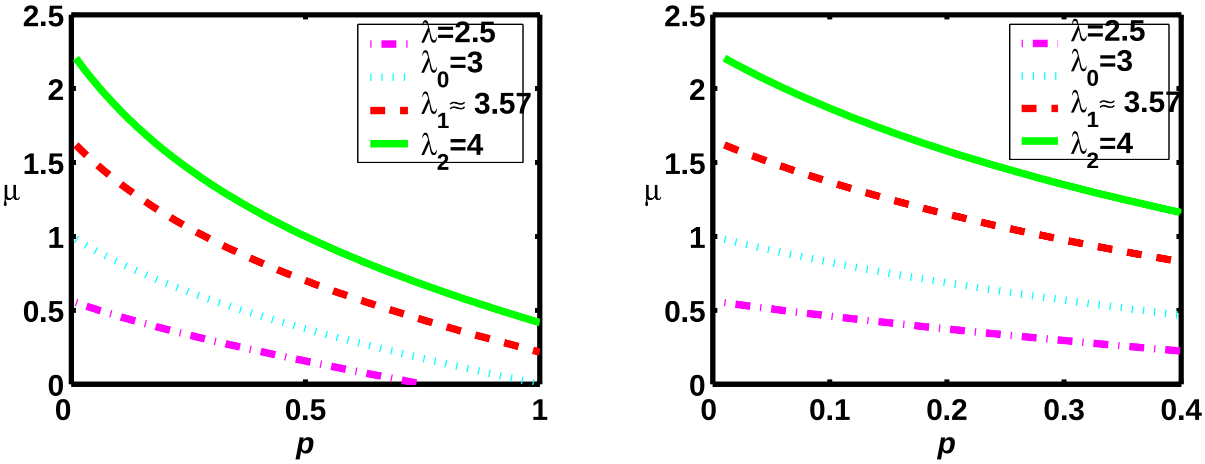

Fig. 1. The relations of μ vs p, with fixed $\lambda = 2.5, 3, 3.5699, 4$ shown by dash-dotted, dotted, dashed, and solid lines, respectively. The left is for $0 \leq p \leq 1$, and the right is for $0 \leq p \leq 0.4$.

$$V(k+2) = 1 + (1+2p) * 1$$
$$-\mu_3 * (1+2p) * 1 = 0,$$

which gives

$$\mu_3 = \frac{2+2p}{1+2p}. \qquad (19)$$

Thus, when $p = 0.1$, one has $\mu_3(0.1) \approx 1.834$ as in Fig. 2. For even bigger $\mu > \mu_3(0.1)$, the spreading volume $V(k)$ still maintains the period-two behavior. It can be observed from Fig. 2 that, for small-world networks, there exists such a μ_3 satisfying $\mu_1 < \mu_3 < \mu_2$, such that the spreading evolution is changing from non-chaotic to chaotic and finally to non-chaotic periodic states.

4. CONCLUSIONS

In this paper, a new discrete spreading model of the N-W small-world type has been proposed, for which some typical dynamical behaviors such as period-doubling bifurcations and chaos have also been analyzed. It was found that the small-world probability and the nonlinear interaction gain constant have significant effects on the network dynamics. As a prototype for modelling the disease spreading phenomenon in various real-life complex small-world networks, this investigation indicates the importance of a suitable strategy for controlling the spreading of diseases, viruses, and disasters alike in various real networks of the small-world type .

REFERENCES

Albert, R. and Barabási, A.L. (2002), Statistical mechanics of complex networks, *Reviews of Modern Physics*, **74**, 47-97.

Li, C.G. and Chen, G. (2004), Local stability and Hopf bifurcation in small-world delayed networks, *Chaos, Solitons and Fractals*, **20**, 353-361.

Li, X. and Chen, G. (2003), Synchronization and de-synchronization of complex dynamical networks: An engineering viewpoint, *IEEE Trans. Circuits and Systems-I*, **50**, 1381-1390.

Li, X. and Chen, G. (2005), Models, dynamics, and control of spreading in complex networks: A survey, *Dynamics of Continuous, Discrete, and Impulsive Systems*, in press.

Li, X., Chen, G. and Li, C.G. (2004a), Stability and bifurcation of disease spreading in complex networks, *Int. J. Systems Sciences*, **35**, 527-536.

Li, X., Wang, X.F. and Chen, G. (2004b), Pinning a complex dynamical network to its equilibrium, *IEEE Trans. Circuits and Systems-I*, **51**, 2074-2087.

Lv, J.H., Yu, X.H. and Chen, G. (2004), Chaos synchronization of general complex dynamical networks, *Physica A*, **334**, 281-304.

Moukarzel, C.F. (1999), Spreading and shortest paths in systems with sparse long-range connections, *Physical Review E*, **60**, 6263-6266.

Newman, M.E.J. and Watts, D.J. (1999), Scaling and percolation in the small-world network model, *Physical Review E*, **60**, 7332-7342.

Wang, X.F. (2002), Complex networks: topology, dynamics and synchronization. *Int. J. Bifurcation and Chaos*, **12**, 885-916.

Wang, X.F. and Chen, G. (2003), Complex networks: Small-world, scale-free, and beyond, *IEEE Circuits and Systems Magazine*, **3**, 6-20.

Watts, D.J. (1999), *Small Worlds*, Princeton University Press.

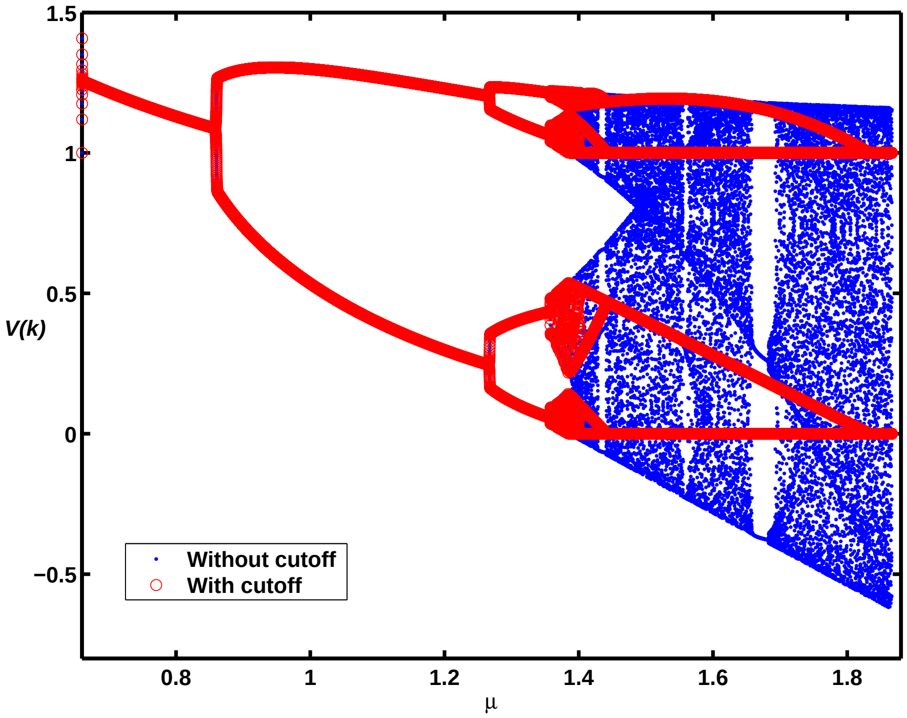

Fig. 2. The bifurcating spreading diagrams w.r.t. μ in the discrete N-W small-world model, where $p = 0.1$ and $V(0) = 1$. Blue dots are the results of the original model (5) without cutoff, and red circles are the results of the modified model (16) with cutoff.

Yang, X.S. (2001), Chaos in small-world networks,
Physical Review E, **63**, 046206.