OPTIMIZING PREDICTION DYNAMICS FOR ROBUST MPC

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Abstract: A convex formulation is derived for optimizing dynamic feedback laws for constrained linear systems with polytopic uncertainty. We show that, when it exists, the maximal invariant ellipsoidal set for the plant state under a dynamic feedback law incorporating any chosen static feedback gain is equal to the maximal invariant ellipsoidal set under any linear feedback law. The dynamic controller and its associated invariant set define a computationally efficient robust MPC law with prediction dynamics belonging to a polytopic uncertainty set.

Keywords: dynamic feedback; predictive control; linear matrix inequalities; constraints

1. INTRODUCTION

The computational burden of model predictive control (MPC) is a key factor limiting the range of its applications. MPC algorithms for the linearly constrained linear-quadratic optimal control problem require the online solution of a quadratic program (QP), and the online computational requirement can be very much higher for MPC algorithms incorporating robustness to model uncertainty (Mayne *et al.*, 2000). In particular, the robust min-max MPC approach of (Kothare *et al.*, 1996) uses an ellipsoidal constraint approximation to formulate the online optimization as a semidefinite program (SDP) with a computational load that is prohibitive for fast sampling applications. The explicit solution of linear min-max MPC (Bemporad *et al.*, 2003) is a piecewise affine feedback law which can be computed offline using multiparametric quadratic programming, however the prohibitive online storage requirement of this approach limits its application to simple low-order systems (Imsland *et al.*, 2004).

Ellipsoidal constraint approximations are the basis of the MPC law of (Kouvaritakis *et al.*, 2000; Kouvaritakis *et al.*, 2002), which provides robustness to polytopic parametric uncertainty but requires a fraction of the online computation of QP-based MPC for linear systems with no model uncertainty. The approximate constraints are computed offline by optimizing an invariant ellipsoidal set for a dynamic state feedback law, and the online computation is therefore reduced to the optimization of the controller state subject to an ellipsoidal constraint. The approach was extended in (Drageset *et al.*, 2003; Imsland *et al.*, 2004) by allowing the parameters of the dynamic feedback law to be variables in the offline optimization. As a result the suboptimality in the size of the stabilizable set associated with the MPC law of (Kouvaritakis *et al.*, 2000; Kouvaritakis *et al.*, 2002) can be reduced, but the formulation of (Drageset *et al.*, 2003; Imsland *et al.*, 2004) leads to a nonconvex optimization with no guarantee of convergence to a solution.

This paper provides a convex formulation for the optimization of prediction dynamics and shows that the resulting maximal invariant ellipsoidal set is equal to the maximal invariant ellipsoidal set under any linear feedback law. The formulation employs a nonlinear transformation of variables similar to that used in dynamic output feedback design problems (Scherer *et al.*, 1997; Skelton *et al.*, 1998), modified to allow for polytopic uncertainty in model parameters and linear input/state constraints. Our approach suggests a generalization of the prediction dynamics of (Kouvaritakis *et al.*, 2000; Kouvaritakis *et al.*, 2002) that enables the dynamics governing the evolution of the predicted controller state to vary depending on the predicted plant state. Despite significant improvements in the size of the associated stabilizable set, this generalization does not increase online computational load.

2. PROBLEM STATEMENT

Consider the discrete-time uncertain linear system

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad y(k) = Cx(k) \quad (1)$$

with constraints on the state $x \in \mathbb{R}^{n_x}$ and input $u \in \mathbb{R}^{n_u}$:

$$Fx(k) + Gu(k) \leq f. \quad (2)$$

The values of $A(k), B(k)$ are unknown but belong to a polytopic uncertainty class Ω at each instant k, i.e.

$[A(k)\ \ B(k)] \in \Omega = \mathrm{Co}\big\{[A_j\ \ B_j],\ j = 0,\ldots,m\big\}$ where $A_j, B_j, j = 0,\ldots,m$ are known constant matrices. The state $x(k)$ is assumed measurable at time k.

We consider a quadratic performance index evaluated for nominal or worst-case predicted performance. Let $\{u(k+i|k), i \geq 0\}$ denote a control sequence predicted at time k, then the worst-case cost index is defined

$$\tilde{J}(k) = \max_{\substack{[A(k+i)\ B(k+i)]\in\Omega \\ i=0,1,\ldots}} \sum_{i=0}^{\infty} \Big[x^T(k+i|k)C^TCx(k+i|k) + u^T(k+i|k)Ru(k+i|k) \Big] \quad (3)$$

where $\{x(k+i|k), i \geq 0\}$ is a predicted trajectory of (1) with $x(k|k) = x(k)$. Alternatively, the nominal performance index corresponding to $[A_0\ \ B_0]$ (e.g. the centre of Ω or expected value of $[A(k)\ \ B(k)]$) is defined

$$J_0(k) = \sum_{i=0}^{\infty} \Big[x_0^T(k+i|k)C^TCx_0(k+i|k) + u^T(k+i|k)Ru(k+i|k) \Big] \quad (4)$$

where $\{x_0(k+i|k), i \geq 0\}$ is the predicted trajectory of

$$x_0(k+i+1|k) = A_0x_0(k+i|k) + B_0u(k+i|k) \quad (5)$$

with $x_0(k|k) = x(k)$.

To ensure closed-loop stability, constraints (2) must be satisfied along predicted trajectories corresponding to all future realizations of model uncertainty. i.e. $\forall i \geq 0$:

$$\max_{\substack{[A(k+j)\ B(k+j)]\in\Omega \\ j=0,1,\ldots,i-1}} Fx(k+i|k) + Gu(k+i|k) \leq f. \quad (6)$$

Applying these constraints to open-loop predicted trajectories leads to highly conservative MPC strategies, and it is therefore preferable to use closed-loop predictions which take into account the mitigating effects of future state measurements. However closed-loop MPC strategies incorporating the robust constraints (6) through either online (Lee and Yu, 1997) or offline (Bemporad *et al.*, 2003) optimization impose a very high computational load. Computational burden can be reduced by approximating the feasible set for the constraints (6). For example (Kothare *et al.*, 1996) optimizes a linear state feedback law $u(k+i|k) = H(k)x(k+i|k)$ online subject to ellipsoidal state constraints that can be formulated as LMI conditions.

The ellipsoidal constraint approximation is constructed online in (Kothare *et al.*, 1996), but very significant reductions in online computational load are obtained if the feasible set approximation is computed offline. This is done in (Kouvaritakis *et al.*, 2000; Kouvaritakis *et al.*, 2002) by expressing the degrees of freedom in predicted inputs as a perturbation sequence $\{v(i|k),\ i = 0,1,\ldots\}$ on a fixed linear feedback law:

$$u(k+i|k) = Hx(k+i|k) + v(i|k), \quad (7)$$

and then constructing an ellipsoidal approximation of the feasible set for the plant state and optimization variables $\{v(i|k)\}$ through an offline optimization subject to LMI constraints. The gain H is required to stabilize the uncertainty class Ω in the absence of constraints, and should also be optimal when constraints are inactive (e.g. LQ-optimal for the nominal model).

Although the offline constraint approximation enables extremely efficient online optimization (Kouvaritakis *et al.*, 2002), it has the effect of limiting the stabilizable set of the MPC law to an ellipsoidal set which is suboptimal relative to the actual feasible set for (6). Moreover the degree of suboptimality depends on the choice of H, the number of free variables, and their parameterization in the sequence $\{v(i|k)\}$. To overcome this problem, this paper optimizes the evolution of the perturbation sequence $\{v(i|k)\}$ in (7) so as to maximize the associated ellipsoidal stabilizable set.

3. OPTIMIZATION OF PREDICTION DYNAMICS

Predicted trajectories of (1) corresponding to (7) can be generated by a dynamic state feedback law:

$$z(k+i+1|k) = \mathscr{A}(k+i|k)z(k+i|k) \quad (8a)$$

$$\mathscr{A}(k+i|k) \in \mathrm{Co}\{\mathscr{A}_j,\ j = 0,\ldots,m\} \quad (8b)$$

$$\mathscr{A}_j = \begin{bmatrix} \Phi_j & B_jC_c \\ 0 & A_c \end{bmatrix},\ \Phi_j = A_j + B_jH \quad (8c)$$

$$z(k|k) = \begin{bmatrix} x(k) \\ c(k) \end{bmatrix},\ \begin{array}{l} x(k+i|k) = \begin{bmatrix} I & 0 \end{bmatrix} z(k+i|k) \\ u(k+i|k) = \begin{bmatrix} H & C_c \end{bmatrix} z(k+i|k) \end{array} \quad (8d)$$

where the controller state $c(k) \in \mathbb{R}^{n_c}$ is the optimization variable to be determined online and $v(i|k) = C_cA_c^ic(k)$ in (7). In (Kouvaritakis *et al.*, 2000) for example, the degrees of freedom in (7) are the perturbations $v(i|k)$ for $i = 0,\ldots,N-1$, with $v(i|k) = 0$ for all $i \geq N$, so that $c(k)$ and the controller parameters C_c, A_c are given (non-uniquely) by:

$$c(k) = \begin{bmatrix} v(0|k) \\ v(1|k) \\ \vdots \\ v(N-1|k) \end{bmatrix}, \quad \begin{array}{l} C_c = \begin{bmatrix} I_{n_u} & 0 & \cdots & 0 \end{bmatrix} \\ A_c = \begin{bmatrix} 0 & I_{n_u} & & \\ \vdots & & \ddots & \\ 0 & & & I_{n_u} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array} \quad (9)$$

Since predictions are governed by the autonomous dynamics (8), an inner bound on the feasible region for (6) in (x,c)-space is given by any invariant set for the state of (8a-c) on which constraints (2) are instantaneously satisfied. For the ellipsoidal set $\mathscr{E} = \{z : z^T\mathscr{P}z \leq 1\}$, it is easy to show (e.g. (Boyd *et al.*, 1994)) that invariance under (8a-c) and feasibility w.r.t. (2) are equivalent to the following conditions:

$$\mathscr{P} - \mathscr{A}_j^T\mathscr{P}\mathscr{A}_j > 0, \quad j = 0,\ldots,m \quad (10a)$$

$$\begin{bmatrix} W & [F+GH\ \ GC_c] \\ * & \mathscr{P} \end{bmatrix} > 0, \quad W_{ii} \leq f_i^2 \quad (10b)$$

($*$ indicates an off-diagonal block of a symmetric matrix). Clearly (10a,b) are LMIs in $\mathscr{P}$, and if the controller parameters A_c, C_c are fixed, $\mathscr{P}$ can therefore be computed offline via a convex optimization with a suitable objective (such as maximization of the projection of $\mathscr{E}$ onto the x-subspace, which is an inner bound on the stabilizable set for x).

Use of fixed A_c and C_c in (9) allow the predictions corresponding to any finite length perturbation sequence $\{v(i|k)\}$ to be realized by (8) with sufficiently large n_c.

In practice, n_c is restricted by limits on the number of free variables in $\mathscr{P}$ that can be handled in the offline optimization problem (the online optimization of $c(k)$ is univariate and its computational load increases only linearly with n_c (Kouvaritakis *et al.*, 2002)). Moreover for any given value of n_c, it is clear that a larger feasible set could be obtained if A_c and C_c were also variables in the optimization of $\mathscr{E}$.

This is the motivation behind the optimization of $\mathscr{E}$ over variables A_c, C_c and $\mathscr{P}$ considered in (Drageset *et al.*, 2003; Imsland *et al.*, 2004). Since (10a) is nonconvex in $A_c, C_c, \mathscr{P}$, a sequential semi-definite programming approach is proposed in (Drageset *et al.*, 2003) for handling nonconvex constraints that enforce (10a,b). However, by defining a transformation of variables similar to that used in dynamic output feedback design problems (Scherer *et al.*, 1997; Skelton *et al.*, 1998), the offline optimization of $A_c, C_c, \mathscr{P}$ can be reformulated as a convex LMI problem, thus eliminating the problems of convergence, feasibility and computational complexity caused by nonconvex constraints in the optimization of $\mathscr{E}$. To show this, let $U, V \in \mathbb{R}^{n_x \times n_c}$, $K \in \mathbb{R}^{n_x \times n_x}$, $M \in \mathbb{R}^{n_u \times n_x}$ and symmetric $X, Y \in \mathbb{R}^{n_x \times n_x}$ be defined by

$$\mathscr{P} = \begin{bmatrix} X^{-1} & X^{-1}U \\ U^T X^{-1} & * \end{bmatrix}, \ \mathscr{P}^{-1} = \begin{bmatrix} Y & V \\ V^T & * \end{bmatrix}, \ \begin{matrix} K = UA_cV^T \\ M = C_cV^T \end{matrix} \tag{11}$$

so that $\mathscr{P}\mathscr{P}^{-1} = I$ implies

$$UV^T = X - Y. \tag{12}$$

The following theorem derives conditions equivalent to (10a,b) that are convex in X, Y, K, M.

Theorem 1. There exist $A_c, C_c, \mathscr{P}, W$ satisfying (10a,b) only if the LMIs in X, Y, K, M, W below are feasible.

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi_j Y + B_j M & \Phi_j X \\ K + \Phi_j Y + B_j M & \Phi_j X \end{bmatrix} \\ * & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} > 0 \ \ j = 0, \ldots, m \tag{13a}$$

$$\begin{bmatrix} W & \begin{bmatrix} (F+GH)Y + GM & (F+GH)X \end{bmatrix} \\ * & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} > 0, \ W_{ii} \le f_i^2 \tag{13b}$$

Furthermore feasibility of (13a,b) is necessary and sufficient for feasibility of (10a,b) if $n_c \ge n_x$.

Proof: Pre- and post-multiplying (10a) respectively by

$$\begin{bmatrix} \Pi^T & 0 \\ 0 & \Pi^T \end{bmatrix}, \quad \begin{bmatrix} \Pi & 0 \\ 0 & \Pi \end{bmatrix}, \quad \text{with } \Pi = \begin{bmatrix} Y & X \\ V^T & 0 \end{bmatrix} \tag{14}$$

and using (12) yields (13a). Similarly, pre- and post-multiplication of (10b) respectively by

$$\begin{bmatrix} I & 0 \\ 0 & \Pi^T \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ 0 & \Pi \end{bmatrix} \tag{15}$$

gives (13b). Therefore (10a,b) can only be feasible if (13a,b) are feasible. Note also that U, V can be assumed full-rank without loss of generality since the inequalities involving $\mathscr{P}$ in (10a,b) are strict, and hence the feasible set for $\mathscr{P}$ is open. From the definitions of K, M in (11), solutions for A_c, C_c therefore exist for given K, M whenever $n_c \ge n_x$, implying that (10a,b) are feasible if and only if (13a,b) are feasible in this case. For the case that $n_c = n_x$, the solutions for A_c, C_c:

$$A_c = U^{-1}KV^{-T}, \quad C_c = MV^{-T}$$

are unique. □

The projection of $\mathscr{E}$ onto the x-subspace is given by $\mathscr{E}_x = \{x : x^T Y^{-1} x \le 1\}$. Therefore the offline maximization of $\mathscr{E}_x$ over $A_c, C_c, \mathscr{P}$ subject to (10a,b) can be performed by solving the SDP problem:

$$\text{maximize} \ \ \log\det Y \ \ \text{subject to} \ \ (13\text{a,b}) \tag{16}$$

in variables X, Y, K, M, W, then factorizing $X - Y$ to determine U, V satisfying (12), and finally using the definitions of K, M in (11) to solve for A_c, C_c.

Corollary 2. The optimal value of Y in (16), and hence also the maximal projection $\mathscr{E}_x$, are independent of the value of n_c if $n_c \ge n_x$.

Remark 3. Nonconvex constraints (i.e. $\text{rank}(X - Y) = n_c$) would be needed in (16) if $n_c < n_x$ to ensure that (12) admits solutions for U, V. Since there is also no advantage to be gained through use of $n_c > n_x$, we assume that $n_c = n_x$ for the remainder of the paper.

It is possible to impose bounds on the predicted cost along trajectories of (8) through LMIs in the variables of (16). For any given bound γ, $\tilde{J}(k) \le \gamma$ is ensured for all initial conditions $z(k|k)$ of (8) in $\mathscr{E}$ if (10a) is replaced by the strengthened invariance condition (see e.g. (Kothare *et al.*, 1996)):

$$\mathscr{P} - \mathscr{A}_j^T \mathscr{P} \mathscr{A}_j > \frac{1}{\gamma} \begin{bmatrix} C^T & H^T \\ 0 & C_c^T \end{bmatrix} \mathscr{D} \begin{bmatrix} C & 0 \\ H & C_c \end{bmatrix} \ \ j = 0, \ldots, m \tag{17}$$

($\mathscr{D} = \text{diag}\{I, R\}$). Using a congruence transformation similar to (14), this condition can be shown to be equivalent to the following LMI in X, Y, K, M.

$$\begin{bmatrix} \gamma I & 0 & \mathscr{D}^{1/2} \begin{bmatrix} CY & CX \\ HY + M & HX \end{bmatrix} \\ * & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi_j Y + B_j M & \Phi_j X \\ K + \Phi_j Y + B_j M & \Phi_j X \end{bmatrix} \\ * & * & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} > 0 \ \ j = 0, \ldots, m \tag{18}$$

Therefore performance bounds can be imposed by including (18) as a constraint in place of (13a) in the offline optimization (16).

4. MAXIMAL STABILIZABLE SET

This section compares the maximal stabilizable set $\mathscr{E}_x$ subject to (13a,b) (or (18) and (13b)) with the maximal invariant ellipsoidal set under any linear feedback law. For the special case that (1) is time-invariant, we show that (13a,b) have solutions iff $\mathscr{E}_x$ is invariant under some linear feedback law. Thus (16) recovers the maximal invariant ellipsoidal set under linear feedback even though predictions (8) are centered on a feedback law $u = Hx$ which provides optimal performance in the absence of constraints but is not designed to be optimal in terms of size of the associated invariant set. We show that the same result applies to LTV systems if (8) is generalized to allow A_c to vary depending on the evolution of $A(k), B(k)$ over the prediction horizon.

Consider the problem of determining $Y, \tilde{H}$ such that the ellipsoidal set $\mathscr{E}_x = \{x : x^T Y^{-1} x \le 1\}$ is invariant under linear feedback $u = \tilde{H}x$. In terms of variables Y and $\tilde{M} = \tilde{H}Y$, conditions for invariance w.r.t. (1),(2) can be expressed as LMIs (Boyd *et al.*, 1994):

$$\begin{bmatrix} Y & A_jY + B_j\tilde{M} \\ * & Y \end{bmatrix} > 0 \quad j = 0,\dots,m \tag{19a}$$

$$\begin{bmatrix} W & FY + G\tilde{M} \\ * & Y \end{bmatrix} > 0, \quad W_{ii} \le f_i^2 \tag{19b}$$

Remark 4. The maximal ellipsoidal invariant set for (1) cannot be enlarged by using dynamic rather than static linear state feedback in the absence of constraints (see e.g. (Boyd *et al.*, 1994)). The same is true when linear input/state constraints (2) are present, i.e. the feasible set for Y in (19a,b) is identical to the set of feasible Y such that $\mathscr{E}_x$ is invariant under a dynamic feedback law of the form $u(k) = \tilde{H}x(k) + C_c c(k)$ with $c(k+1) = A_c c(k) + B_c x(k)$, where $\tilde{H}, C_c, A_c, B_c$ are variables. Therefore all ellipsoidal sets $\mathscr{E}_x$ that are invariant under linear feedback must satisfy (19a,b).

The following theorem shows that the feasible sets for Y in (19a,b) and (13a,b) are identical for the LTI case.

Theorem 5. Let $A(k) = A_0$, $B(k) = B_0$ in (1) for all k. Then there exist X, Y, K, M, W satisfying (13a,b) if and only if Y, W are solutions of (19a,b) for some $\tilde{M}$.

Proof: We first eliminate K in (13a,b) by showing that (13a) is equivalent to the following conditions:

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi_0 X \\ \Phi_0 X \end{bmatrix} \\ * & X \end{bmatrix} > 0 \tag{20a}$$

$$\begin{bmatrix} Y & \begin{bmatrix} \Phi_0 Y + B_0 M & \Phi_0 X \end{bmatrix} \\ * & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} > 0. \tag{20b}$$

Necessity of (20a,b) follows directly from cancelling the second or third block row/column in (13a). Sufficiency is shown by writing equivalent conditions for (13a) as:

$$\begin{bmatrix} Y & \Phi_0 X \\ * & X \end{bmatrix} > 0 \tag{21a}$$

$$\begin{bmatrix} X & K + \Phi_0 Y + B_0 M \\ * & Y \end{bmatrix} - \begin{bmatrix} X & \Phi_0 X \\ (\Phi_0 Y + B_0 M)^T & X \end{bmatrix} \cdot \begin{bmatrix} Y & \Phi_0 X \\ * & X \end{bmatrix}^{-1} \begin{bmatrix} X & \Phi_0 Y + B_0 M \\ X\Phi_0^T & X \end{bmatrix} > 0 \tag{21b}$$

so if K is chosen so as to make (21b) block-diagonal:

$$K = \begin{bmatrix} X & \Phi_0 X \end{bmatrix} \begin{bmatrix} Y & \Phi_0 X \\ * & X \end{bmatrix}^{-1} \begin{bmatrix} \Phi_0 Y + B_0 M \\ X \end{bmatrix} - \Phi_0 Y - B_0 M, \tag{22}$$

then (21b) gives the conditions

$$X - \begin{bmatrix} X & \Phi_0 X \end{bmatrix} \begin{bmatrix} Y & \Phi_0 X \\ * & X \end{bmatrix}^{-1} \begin{bmatrix} X \\ X\Phi_0^T \end{bmatrix} > 0,$$

$$Y - \begin{bmatrix} (\Phi_0 Y + B_0 M)^T & X \end{bmatrix} \begin{bmatrix} Y & \Phi_0 X \\ * & X \end{bmatrix}^{-1} \begin{bmatrix} \Phi_0 Y + B_0 M \\ X \end{bmatrix} > 0,$$

which are the Schur complements of (21a) in (20a) and (20b) respectively. Hence solutions of (20a,b) for X, Y, M also satisfy (13a) if K is given by (22).

Next we show that (20a,b),(13b) are feasible iff (19a,b) are feasible for $j = 0$. Condition (20a) is equivalent to

$$\begin{bmatrix} X & \Phi_0 X \\ * & X \end{bmatrix} > 0, \quad Y - X > 0 \tag{23}$$

whereas (20b) and (13b) can be written

$$\begin{bmatrix} Y & \Phi_0 Y + B_0 M \\ * & Y \end{bmatrix} - \begin{bmatrix} \Phi_0 X \\ X \end{bmatrix} X^{-1} \begin{bmatrix} X\Phi_0^T & X \end{bmatrix} > 0, \; X > 0 \tag{24a}$$

$$\begin{bmatrix} W & (F + GH)Y + GM \\ * & Y \end{bmatrix} - \begin{bmatrix} (F + GH)X \\ X \end{bmatrix} X^{-1} \cdot \begin{bmatrix} X(F + GH)^T & X \end{bmatrix} > 0, \quad W_{ii} \le f_i^2. \tag{24b}$$

Therefore (19a,b) for $j = 0$ are implied by (24a,b) with $\tilde{M} = M + HY$. Alternatively, given $Y, \tilde{M}, W$ satisfying (19a,b) for $j = 0$ and any solution X_0 of the first LMI in (23), a solution of (24a,b) can be constructed by setting $M = \tilde{M} - HY$ and $X = \varepsilon X_0$ for sufficiently small $\varepsilon > 0$. Finally note that the first LMI in (23) is necessarily feasible due to the assumption that H is stabilizing in the absence of constraints. □

For the case of LTI models, all solutions of (13a,b) for Y are also valid solutions of (19a,b) because A_c is available to place eigenvalues of $\mathscr{A}_0$ at the eigenvalues of $A_0 + B_0\tilde{H}$, thus enabling (8a-c) to generate the predictions that would be obtained with any given static feedback gain $\tilde{H}$. However it is not possible to generate all predicted trajectories of an uncertain LTV model under $u = \tilde{H}x$ with a single value of A_c (or equivalently to satisfy (22) with a single value of K if Φ_0, B_0 are replaced by Φ_j, B_j, $j = 0,\dots,m$). Hence the maximal stabilizable set $\mathscr{E}_x$ subject to (13a,b) is necessarily smaller than the maximal invariant $\mathscr{E}_x$ constrained by (19a,b) in the uncertain LTV case.

To extend Theorem 5 to the case of polytopic uncertainty, A_c must be allowed to take any value in a polytopic set with as many vertices as Ω. This is achieved by replacing (8b) with the modified prediction system:

$$\begin{aligned} &\mathscr{A}(k+i|k) \in \mathrm{Co}\{\mathscr{A}_j, \; j = 0,\dots,m\}, \\ &\mathscr{A}_j = \begin{bmatrix} \Phi_j & B_j C_c \\ 0 & A_{c,j} \end{bmatrix}, \; \Phi_j = A_j + B_j H. \end{aligned} \tag{25}$$

With $\mathscr{A}_j$ as defined above, the invariance conditions (10a,b) ensure that, for any $A(k), B(k)$ within the uncertainty class Ω, there exists $A_c(k) \in \mathrm{Co}\{A_{c,j}, \; j = 0,\dots,m\}$ so that $\mathscr{E} = \{z : z^T \mathscr{P} z \le 1\}$ is invariant.

By defining transformed variables U, V, X, Y, M as in (11) and $K_j = UA_{c,j}V^T$ $j = 0,\dots,m$, it can be shown using congruence transformations (14) and (15) that feasibility of (10a,b) is equivalent to feasibility of:

$$\begin{bmatrix} \begin{bmatrix} Y & X \\ X & X \end{bmatrix} & \begin{bmatrix} \Phi_j Y + B_j M & \Phi_j X \\ K_j + \Phi_j Y + B_j M & \Phi_j X \end{bmatrix} \\ * & \begin{bmatrix} Y & X \\ X & X \end{bmatrix} \end{bmatrix} > 0 \;\; j = 0,\ldots,m \tag{26}$$

and (13b). Hence the maximization of $\mathscr{E}_x$ can be performed via a convex optimization of the form (16), and the solutions for C_c and $A_{c,j}$, $j = 0,\ldots,m$ are unique if $n_c = n_x$. The generalization of Theorem 5 for the LTV case is stated below.

Corollary 6. The feasible sets for Y in (19a,b) and (26),(13b) are identical.

Remark 7. Theorem 5 and Corollary 6 also apply if bounds are imposed on performance by replacing (10a) with (17). Thus for the LTI case the maximal $\mathscr{E}_x$ subject to (18),(13b) is identical to the maximal ellipsoidal invariant set under linear feedback satisfying the given cost bound. The same property holds for LTV models if the modified prediction dynamics (25) are used, and K is replaced by K_j in (18).

5. PREDICTION COST AND CONTROL LAW

Although the use of a prediction system incorporating (25) introduces feedback into the perturbation sequence in (7) by letting $\{v(i|k)\}$ vary depending on the uncertain plant model, thus allowing a larger stabilizable set $\mathscr{E}_x$ than (8b), it does not require an increase in online computation. This is because an MPC law based on optimizing $c(k)$ online can be implemented without computing the implied sequence $\mathscr{A}(k+i|k)$. Furthermore the predicted cost index is quadratic and convex in $c(k)$, and the online computational requirement is therefore the same as that of (Kouvaritakis *et al.*, 2000; Kouvaritakis *et al.*, 2002).

For the prediction system (8a-c) with unique A_c, the nominal cost (4),(5) is given by $J_0(k) = z^T(k|k)\mathscr{W}z(k|k)$, where $\mathscr{W}$ is the solution of the Lyapunov equation:

$$\mathscr{W} - \mathscr{A}_0^T \mathscr{W} \mathscr{A}_0 = \begin{bmatrix} C^T & H^T \\ 0 & C_c^T \end{bmatrix} \mathscr{D} \begin{bmatrix} C & 0 \\ H & C_c \end{bmatrix} \tag{27}$$

($\mathscr{D} = \mathrm{diag}\{I, R\}$). If H is unconstrained LQ-optimal for J_0, then $J_0(k) = x^T(k)W_x x(k) + c^T(k)W_c c(k)$,

$$W_x - \Phi_0^T W_x \Phi_0 = C^T C + H^T R H \tag{28a}$$

$$W_c - A_c^T W_c A_c = C_c^T (R + B_0^T W_x B_0) C_c. \tag{28b}$$

For a prediction system incorporating (25), where A_c can take any value in a polytopic set, we define J_0 as the maximum predicted cost for the nominal model,

$$J_0(k) = \max_{\substack{A_c(k+i) \in \mathrm{Co}\{A_{c,j}\} \\ i=0,1,\ldots}} \sum_{i=0}^{\infty} \Big[x_0^T(k+i|k) C^T C x_0(k+i|k) + u^T(k+i|k) R u(k+i|k) \Big] \tag{29}$$

where x_0 satisfies (5). A quadratic bound for (29) is stated below.

Lemma 8. If H is unconstrained LQ-optimal for J_0, then $J_0(k) \le x^T(k)W_x x(k) + c^T(k)W_c c(k)$ where

$$W_c - A_{c,j}^T W_c A_{c,j} \ge C_c^T (R + B_0^T W_x B_0) C_c \quad j = 0,\ldots,m \tag{30}$$

Furthermore solutions for W_c exist if (10a) is feasible.

In either case of a unique A_c or A_c chosen from a polytopic set, the MPC law defined by the online minimization of J_0 subject to $z(k|k) \in \mathscr{E}$ has the form

$$u(k) = Hx(k) + C_c c(k), \;\; c(k) = \underset{c \in \mathscr{E}_c(x(k))}{\mathrm{argmin}} \; c^T W_c c \tag{31}$$

where $\mathscr{E}_c(x(k)) = \{c : [x^T(k) \;\; c^T]^T \in \mathscr{E}\}$ is the ellipsoidal feasible set approximation corresponding to $z(k|k) \in \mathscr{E}$. The online optimization (31) can be formulated as a univariate search with complexity $O(n_x)$ if W_c and $\mathscr{P}$ are factorized offline (Kouvaritakis *et al.*, 2002). The closed-loop stability properties of this control law follow from the finite l_2-gain of the closed-loop dynamics mapping the perturbation sequence $\{C_c c(k)\}$ to $\{x(k)\}$, stated below.

Lemma 9. If $\mathscr{P}$ is a solution of (10a) or (17), then the closed-loop trajectories under (31) satisfy

$$\sum_{k=0}^{\infty} x^T(k)(C^T C + H^T R H)x(k) \le \beta \sum_{k=0}^{\infty} c^T(k) C_c^T C_c c(k) + \gamma x^T(0) X^{-1} x(0) \tag{32}$$

for some $\beta, \gamma > 0$, where X^{-1} is the 1,1-block of $\mathscr{P}$.

Theorem 10. The origin of (1) is asymptotically stable under (31) for any initial condition $x(0) \in \mathscr{E}_x$.

Proof: If $x(0) \in \mathscr{E}_x$, then $\mathscr{E}_c(x(0)) \ne \emptyset$, and the invariance of $\mathscr{E}$ ensures that (31) remains feasible at all future times. Also $A_c(k)c(k)$ defines a feasible solution for $c(k+1)$ (where $A_c(k) = A_c$ if a unique A_c is optimized offline, and $A_c(k) \in \mathrm{Co}\{A_{c,j} \;\; j = 0,\ldots,m\}$ otherwise), so using (28b) or (30) and the bound (32), and defining $\lambda = \lambda_{\min}(R + B_0^T W_x B_0) > 0$, we have

$$\sum_{k=0}^{\infty} x^T(k)(C^T C + H^T R H)x(k) \le \frac{\beta}{\lambda} c^T(0) W_c c(0) + \gamma x^T(0) X^{-1} x(0)$$

which implies that $[Cx(k) \;\; R^{1/2}Hx(k)] \to 0$ uniformly as $k \to \infty$, and therefore $x(k)$ converges asymptotically to zero under the usual observability assumptions.

Remark 11. The above approach can also be used to construct a min-max MPC law if an upper bound on the worst-case cost is computed as $\tilde{J}(k) \le z^T(k|k)\mathscr{W}z(k|k)$ where $\mathscr{W}$ is a solution of

$$\mathscr{W} - \mathscr{A}_j^T \mathscr{W} \mathscr{A}_j \ge \begin{bmatrix} C^T & H^T \\ 0 & C_c^T \end{bmatrix} \mathscr{D} \begin{bmatrix} C & 0 \\ H & C_c \end{bmatrix}, \quad j = 0,\ldots,m$$

with $\mathscr{A}_j$ as defined in (8b) or (25). The online MPC optimization: $\min_{z \in \mathscr{E}} z^T \mathscr{W} z$ retains the computational advantages of (31), and closed-loop asymptotic stability follows from the rate of decrease of the optimal cost $\tilde{J}(k)$ along closed-loop trajectories as in conventional robust MPC (see e.g. (Mayne *et al.*, 2000)).

6. IMPLEMENTATION AND EXAMPLES

From the proof of Theorem 5 it follows that solutions of (16) for X are non-unique in general. However, in the interest of good closed-loop performance under the MPC law (31), the ellipsoidal set defined $\mathcal{E}_{xx} = \{x : \; x^T X^{-1} x \leq 1\}$ should be maximized since this is the region on which the unconstrained LQ-optimal feedback law is feasible in (31). To address this secondary objective, the offline optimization of prediction dynamics can be split into two SDP problems:

(i). maximize $\log\det Y$ subject to (19a,b) (ii). maximize $\log\det X$ subject to (13a,b)

where Y and $M = \tilde{M} - HY$ in the constraints of step (ii) are fixed at the values computed in step (i). An alternative approach (avoiding numerical ill-conditioning in step (ii) above) is to use a single optimization:

$$\text{maximize } (\det Y)^{1/n_x} + \alpha(\det X)^{1/n_x} \text{ s.t. (13a,b)} \quad (33)$$

for some small constant $\alpha > 0$. This can be formulated as an SDP problem (Nesterov and Nemirovskii, 1994).

Example 1 The constrained LTI double integrator considered in (Imsland *et al.*, 2004) has

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix}, \; C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{array}{l} |u| \leq 1 \\ \left|\begin{bmatrix} 0 & 1 \end{bmatrix} x\right| \leq 1 \end{array}$$

with $T_s = 0.05$. The sizes of the maximal invariant sets $\mathcal{E}_x$ computed using the objective of (33) with $\alpha = 0.01$ subject to constraints (18),(13b) are indicated in Table 1. These are identical to the maximal invariant sets for the same cost bounds under variable static feedback. The sizes of $\mathcal{E}_{xx}$ are comparable to the maximal invariant ellipsoidal set for $n_c = 0$ under LQ-optimal feedback $u = Hx$, for which $\det(Y)^{1/2} = 1.05$.

Table 1. Invariant set size vs. γ

γ	10^2	10^4	10^6	10^8
$\det(Y)^{1/2}$	1.46	6.77	31.2	6573
$\det(X)^{1/2}$	1.02	1.01	0.97	0.99

Closed-loop costs under (31) are given in Table 2. Improved performance is obtained by switching online between ellipsoidal sets and optimized prediction dynamics corresponding to successively lower performance bounds. The switching criteria used here require the predicted cost to be lower and more rapidly decreasing before switching to a candidate prediction system. Further reductions in suboptimality are obtained using scaling (Kouvaritakis *et al.*, 2002).

Table 2. Closed-loop costs

initial condition	(31)	(31)+ switching	(31)+ switching & scaling	infinite horizon MPC
$(-4, 0)$	637.8	631.3	623.2	609.4
$(-2, -0.6)$	222.5	210.9	206.3	204.5

Example 2 For uncertain LTV systems, optimization of a polytopic set rather than a unique value for A_c necessarily results in larger maximal invariant sets, and can provide improvements of several orders of magnitude in the volume of $\mathcal{E}_x$. For example, with

$$A_0 = \begin{bmatrix} 0.3 & 0 & -0.5 & -0.5 \\ 0 & 0 & 0.6 & -0.4 \\ -0.5 & 0.6 & 0.2 & 0.2 \\ -0.5 & -0.4 & 0.2 & -0.7 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.4 & 0.3 & 0 & 0 \\ 0.3 & 0.3 & 0.1 & 0.2 \\ 0 & 0.1 & 0.7 & -0.1 \\ 0 & 0.2 & -0.1 & 0.6 \end{bmatrix},$$

$B_0 = [0.6 \; 1.0 \; 0 \; -1.1]^T$, $B_1 = [0.1 \; 0.2 \; -0.8 \; 0]^T$, $C = [0.8 \; 0 \; 0 \; 0]$, and input constraints $|u| \leq 1$, the maximal $\mathcal{E}_x$ subject to (13a,b) has $(\det Y)^{1/2} = 823$, whereas including $A_{c,j}$ $j = 0, 1$ as d.o.f. through constraints (26), (13b) yields the maximal ellipsoidal invariant set under any linear feedback law, with $(\det Y)^{1/2} = 78500$.

This increase in the stabilizable set for (31) is obtained for no increase in computational load and insignificant performance degradation. For the example above with nominal cost, the average increase in closed-loop cost over 100 random feasible initial conditions relative to the case of unique A_c is only 0.73%. Use of a worst-case cost in (31) results in a reduction in closed-loop cost (averaged over the same 100 initial conditions) of 12.5% relative to the min-max MPC of (Kothare *et al.*, 1996). This improvement in performance is obtained despite a reduction of several orders of magnitude in online computation. Each simulation used the same sequence of time-varying system matrices, with $A(k), B(k)$ uniformly distributed within Ω.

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