ON SLIDING MODE OBSERVERS FOR NON-LINEAR SYSTEMS UNDER EXTERNAL DISTURBANCES

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Abstract: The paper introduces the cascade principle of sliding mode observer design for non-linear systems under external disturbances. An equivalent control approach will be utilized for step-by-step reconstruction of the non-measured components of the state and external disturbances also.

Keywords: observability of non-linear systems, disturbances, sliding mode observer.

1. INTRODUCTION

New perspectives to solve of the observation problem for dynamic systems under external disturbances are concerned with cascade design of state observers. This approach is based on transformations of the original system to the block-observable form being regular with respect to the disturbances, that allows one to reveal structural properties of the observability and to select an observable subspace of the state of a possibly large dimension. The state observer is designed as a replica of the block-observable form with an additional innovation correcting actions. The decomposition, and also invariance to available unknown-but-bounded perturbations are ensured by motions separation method in the class of systems with high-gain feedback or discontinuous correcting actions operating in the sliding modes. Within the framework of the given approach, the observation problem for linear systems under external disturbances has been solved (Utkin, 1990; Krasnova, 2002). An attempt to expand the approach to non-linear systems with additive, unobservable disturbances has shown, that for representation of observable blocks of a nonlinear system in the regular forms concerning the disturbances, it is required to split differential equations and, therefore, to fulfill multiply integral conversions by a method of the differential geometry (Krasnova, 2003).

This paper introduces a step-by-step design of the sliding mode observer for non-linear multivariate systems under external disturbances, based on equivalent control approach. In comparison to the previous results, this approach allows one to avoid integral transformations, since it is required to transform into the regular form concerning the disturbances the algebraic equations. Besides, there is a possibility to reconstruct the external disturbances also.

2. SYSTEM DEFINITION AND PROBLEM STATEMENT

Consider a non-linear multivariate system

$$\dot{x} = f(x,u) + Q(x)\eta(t), \ y_1 = h(x,u), \qquad (1)$$

where $x \in X \subset R^n$ is the state, $y_1 \in Y_1 \subset R^{m_1}$ is the output, $u \in R^r$ is the control (known functions in time), $\eta(t) \in R^p$ is an external disturbance. The vector-functions $f(x,u)$, $h(x,u)$ and the columns $q_i(x)$, $i = \overline{1,p}$ of a non-linear matrix $Q(x)$ are certain times (according to the reduced below procedure) differentiable in all arguments in some local frame $\widetilde{X} = X \oplus W$. So as not to restrict the class of admissible controls of a class of multiply differentiable functions, let us add system (1) by a dynamic compensator (Ciccarella, *et al.,* 1993) $\dot{\varpi}_i = -\varpi_i + \varpi_{i+1}$, $i = \overline{1,\upsilon}$, where $\varpi_i \in R^r$, $u = \varpi_1$, $\varpi_{\upsilon+1}$ is a new control, $\varpi = \mathrm{col}(\varpi_1,...,\varpi_\upsilon)$, $\varpi \in W \subset R^{r\upsilon}$. The number of blocks of the compensator is equal to the observability index in view of the disturbances of system (1) and υ is defined by the following procedure. Further, for a notation of vectors of control derivatives, the notations of the following type will be used

$$\varpi_i^* = \mathrm{col}(\varpi_1,\varpi_2,...,\varpi_{i+1}) \, , \; i = \overline{0,\upsilon} \, . \qquad (2)$$

The disturbances are unknown-but-bounded functions in time

$$|\eta_i(t)| \le N_i \, , \; i = \overline{1,p} \, , \forall t \in [0,\infty) \, , \qquad (3)$$

where N_i are known constants. The class of admissible disturbances is extended to non-smooth functions meeting (3).

As known, the observability of a non-perturbed non-linear system depends on a couple $\{h(x,u),\ f(x,u)\}$ (Isidori, 1995; Krasnova, *et al.,* 2001). Hereinafter, the observability is understood in the local sense. In system (1), presence of external disturbances can lead to decreasing of dimension of the observable subspace of the state up to complete loss of the observability. For system (1), the task of step-by-step investigation of thee matrix of the partial derivatives of the triple $\{h(x,u), \{f(x,u), Q(x)\}\}$ is set. The purpose of the research will be consist in determining the observable subspace of the state of a possibly large dimension, and also linear combinations of external disturbances subject to an estimation. In the given problem statement, no class of problems based on design of dynamic model of disturbances is considered. It is assumed that the noises in channels of measurement are missed.

3. THE PROCEDURE OF SYNCHRONOUS ANALYSIS AND SYNTHESIS

The proposed procedure consists of step-by-step transformations of the same type. The essence of these algorithms consists in synchronizing the analysis and synthesis, since each step of the analysis of the structural properties of the observability is accompanied by design of an appropriate block of the sliding mode observer, whose dimension corresponds to the next index of the observability in view of the disturbances of original non-linear system (1).

Step *1*. Without loss of generality it is assumed, that

$$\mathrm{rank}\{\partial h(x,\varpi_0^*)/\partial x\} = \dim y_1 = m_1 \, , \; m_1 < n \, .$$

(a) Then the state vector of system (1) can be represented as follows $x = \mathrm{col}(x_1^*, x_1)$, $x_1 \in R^{n-m_1}$, $x_1^* \in R^{m_1}$, so that $\mathrm{rank}\{\partial h(x_1^*, x_1, \varpi_0^*)/\partial x_1^*\} = m_1$. Let us make a diffeomorphic replacement of the local coordinates $x_1^* \to y_1$ and represent system (1) within notations (2) concerning new the variables as follows:

$$\begin{aligned} \dot{y}_1 &= h_1(y_1,x_1,\varpi_1^*) + Q_{y1}(y_1,x_1,\varpi_0^*)\eta, \\ \dot{x}_1 &= f_1(y_1,x_1,\varpi_0^*) + Q_1(y_1,x_1,\varpi_0^*)\eta. \end{aligned} , \qquad (4)$$

(b) For the first equation of system (4), let us construct the following state observer

$$\dot{z}_1 = v_1 \, , \qquad (5)$$

where $z_1 \in R^{m_1}$ is the state, $v_1 \in R^{m_1}$ are correcting actions of the observer. Let us rewrite systems (4)-(5) with respect to the mismatches $\varepsilon_1 = y_1 - z_1$ as follows:

$$\dot{\varepsilon}_1 = g_1(y_1,x_1,\varpi_1^*,\eta) - v_1 \, , \qquad (6)$$

where $\varepsilon_1 \in R^{m_1}$,

$$g_1 = h_1(y_1,x_1,\varpi_1^*) + Q_{y1}(y_1,x_1,\varpi_0^*)\eta \, . \qquad (7)$$

It is assumed, that in a local domain $\widetilde{X}$ the conditions

$$|g_{1j}| \le G_{1j} \, , \; j = \overline{1,m_1} \qquad (8)$$

are fulfilled by virtue of (3), where $G_{1j} > 0$ are known constants. Let us generate correcting actions as discontinuous functions of mismatches

$$v_1 = M_1 \mathrm{sign}\varepsilon_1 \, , \qquad (9)$$

where $\mathrm{sign}\varepsilon_1 = \mathrm{col}(\mathrm{sign}\varepsilon_{11}, \mathrm{sign}\varepsilon_{12},..., \mathrm{sign}\varepsilon_{1m_1})$, $M_1 = \mathrm{diag}(M_{11}, M_{12},..., M_{1m_1})$. The diagonal elements

$$M_{1j} > G_{1j} \, , \; j = \overline{1,m_1} \qquad (10)$$

are the amplitudes of the discontinuous corrections. Then in the closed-loop system $\dot{\varepsilon}_1 = g_1(y_1,x_1,\varpi_1^*,\eta) - M_1\mathrm{sign}\varepsilon_1$ (6)-(10) the sliding mode takes place on the manifold $S_1 = \{\varepsilon_1 = 0\}$, $\varepsilon_1 = 0 \Rightarrow y_1 = z_1$ in finite time t_1 .

Remark 1. Let us assume, that in the first equation of system (4) it is possible to select an addend depending only on known, at present, coordinates y_1,ϖ_1^* , namely,

$$h_1 = \varphi_1(y_1,\varpi_1^*) + \breve{h}_1(y_1,x_1,\varpi_1^*) \, .$$

Such a construction reflects non-linear properties of the original system in a more extent. If parameters of a vector-function $\varphi_1(y_1,\varpi_1^*)$ are precisely known, the observer is possible to be constructed as follows: $\dot{z}_1=\varphi_1(y_1,\varpi_1^*)+M_1\mathrm{sign}\varepsilon_1$. It is possible to decrease the amplitudes M_{1j}, if in the system $\dot{\varepsilon}_1=\breve{h}_1(y_1,x_1,\varpi_1^*)+Q_{y1}(y_1,x_1,\varpi_0^*)\eta-M_1\mathrm{sign}\varepsilon_1$ the following conditions are fulfilled $|\breve{h}_{1j}+Q_{y1j}\eta|\le\breve{G}_{1j}<G_{1j}$, $j=\overline{1,m_1}$.

(c) In accordance to the equivalent control approach (Utkin, 1992), from static equation of system (6) $\dot{\varepsilon}_1=g_1(y_1,x_1,\varpi_1^*,\eta)-v_{1\mathrm{eq}}=0$ it is obtained

$$v_{1\mathrm{eq}}=g_1. \tag{11}$$

Let us realize the equivalent value operator as a first order low pass filter

$$\tau_{1j}\dot{\mu}_{1j}=-\mu_{1j}+v_{1j},\ j=\overline{1,m_1}. \tag{12}$$

Under small time constants $\tau_{1j}\to 0$, it will be possible to receive asymptotically equivalent values of the discontinuous corrections as follows:

$$\lim_{\tau_{1j}\to 0}\mu_{1j}=v_{1j(\mathrm{eq})},\ j=\overline{1,m_1}. \tag{13}$$

Neglecting the dynamics of filters (12), let us suppose, that relations (11), (13) are fulfilled for theoretically finite time. Sliding mode observer (5) with discontinuous corrections (9)-(10) and filters (12)-(13) allows one to stabilize variable ε_1 in block (6) and to reconstruct a vector g_1 (7), being estimates of the derivatives $\dot{y}_1$. As a matter of fact, this observer is a differentiator on the sliding modes. The fact, that current estimate of equivalent control (11) is obtained by filter (12), allows one to implement auto-shelfs and to reduce to minimum the amplitudes of discontinuous corrections (9) after occurring a sliding mode, that diminishes auto-oscillations in the steady-state mode and increase the quality of the estimation. By virtue of (10)-(13), the algorithm of tuning amplitudes in the sliding mode in $t>t_1$ is as follows

$$M_{1j}=\mu_{1j}+\alpha_{1j},\ j=\overline{1,m_1}, \tag{14}$$

where α_{1j} are arbitrary, positive, as small as required constants.

(d) The transformations of this item consist in transforming into the regular form concerning disturbances of algebraic equations (7). Let $\mathrm{rank}Q_1=p_1$, $\mathrm{rank}\{Q_{y1},\partial h_1/\partial x_1\}=d_1$. If the condition

$$p_1=d_1 \tag{15}$$

has been met, the procedure is finished, since strings of the vector-function h_1, observed independently on the external disturbances, are absent in system (7), and the second equation of system (4) is unobservable with respect to the output y_1.

Otherwise $0<p_1<d_1\le m_1$ by permutation of the strings, let us split equation (7) into two subsystems

$$g_1=\begin{pmatrix}\widehat{g}_1\\ \hat{g}_1\end{pmatrix}=\begin{pmatrix}\widehat{h}_1+\widehat{Q}_1\eta\\ \hat{h}_1+\hat{Q}_1\eta\end{pmatrix}, \tag{16}$$

where $\widehat{g}_1\in R^{m_1-p_1}$, $\dim\widehat{Q}_1=(m_1-p_1)\times p$, $\hat{g}_1\in R^{p_1}$, $\dim\hat{Q}_1=p_1\times p$, $\mathrm{rank}Q_{y1}=\mathrm{rank}\hat{Q}_1=p_1$. Then the diffeomorphic replacement of local variables

$$g_1^{'}=\widehat{g}_1-Q_1^*\hat{g}_1, \tag{17}$$

where $g_1^{'}\in R^{m_1-p_1}$, $Q_1^*=\widehat{Q}_1\hat{Q}_1^+$, and $\hat{Q}_1^+=\hat{Q}_1^T(\hat{Q}_1\hat{Q}_1^T)^{-1}$ is the pseudoinverse matrix, allows one to represent system (16) in the regular form concerning disturbances as follows:

$$g_1^{'}=h_1^{'}(y_1,x_1,\varpi_1^*), \tag{18}$$

$$\hat{g}_1=\hat{h}_1(y_1,x_1,\varpi_1^*)+\hat{Q}_1(y_1,x_1,\varpi_0^*)\eta.$$

Note, that for such a decomposition of the differential equation $\dot{y}_1=h_1+Q_{y1}\eta$ it would be necessary to fulfil integral transformations (Krasnova, 2003). If $p_1=0$, conversions (17) it is not required.

The specificity of the observation problem (necessity of physical realizability) demands transformation (17) to depend only on known, at present, variables. Therefore, if all strings of the matrix product

$$Q_1^*(y_1,x_1,\varpi_0^*)=\widehat{Q}_1(y_1,x_1,\varpi_0^*)\hat{Q}_1^+(y_1,x_1,\varpi_0^*) \tag{19}$$

depend on x_1, the procedure is also finished and, in the terms of this problem, the vector $x_1\in R^{n-m_1}$ is unobservable.

(e) Assume, that x_1 misses among arguments of a non-linear matrix $Q_1^*(y_1,\varpi_0^*)$. If $d_1<m_1$, then the first equation (18) by permutation of strings is split into two subsystems

$$\widetilde{g}_1=\widetilde{h}_1(y_1,x_1,\varpi_1^*),\ \overline{g}_1=\overline{h}_1(y_1,x_1,\varpi_1^*), \tag{20}$$

where $\dim\widetilde{g}_1=\mathrm{rank}\{\partial\widetilde{h}_1/\partial x_1\}=d_1-p_1=m_2$, $\overline{g}_1\in R^{m_1-d_1}$, $g_1^{'}=\mathrm{col}(\widetilde{g}_1,\overline{g}_1)$. The vector $y_2=\widetilde{g}_1$ with known components is set as a virtual output for the second equation of system (4). In the terms of the geometrical approach, the transformations of the first step allow one to select the second addend of direct sum $Y_1\oplus Y_2$ of maximal dimension $y_2\in Y_2\subset R^{m_2}$, whose components are observable independently on external disturbances.

If $m_2<\dim x_1$, we go to the second step of the procedure, where the operations and the transformations circumscribed on the first step are applied to system $\dot{x}_1=f_1(y_1,x_1,\varpi_0^*)+Q_1(y_1,x_1,\varpi_0^*)\eta$ with virtual output $y_2=\widetilde{h}_1(y_1,x_1,\varpi_1^*)$, and so on.

Remark 2. In that specific case, when there are only some number p_1''', $0 < p_1''' < p_1$ of strings of matrix Q_1^* depending on x_1, system (16) can be transformed partially into the regular form concerning disturbances as follows:

$$g_1'' = h_1'' ,\ g_1''' = h_1''' + Q_1'''\eta ,\ \hat{g}_1 = \hat{h}_1 + \hat{Q}_1\eta \qquad (21)$$

where $g_1'' \in R^{m_1-p_1-p_1'''}$, $g_1''' \in R^{p_1'''}$, $\hat{g}_1 \in R^{p_1}$. In system (21), the dimension of the vector g_1'', which is not depending on disturbances is less on the magnitude p_1''' than the dimension of the corresponding vector $p_1' \in R^{m_1-p_1}$ in system (18).

Form (21), it is gained by the diffeomorphic transformation $g_1'' = \bar{g}_1'' - \bar{Q}_1''\hat{Q}_1^{+}\hat{g}_1$, where $\bar{Q}_1 = \begin{pmatrix} \bar{Q}_1'' \\ Q_1''' \end{pmatrix}$, $\bar{g}_1 = \begin{pmatrix} \bar{g}_1'' \\ g_1''' \end{pmatrix}$, $\bar{g}_1'' \in R^{m_1-p_1-p_1'''}$, the matrix product $\bar{Q}_1''\hat{Q}_1^{+}$ does not depend on x_1. In system (21) the third condition of finishing the procedure can be met, namely, if $d_1 < m_1$ and

$$\mathrm{rank}\{\partial h_1'' / \partial x_1\} = 0 . \qquad (22)$$

Thus, if the procedure is finished on the first step under conditions (15), (19) or (22), there is only current estimate of the derivative $\dot{y}_1 = g_1$ obtained via observer (5)-(12). It is that maximum of the estimation, following from the structure of the observability by virtue of disturbances of system (1)-(3). Note, that the conventional observers with continuous corrections do not give even such a possibility, and to reconstruct $\dot{y}_1$ (if it is necessary for the purposes of control) it is required to differentiate output signals really.

Step i. Repeating this procedure, at the *i*-th step the following system is considered

$$\dot{x}_{i-1} = f_i(y_{i-1}^*, x_{i-1}, \varpi_{i-1}^*) + Q_i(y_{i-1}^*, x_{i-1}, \varpi_{i-2}^*)\eta, \qquad (23)$$
$$y_i = \widetilde{g}_{i-1} = \widetilde{h}_{i-1}(y_{i-1}^*, x_{i-1}, \varpi_{i-1}^*),$$

where $x_{i-1} \in R^{n-m_1-\ldots-m_{i-1}}$, $\dim y_i = m_i = \mathrm{rank}\{\partial\widetilde{h}_{i-1}/\partial x_{i-1}\}$, and $y_{i-1}^* = \mathrm{col}(y_1, y_2, \ldots, y_{i-1})$, y_i are already reconstructed.

(a) Let $\dim x_{i-1} > m_i$, then system (23) can be transformed into the following form

$$\dot{y}_i = h_i(y_i^*, x_i, \varpi_i^*) + Q_{yi}(y_i^*, x_i, \varpi_{i-1}^*)\eta, \qquad (24)$$
$$\dot{x}_i = f_i(y_i^*, x_i, \varpi_{i-1}^*) + Q_i(y_i^*, x_i, \varpi_{i-1}^*)\eta.$$

by diffeomorphic replacement of the local coordinates $x_i^* \to y_i$, where $x_{i-1} = \mathrm{col}(x_i^*, x_i)$, $x_i \in R^{n-m_1-\ldots-m_i}$, $\dim x_i^* = \mathrm{rank}\{\partial\widetilde{h}_{i-1}/\partial x_i^*\} = m_i$, $y_i^* = \mathrm{col}(y_1, \ldots, y_i)$.

(b) For the first equation of system (21), let us design the state observer $\dot{z}_i = v_i$, where $z_i \in R^{m_i}$ is the state, $v_i = M_i \mathrm{sign}(y_i\text{-}z_i)$ is discontinuous correcting action. The system rewritten with respect to the mismatches $\varepsilon_i = y_i - z_i$, $\varepsilon_i \in R^{m_i}$ is as follows

$$\dot{\varepsilon}_i = g_i(y_i^*, x_i, \varpi_i^*, \eta) - M_i \mathrm{sign}\varepsilon_i , \qquad (25)$$

where $g_i \in R^{m_i}$,

$$g_i = h_i(y_i^*, x_i, \varpi_i^*) + Q_{yi}(y_i^*, x_i, \varpi_{i-1}^*)\eta . \qquad (26)$$

It is assumed, that condition $|g_{ij}| \le G_{ij}$, $j = \overline{1, m_i}$, where $G_{ij} > 0$ are known constants, has been met in a local domain $\breve{X}$ by virtue of (3). Under the condition of $M_{ij} > G_{ij}$, $j = \overline{1, m_i}$, $M_i = \mathrm{diag}(M_{i1}, \ldots, M_{im_i})$, in closed-loop system (25) the sliding mode takes place on the manifold $S_i = \{\varepsilon_i = 0 \cap S_{i-1}\}$, $S_{i-1} = \{\varepsilon_{i-1} = 0 \cap \ldots \cap \varepsilon_1 = 0\}$, $\varepsilon_i = 0 \Rightarrow y_i = z_i$ in finite time $t_i > t_{i-1} > \ldots > t_1$.

(c) From static equation $\dot{\varepsilon}_i = 0$ it is obtained

$$v_{i\mathrm{eq}} = g_i . \qquad (27)$$

Let us realize the equivalent value operator as a first order low pass filter $\tau_{ij}\dot{\mu}_{ij} = -\mu_{ij} + v_{ij}$, $j = \overline{1, m_i}$. It will allow one to receive equivalent values of discontinuous corrections

$$\lim_{\tau_{ij} \to 0} \mu_{ij} = v_{ij(\mathrm{eq})} ,\ j = \overline{1, m_i} \qquad (28)$$

for theoretically finite time as $\tau_{ij} \to 0$.

If it is possible to represent the first equation of system (24) in the following manner

$$\dot{y}_i = \varphi_i(y_i^*, \varpi_i^*) + \breve{h}_i(y_i^*, x_i, \varpi_i^*) + Q_{yi}(y_i^*, x_i, \varpi_{i-1}^*)\eta ,$$

where the parameters of the vector-function $\varphi_{i1}(y_i^*, \eta_i^*)$ are precisely known, then the state observer is designed as follows $\dot{z}_i = \varphi_i(z_i^*, \varpi_i^*) + v_i$ (see remark 1).

(d) Let $\mathrm{rank}\{Q_{yi}, \partial h_i / \partial x_i\} = d_i$, $\mathrm{rank} Q_{yi} = p_i$, $0 < p_i < d_i \le m_i$, then system (26) can be transformed into the following form

$$\bar{g}_i = \bar{h}_i(y_i^*, x_i, \varpi_i^*) + \bar{Q}_i(y_i^*, x_i, \varpi_{i-1}^*)\eta, \qquad (29)$$
$$\hat{g}_i = \hat{h}_i(y_i^*, x_i, \varpi_i^*) + \hat{Q}_i(y_i^*, x_i, \varpi_{i-1}^*)\eta,$$

$$g_i = \begin{pmatrix} \bar{g}_i \\ \hat{g}_i \end{pmatrix},\ \bar{g}_i \in R^{m_i-p_i},\ Q_{yi} = \begin{pmatrix} \bar{Q}_i \\ \hat{Q}_i \end{pmatrix}$$

so that $\dim \hat{g}_i = \mathrm{rank} Q_{yi} = \mathrm{rank}\hat{Q}_i = p_i$, where

$\dim\bar{Q}_i = (m_i - p_i)\times p$, $\dim\hat{Q}_i = p_i \times p$. Let us consider the matrix product

$$Q_i^* = \bar{Q}_i(y_i^*, x_i, \varpi_{i-1}^*)\hat{Q}_i^+(y_i^*, x_i, \varpi_{i-1}^*), \quad (30)$$

where $\hat{Q}_i^+ = \hat{Q}_i^T(\hat{Q}_i\hat{Q}_i^T)^{-1}$ is the pseudoinverse matrix. If x_i misses among arguments of a non-linear matrix $Q_i^*(y_i^*, \varpi_{i-1}^*)$ (30), let us generate a vector

$$g_i^{'} = \bar{g}_i - \bar{Q}_i\hat{Q}_i^+\hat{g}_i \;, \quad (31)$$

where $g_i^{'} \in R^{m_i - p_i}$ does not depend on the disturbances and consists of known at present components. Non-special replacement (31) allows one to represent system (29) in the complete regular form concerning disturbances as follows:

$$g_i^{'} = h_i^{'}(y_i^*, x_i, \varpi_i^*)\,, \quad (32)$$
$$\hat{g}_i = \hat{h}_i(y_i^*, x_i, \varpi_i^*) + \hat{Q}_i(y_i^*, x_i, \varpi_{i-1}^*)\eta\,.$$

Note, that if only $p_i^{'''}$, $0 < p_i^{'''} < p_i$ strings of products (30) depend on x_i (see remark 2), than system (29) can be transformed partially into the regular form concerning disturbances as follows:

$$g_i^{''} = h_i^{''}\,,\; g_i^{'''} = h_i^{'''} + Q_i^{'''}\eta\,,\; \hat{g}_i = \hat{h}_i + \hat{Q}_i\eta \quad (33)$$

where $g_i^{''} \in R^{m_i - p_i - p_i^{'''}}$, $g_i^{'''} \in R^{p_i^{'''}}$, and the procedure can be prolonged, if $\mathrm{rank}\{\partial h_i^{''}/\partial x_i\} \neq 0$.

If $d_i < m_i$, then by permutation of strings the first equation of system (32) (or (33)) can be transformed into the following form $\tilde{g}_i = \tilde{h}_i(y_i^*, x_i, \varpi_i^*)$, $\bar{g}_i = \bar{h}_i(y_i^*, x_i, \varpi_i^*)$, where $g_i^{'} = \mathrm{col}(\tilde{g}_i, \bar{g}_i)$, $\bar{g}_i \in R^{m_i - d_i}$, $\dim\tilde{g}_i = \mathrm{rank}\{\partial\tilde{h}_i/\partial x_i\} = d_i - p_i =$ $= m_{i+1}$. If $m_{i+1} < \dim x_i$, then we go to the next step, where circumscribed transformations are applied to

$$\dot{x}_i = f_i(y_i^*, x_i, \varpi_i^*) + Q_i(y_i^*, x_i, \varpi_{i-1}^*)\eta\,, \quad (34)$$

and known at present vector $y_{i+1} = \tilde{g}_i \in R^{m_{i+1}}$ is considered as a virtual output, etc. At each step, the dimension of the state vector of the new system becomes less then that of the previous system. Therefore, this process is presumed to terminate in a finite number of steps. In particular, if at *i*-th step the condition $\dim y_{i+1} = \dim x_i = m_{i+1}$ has been met, it means that the state vector is completely decomposed into the direct sum $X \to Y = Y_1 \oplus Y_2 \oplus ... \oplus Y_{i+1}$, where $y_j \in Y_j \subset R^{m_j}$, $\sum_{j=1}^{i+1} m_j = n$, and initial system (1) is locally observable. At this stage $y_{i+1}^* \in Y \subset R^n$ is known. The coordinates of y_j are coupled by one-to-one dependence to the original coordinates of x_j^*, as $\dim x_j^* = \mathrm{rank}\{\partial\tilde{h}_{j-1}/\partial x_j^*\} = m_j$, and

$$x = \mathrm{col}(x_1^*, x_1)\,,$$
$$x_j = \mathrm{col}(x_{j+1}^*, x_{j+1}) = \mathrm{col}(x_{j+1}^*, x_{j+2}^*, ..., x_{i+1}^*)\,,$$

$j = \overline{1, i-1}$, $x_i = x_{i+1}^*$, which can be reconstructed by sequential reconversions

$$x_{i+1}^* = \tilde{h}_i^{-1}(y_{i+1}^*, \varpi_i^*)\,,$$
$$x_j^* = \tilde{h}_{j-1}^{-1}(y_j^*, x_{j+1}^*, ..., x_{i+1}^*, \varpi_{j-1}^*)\,,\; j = \overline{i,2}\,;$$
$$x_1^* = h^{-1}(y_1, x_2^*, ..., x_{i+1}^*, \varpi_0^*)\,.$$

Since also the task of reconstruction of the linear combinations of the external disturbances has been stated, let us go to transformations. The last block of system (34) can be obtained by diffeomorphic replacement of local coordinates $x_{i+1}^* \to y_{i+1}$ as follows

$$\dot{y}_{i+1} = h_{i+1}(y_{i+1}^*, \varpi_{i+1}^*) + Q_{y(i+1)}(y_{i+1}^*, \varpi_i^*)\eta\,, \quad (35)$$

where only disturbance η is unknown. Let us design sliding mode observer for system (35) as follows $\dot{z}_{i+1} = h_{i+1}(z_{i+1}^*, \varpi_{i+1}^*) + M_{i+1}\mathrm{sign}\varepsilon_{i+1}$, where $| h_{i+1}(y_{i+1}^*, \varpi_{i+1}^*) - h_{i+1}(z_{i+1}^*, \varpi_{i+1}^*) + Q_{y(i+1)_j}\eta | < M_{i+1,j}$ $\varepsilon_{i+1} = y_{i+1} - z_{i+1}$, $\tau_{i+1,j}\dot{\mu}_{i+1,j} = -\mu_{i+1,j} + v_{i+1,j}$, $\lim_{\tau_{i+1,j}\to 0} \mu_{i+1,j} = v_{i+1,j(\mathrm{eq})}$, $j = \overline{1, m_{i+1}}$. Then the following relations will be provided for theoretically finite time $t_{i+1} > t_i$:

$$\varepsilon_{i+1} = 0 \Rightarrow y_{i+1} = z_{i+1}\,,\; v_{i+1(\mathrm{eq})} = Q_{y(i+1)}\eta\,, \quad (36)$$

From the second equations of systems such as (32) in view of made replacements of the variables, the estimates of other linear combinations of disturbances $\hat{Q}_j(y_{i+1}^*, \varpi_i^*)\eta = \hat{g}_j - \hat{h}_j(y_{i+1}^*, \varpi_i^*)$, $\hat{g}_j = \hat{v}_{j(\mathrm{eq})}$, $j = \overline{1, i}$ can be obtained, if parameters of the vector-functions $\hat{h}_j(y_{i+1}^*, \varpi_i^*)$ are known (otherwise, we have estimates of the sums $\hat{h}_j(y_{i+1}^*, \varpi_i^*) + \hat{Q}_j(y_{i+1}^*, \varpi_i^*)\eta$, $j = \overline{1, i}$) only. In the specific case, when $p \le n$ and $\mathrm{rank}Q_\Sigma = p$, where

$$Q_\Sigma = \begin{pmatrix} \hat{Q}_1(y_{i+1}^*, \varpi_i^*) \\ ... \\ \hat{Q}_i(y_{i+1}^*, \varpi_i^*) \\ Q_{y(i+1)}(y_{i+1}^*, \varpi_i^*) \end{pmatrix},$$

it is possible to reconstruct the disturbances by equation $\eta = Q_\Sigma^+(\hat{g}_j - \hat{h}_j)$.

Now let us consider other conditions of the termination of the procedure. Assume, that at the *i*-th step in system (26) $p_i = d_i$; or all strings of the matrix product (30) depend on x_i, $Q_i^*(y_i^*, x_i, \varpi_{i-1}^*)$; or in partially regular form (33) the condition $\text{rank}\{\partial h_i'' / \partial x_i\} = 0$ has been met. In contrast to the first step, at the *i*-th step in these cases possibilities of the estimation are not yet exhausted. In view of the done changes of variables, let us introduce the notation

$$\hat{g} = \hat{h}(y_i^*, x_i, \varpi_i^*) + \hat{Q}_\Sigma\ (y_i^*, x_i, \varpi_{i-1}^*)\eta, \quad (37)$$

where $\hat{h} = \begin{pmatrix} \hat{h}_1 \\ \dots \\ \hat{h}_{i-1} \\ h_i \end{pmatrix}$, $\hat{Q}_\Sigma = \begin{pmatrix} \hat{Q}_1 \\ \dots \\ \hat{Q}_{i-1} \\ Q_{yi} \end{pmatrix}$, $\text{rank}\hat{Q}_\Sigma = \hat{p}$, $\text{rank}\{\hat{Q}_\Sigma\ , \partial\hat{h}/\partial x_i\} = \hat{d}$. The final completion of the procedure will happen, if in system (37) one of the following conditions will be met:

$$\hat{p} = \hat{d}\ ; \quad (38)$$

or under $\hat{p} < \hat{d}$ all string of the matrix product

$$Q^* = \widehat{Q}\hat{Q}^+,\ \hat{Q}_\Sigma = \begin{pmatrix} \widehat{Q} \\ \hat{Q} \end{pmatrix}, \quad (39)$$

where $\text{rank}\hat{Q}_\Sigma = \text{rank}\hat{Q} = \hat{p}$, depend on x_i, $Q^*(y_i^*, x_i, \varpi_{i-1}^*)$; or in partially regular form such as (33), when only $\hat{p}''' < \hat{p}$ strings depend on x_i and $g_i'' = \hat{h}''(y_i^*, x_i, \varpi_i^*)$, $g_i'' \in R^{m_i - p_i - p_i'''}$,

$$\text{rank}\{\partial\hat{h}'' / \partial x_i\} = 0\,. \quad (40)$$

In these cases we have only earlier obtained estimates of sums (37) and converted coordinates

$$y_1 = h(x_1^*, ..., x_{i-1}^*, x_{i-1}, \varpi_0^*)\,, \quad (41)$$
$$y_j = \widetilde{h}_{j-1}(y_{j-1}^*, x_j^*, ..., x_{i-1}^*, x_{i-1}, \varpi_{j-1}^*)\,,\ j = \overline{2, i}\,,$$

where $\text{rank}\{\partial h / \partial x_1^*\} = m_1$, $\dim y_j = \dim x_j^* = \text{rank}\{\partial\widetilde{h}_{j-1} / \partial x_j^*\} = m_j$, $m_1 + ... + m_i < n$. Since $x_i \in R^{n - m_1 - ... - m_i}$ cannot be obtained from system (41), it is considered unobservable within the framework of the given problem.

Otherwise (if such conditions as (38), (39) or (40) were not fulfilled), as was shown above, the vector $y_{i+1} = \widetilde{\hat{h}}(y_i^*, x_i, \varpi_i^*) \in R^{m_{i+1}}$, $\text{rank}\{\partial\widetilde{\hat{h}} / \partial x_i\} = m_{i+1}$, observed independently on the disturbances, is selected from system (37). This vector is considered as a virtual output for system (34), and the procedure is prolonged so long as at the υ-th step the condition $m_{\upsilon+1} = n - m_1 - ... - m_\upsilon$ will be met, and the observation problem will be solved completely; or under forming the next vector as (37), the conditions (38) either (39) or (40) will not be met (then x_υ will be unobservable). Since each stage of the synthesis of the sliding mode observer was accompanied by the analysis of structural properties of the observability and blocks of the observer are physically realizable, the given procedure allows one to select locally observable subspace of the state of the maximally possible dimension. Condition $\dim y_{\upsilon+1} = \dim x_\upsilon$ is a necessary and sufficient condition of the local observability of system (1)-(3) by virtue of the given procedure.

4. CONCLUSIONS

A sliding mode observer design method based on the equivalent control methodology have been studied for non-linear system under unmonitored disturbances. The presented technique can be implemented also in a narrower setting, when it is assumed that the external disturbances are differentiable functions in time or generated by some dynamic model. The proposed procedure of step-by-step analysis of the structural properties of observability (in this case, the initial system is extended due to the model of the external disturbances) and the state observer design will allow one to bypass the large dimension problem. This technique, which allows one to reduce essentially body of the prior information on the operator of a control plant and medium of its operation, can find an application within the estimation problems for a wide class of modern complex industrial plants.

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