NONLINEAR SAMPLED-DATA OBSERVER DESIGN VIA APPROXIMATE DISCRETE-TIME MODELS AND EMULATION

Murat Arcak [*] **Dragan Nešić** [**]

[*] *Department of Electrical, Computer, and Systems Engineering*
Rensselaer Polytechnic Institute
Troy, NY 12180, U.S.A.
arcakm@rpi.edu

[**] *Department of Electrical and Electronic Engineering*
The University of Melbourne
Parkville, 3052, Victoria, Australia
d.nesic@ee.mu.oz.au

Abstract: We study observer design for sampled-data nonlinear systems using two approaches: (i) the observer is designed via an approximate discrete-time model of the plant; (ii) the observer is designed based on the continuous-time plant model and then discretized for sampled-data implementation (emulation). In each case we present Lyapunov conditions under which the observer design guarantees semiglobal practical convergence for the unknown exact discrete-time model. The semiglobal region of attraction is expanded by decreasing the sampling period. The practical convergence set is shrunk by decreasing either the sampling period, or a modelling parameter which refines the accuracy of the approximate model.

Keywords: Observers; Sampled-data systems; Nonlinear systems.

1. INTRODUCTION

A shortcoming of the existing sampled-data observer theory is that the availability of exact discrete-time models is assumed, which is usually unrealistic. A more practical approach pursued in this paper is to employ approximate discrete-time models for design, and to study how robust such approximate designs would be when implemented on the exact model. This approach may be advantageous even when an exact model is available, because the very few constructive tools for nonlinear observer design may be applicable only to an approximate model which preserves structural properties of the continuous-time model.

Most results on discrete-time observer design in the literature rely on an exact model. Several of them, such as (Boutayeb and Darouach 2000), present examples where the models are obtained from Euler approximations, but do not discuss the effect of the exact-approximate mismatch. For a class of state-affine systems, (Nadri and Hammouri 2003) pursue a mixed continuous- and discrete-time design which circumvents the need

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for an explicit exact model with the help of continuous-time updates.

In (Arcak and Nešić 2003) we have shown with examples that a design based on an approximate model may fail to produce a stable observer for the exact discrete-time model regardless of how small the sampling period is. We have then given sufficient Lyapunov conditions which rule out this possibility, and guarantee convergence for the exact model in a *semiglobal practical* sense. In this paper we extend this results in two directions: First, we introduce a "modelling parameter" which improves the accuracy of the approximate model for a constant sampling period, and show that we can achieve practical convergence by tuning this parameter instead of the sampling period. Next, we address *emulation* design of observers, not studied in (Arcak and Nešić 2003), in which the discrete-time observer is obtained from a continuous-time design via an approximate discretization. With appropriate conditions on the approximation, and on the underlying continuous-time observer, we again achieve semiglobal practical convergence. A distinction of the emulation design, however, is that we cannot achieve practical convergence by decreasing the modelling parameter alone. Indeed, as we show with an example, decreasing this parameter does not reduce the size of the practical convergence set arbitrarily, but to a level dictated by the sampling period. Due to space limitations we omit the proofs and present them in the report (Arcak and Nešić 2004).

2. PRELIMINARIES AND PROBLEM STATEMENT

We consider the system

$$\dot{x} = f(x,u), \qquad y = h(x), \tag{1}$$

where $x \in I\!R^n, u \in I\!R^m, y \in I\!R^p$, and $f(x,u)$ is locally Lipschitz. Given a sampling period $T > 0$, we assume that the control u is constant during sampling intervals $[kT,(k+1)T)$ and that the output y is measured at sampling instants kT; that is, $y(k) := y(kT)$. The family of exact discrete-time models of (1) is:

$$x(k+1) = F^e_T(x(k),u(k)), \qquad y(k) = h(x(k)), \tag{2}$$

where $F^e_T(x,u)$ is the solution of (1) at time T starting at x, with the constant input u. This model is well-defined when the continuous model (1) does not exhibit finite-escape time. When there is finite escape time, (2) is valid on compact sets which can be rendered arbitrarily large by reducing T.

To compute (2) we need a closed-form solution to (1) over one sampling interval $[kT,(k+1)T)$, which is impossible to obtain in general. It is realistic, however, to assume that a family of approximate discrete-time models is available:

$$x(k+1) = F^a_{T,\delta}(x(k),u(k)), \quad y(k) = h(x(k)). \tag{3}$$

This family is parameterized by the sampling period T, and a "modelling parameter" δ which will be used to refine the approximate model when T is fixed. It can be interpreted as the *integration period* in numerical schemes for solving differential equations. The case where $\delta = T$ is of separate interest because several approximations of this type (such as Euler approximation) preserve the structure and types of nonlinearities of the continuous-time system and, hence, may be preferable to the designer. When $\delta = T$ we use the short-hand notation

$$F^a_T(x,u) := F^a_{T,T}(x,u). \tag{4}$$

For the linear system $\dot{x} = Ax$, the forward Euler numerical scheme $x(t+\delta) = (I+\delta A)x(t)$ can be used to generate an approximate model by dividing the sampling period T into N integration periods $\delta = T/N$, and by applying $(I+\delta A)$ for each integration period; that is, $F^a_{T,\delta} = (I+\delta A)^{T/\delta}x$. As $\delta \to 0$, this $F^a_{T,\delta}$ converges to the exact model $F^e_T = \exp(AT)x$. If $\delta = T$, then we obtain $F^a_T = (I+TA)x$.

Throughout the paper we assume that the approximate model (3) is *consistent* with the exact model, as defined in (Nešić *et al.* 1999), and (Nešić and Teel 2004):

Definition 1.
a) When $\delta = T$ the family $F^a_T(x,u)$ is said to be consistent with $F^e_T(x,u)$ if for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a class-$\mathcal{K}$ function $\rho(\cdot)$ and a constant $T_0 > 0$ such that, for all $(x,u) \in \Omega$ and all $T \in (0,T_0]$,

$$|F^e_T(x,u) - F^a_T(x,u)| \le T\rho(T). \tag{5}$$

b) When δ is independent of T, $F^a_{T,\delta}(x,u)$ is said to be *consistent* with $F^e_T(x,u)$ if, for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a class-$\mathcal{K}$ function $\rho(\cdot)$ and a constant $T_0 > 0$, and for each fixed $T \in (0,T_0]$ there exists $\delta_0 \in (0,T]$ such that, for all $(x,u) \in \Omega$ and $\delta \in (0,\delta_0]$,

$$|F^e_T(x,u) - F^a_{T,\delta}(x,u)| \le T\rho(\delta). \tag{6}$$

It is not necessary to know the exact model $F^e_T(x,u)$ to verify the consistency property. Verifiable conditions to check (5) and (6) are given in (Nešić *et al.* 1999) and (Nešić and Teel 2004). For the approximate model (3), we design a family of observers (depending on T and δ) of the form

$$\hat{x}(k+1) = F^a_{T,\delta}(\hat{x}(k),u(k)) + \ell_{T,\delta}(\hat{x}(k),y(k),u(k)), \tag{7}$$

and analyze under what conditions, and in what sense, this design guarantees convergence when applied to the exact model (2). Due to the mismatch of the exact and approximate models, the observer error system is now driven by the plant trajectories $x(t)$ and controls $u(t)$, which act as disturbance inputs. When these inputs are bounded, we want the observer to guarantee *semiglobal practical* convergence, as defined next:

Definition 2.
a) When $\delta = T$ we say that the convergence of the observer (7) is *semiglobal practical in* T, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any $D > d > 0$ and compact sets $\mathcal{X} \subset I\!\!R^n, \mathcal{U} \subset I\!\!R^m$, we can find a $T^* > 0$ with the property that for all $T \in (0, T^*]$,

$$|\hat{x}(0) - x(0)| \le D, \quad \text{and} \quad x(k) \in \mathcal{X}, u(k) \in \mathcal{U}, \tag{8}$$

$\forall k \ge 0$, imply

$$|\hat{x}(k) - x(k)| \le \beta(|\hat{x}(0) - x(0)|, kT) + d. \tag{9}$$

b) When δ is independent of T we say that the convergence of the observer (7) is *semiglobal in* T and *practical in* δ, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any given real number $D > 0$, and compact sets $\mathcal{X} \subset I\!\!R^n, \mathcal{U} \subset I\!\!R^m$, we can find and a $T^* > 0$, and for any $T \in (0, T^*]$ and $d \in (0, D)$, we can find $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, (8) implies (9).

c) We say that the convergence of the observer (7) is *semiglobal in* T and *practical in* T *and* δ, if there exists a class-$\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that for any $D > d_1 > 0$, and compact sets $\mathcal{X} \subset I\!\!R^n, \mathcal{U} \subset I\!\!R^m$, we can find a $T^* > 0$, and for any $T \in (0, T^*]$ and $d_2 \in (0, D - d_1)$, we can find $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$, (8) implies

$$|\hat{x}(k) - x(k)| \le \beta(|\hat{x}(0) - x(0)|, kT) + d_1 + d_2. \tag{10}$$

Unlike Definition 2(b) where we can arbitrarily reduce the residual observer error d in (9) by decreasing δ, in Definition 2(c) we can only reduce d_2 with δ, while d_1 is dictated by the sampling period T. As we shall see in Section 4, this situation arises in emulation design where, decreasing δ can reduce the residual observer error, but cannot eliminate it completely if T is held constant.

3. OBSERVER DESIGN VIA APPROXIMATE DISCRETE-TIME MODELS

We now derive conditions which guarantee semiglobal practical observer convergence for the exact model. For our analysis we first note from (2) and (7) that the observer error $e := \hat{x} - x$ satisfies

$$e(k+1) = F^a_{T,\delta}(\hat{x}(k), u(k)) + \ell_{T,\delta}(\hat{x}(k), y(k), u(k)) - F^e_T(x(k), u(k)). \tag{11}$$

Adding and subtracting the approximate model $F^a_{T,\delta}(x(k), u(k))$, we rewrite (11) as

$$e(k+1) = E_{T,\delta}(e(k), x(k), u(k)) + F^a_{T,\delta}(x(k), u(k)) - F^e_T(x(k), u(k)), \tag{12}$$

where

$$E_{T,\delta}(e, x, u) := F^a_{T,\delta}(\hat{x}, u) + \ell_{T,\delta}(\hat{x}, y, u) - F^a_{T,\delta}(x, u) \tag{13}$$

represents the nominal observer error dynamics for the approximate design, and $F^a_{T,\delta}(x(k), u(k)) - F^e_T(x(k), u(k))$ is the mismatch between the approximate and exact plant models.

In Theorem 1 below, we address the situation $\delta = T$, which was studied in (Arcak and Nešić 2003). We repeat this result here for comparison with the subsequent theorems:

Theorem 1. **(**$\delta = T$**)** The observer (7) is semiglobal practical in T as in Definition 2(a) if the following conditions hold:

(i) $\delta = T$.

(ii) F^a_T is consistent with F^e_T as in Definition 1(a).

(iii) There exists a family of Lyapunov functions $V_T(x, \hat{x})$, class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, $\rho_0(\cdot)$, and nondecreasing functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, with the following property:

For any compact sets $\mathcal{X} \subset I\!\!R^n$, $\hat{\mathcal{X}} \subset I\!\!R^n$, $\mathcal{U} \subset I\!\!R^m$, there exist constants $T^* > 0$ and $M > 0$, such that, for all x, x_1, $x_2 \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$, $u \in \mathcal{U}$, and $T \in (0, T^*]$,

$$|V_T(x_1, \hat{x}) - V_T(x_2, \hat{x})| \le M|x_1 - x_2| \tag{14}$$

$$\alpha_1(|e|) \le V_T(x, \hat{x}) \le \alpha_2(|e|) \tag{15}$$

$$\frac{V_T(F^a_T(x,u), F^a_T(\hat{x},u) + \ell_T(\hat{x},y,u)) - V_T(x,\hat{x})}{T} \le -\alpha_3(|e|) + \rho_0(T)[\gamma_0(|e|) + \gamma_1(|x|) + \gamma_2(|u|)]. \tag{16}$$

□

Theorem 1 establishes semiglobal practical convergence by reducing the sampling period T. When T is fixed and cannot be reduced, it is still possible to achieve practical convergence by, instead, refining the accuracy of the approximate models with the parameter δ:

Theorem 2. **(**δ **independent of** T**)** The observer (7) is semiglobal in T and practical in δ as in Definition 2(b) if the following conditions hold:

(i) δ can be adjusted independently of T.

(ii) $F^a_{T,\delta}(x, u)$ is consistent with the exact model $F^e_T(x, u)$ as in Definition 1(b).

(iii) There exists a family of Lyapunov functions $V_{T,\delta}(x,\hat{x})$, class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, $\rho_0(\cdot)$, and nondecreasing functions $\gamma_0(\cdot)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, with the following property:

For any compact sets $\mathcal{X} \subset \mathbb{R}^n$, $\hat{\mathcal{X}} \subset \mathbb{R}^n$, $\mathcal{U} \subset \mathbb{R}^m$, there exists a constant $T^* > 0$, and for any fixed $T \in (0, T^*]$ there exists $\delta^* > 0$, and for any $\varepsilon_1 > 0$ there exists $c > 0$, such that, for all x, x_1, $x_2 \in \mathcal{X}$, $\hat{x} \in \hat{\mathcal{X}}$, $u \in \mathcal{U}$, and $\delta \in (0, \delta^*]$,

$$|x_1 - x_2| \le c \quad \Rightarrow \quad |V_{T,\delta}(x_1,\hat{x}) - V_{T,\delta}(x_2,\hat{x})| \le \varepsilon_1 \tag{17}$$

$$\alpha_1(|e|) \le V_{T,\delta}(x,\hat{x}) \le \alpha_2(|e|) \tag{18}$$

$$\frac{V_{T,\delta}(F^a_{T,\delta}(x,u), F^a_{T,\delta}(\hat{x},u) + \ell_{T,\delta}(\hat{x},y,u))}{T} - \frac{V_{T,\delta}(x,\hat{x})}{T} \le -\alpha_3(|e|) + \rho_0(\delta)[\gamma_0(|e|) + \gamma_1(|x|) + \gamma_2(|u|)]. \tag{19}$$

□

Example 1. Theorems 1-2 are also applicable to reduced-order observers when e is interpreted as the difference between the unmeasured components of x, and their observer estimates. We now design such a reduced-order observer for the Duffing oscillator

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_1^3, \tag{20}$$

from sampled measurements of its output $y = x_1$. For our first design we use the Euler approximation with $\delta = T$:

$$x_1(k+1) = x_1(k) + Tx_2(k)$$
$$x_2(k+1) = x_2(k) + T(-x_1(k) - x_1(k)^3). \tag{21}$$

Observer design for this model is straightforward because the nonlinearity depends only on the output $y = x_1$. Defining the new variable $\chi := x_2 - y$, which is governed by

$$\chi(k+1) = (1-T)\chi(k) + T[-2y(k) - y(k)^3], \tag{22}$$

we employ the observer

$$\hat{\chi}(k+1) = (1-T)\hat{\chi}(k) + T[-2y(k) - y(k)^3] \tag{23}$$
$$\hat{x}_2 = \hat{\chi} + y, \tag{24}$$

and note that the error variable $e_2 = \hat{x}_2 - x_2 = \hat{\chi} - \chi$ satisfies

$$e(k+1) = (1-T)e(k). \tag{25}$$

The assumptions of Theorem 1 hold because the Lyapunov function

$$V_T(e) = \frac{1}{2}e^2 \tag{26}$$

satisfies, along the trajectories of the approximate error model (25),

$$\frac{V_T(e(k+1)) - V_T(e(k))}{T} = -e(k)^2 + \frac{T}{2}e(k)^2, \tag{27}$$

which is as in (16). Since the trajectories of the Duffing oscillator are bounded, we conclude from Theorem 1 that the observer (23)-(24) achieves semiglobal practical convergence for the unknown exact model. This is illustrated with numerical simulations in Figure 1 below, where the residual error between the solid trajectories and the dashed observer estimates becomes smaller as the sampling period T is reduced.

We next investigate the effect of refining the approximate models as in Theorem 2. A disadvantage of this approach is that, unlike the Euler approximation with $\delta = T$ above, the refined models do not preserve the structure of the continuous model (20), where the nonlinearity depends only on the output. In fact, an application of (Chung and Grizzle 1990, Theorem 7) to (20) shows that, even its exact discrete-time model does not possess this structure. Instead, we pursue a less ambitious design where we modify the observer to account for the $\mathcal{O}(T^2)$ nonlinear terms in the higher-order approximation

$$\chi(k+1) = (1-T)\chi(k) + T[-2y(k) - y(k)^3] + T^2\left[-y(k)^3 - \frac{3}{2}y(k)^2\chi(k) - \frac{1}{2}\chi(k)\right] + \mathcal{O}(T^3). \tag{28}$$

Because the first of the bracketed $\mathcal{O}(T^2)$ terms only depends on the output, and the latter two have negative signs, we incorporate them in the observer (23)-(24):

$$\hat{\chi}(k+1) = (1-T)\hat{\chi}(k) + T[-2y(k) - y(k)^3] + T^2\left[-y(k)^3 - \frac{3}{2}y(k)^2\hat{\chi}(k) - \frac{1}{2}\hat{\chi}(k)\right] \tag{29}$$

With this modification the $\mathcal{O}(T)$ terms in the exact Lyapunov difference equation are unchanged, while the sign-indefinite $\mathcal{O}(T^2)$ terms are now negative definite. Thus, it is plausible to expect a reduction in the size of the residual observer error. Simulations with the modified observer (29) in Figure 2 indeed show such an improvement over the original design in Figure 1.

4. OBSERVER DESIGN VIA EMULATION

A common method for digital implementation of controllers and observers, known as "emulation", is to discretize continuous-time designs using approximate techniques. We assume that a continuous-time observer of the form

$$\dot{\hat{x}} = g(\hat{x}, y, u) \tag{30}$$

is available, and implement it with the approximate discrete-time equation:

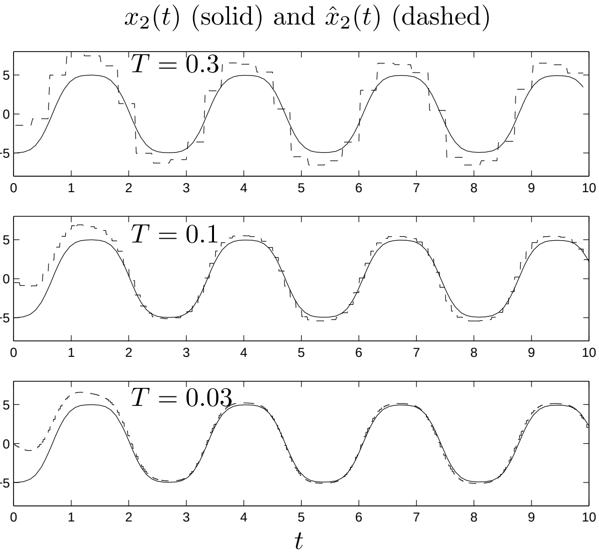

Fig. 1. Simulation results for $x_2(t)$ (solid) from the Duffing oscillator (20), and $\hat{x}_2(t)$ (dashed) from the observer (23)-(24). As predicted by Theorem 1, the residual observer error diminishes as T is decreased.

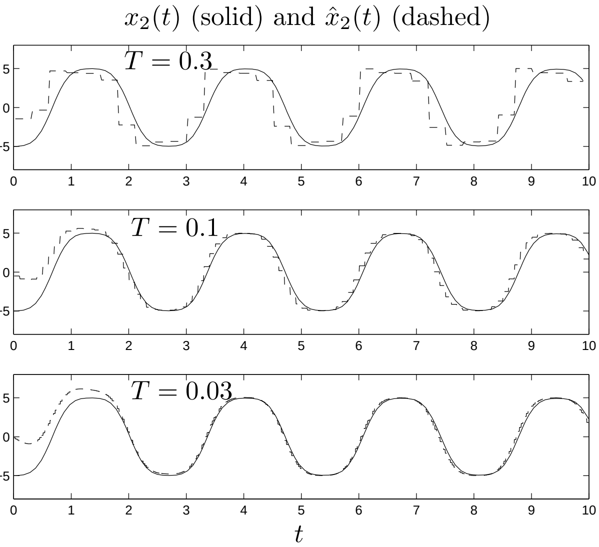

Fig. 2. Simulation results for $x_2(t)$ (solid) from the Duffing oscillator (20), and $\hat{x}_2(t)$ (dashed) from the modified observer (29). The residual errors are smaller than those with the original observer in Figure 1.

$$\hat{x}(k+1) = G^a_{T,\delta}(\hat{x}(k), y(k), u(k)). \qquad (31)$$

We assume in this section that functions $f(\cdot,\cdot)$ and $g(\cdot, h(\cdot), \cdot)$ are locally Lipschitz in all their arguments.

When $\delta = T$ we establish semiglobal practical convergence in T under a Lyapunov condition on the continuous-time observer (30), and a consistency property of the approximate discretization in (31):

Theorem 3. **($\delta = T$)** The observer (31) is semiglobal practical in T as in Definition 2(a) if the following conditions hold:

(i) $\delta = T$.

(ii) G^a_T is consistent with G^e_T as in Definition 1(a), with (y, u) interpreted as constant inputs during sampling intervals.

(iii) The continuous-time observer (30) ensures convergence with a C^1 function $V(x, \hat{x})$ satisfying, for all $x, \hat{x} \in I\!R^n$ and for all $u \in I\!R^m$,

$$\alpha_1(|e|) \leq V(x, \hat{x}) \leq \alpha_2(|e|) \qquad (32)$$

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial \hat{x}} g(\hat{x}, y, u) \leq -\alpha_3(|e|). \qquad (33)$$

□

Finally we study the situation where δ can be tuned independently of T. Theorem 4 below shows that by tuning δ we can reduce d in (9) not arbitrarily, but to a level dictated by T as in Definition 2(c):

Theorem 4. **(δ independent of T)** The observer (31) is semiglobal in T, and practical in T and δ as in Definition 2(c), if the following conditions hold:

(i) δ can be adjusted independently of T.

(ii) $G^a_{T,\delta}$ is consistent with G^e_T as in Definition 1(b), with (y, u) interpreted as constant inputs during sampling intervals.

(iii) The continuous-time observer (30) satisfies condition (iii) of Theorem 3. □

Example 2. For the Duffing oscillator (20) in Example 1, a continuous-time reduced-order observer is

$$\hat{x}_2 = \chi + y \qquad \dot{\chi} = -\chi - 2y - y^3. \qquad (34)$$

Because the observer error $e := \hat{x}_2 - x_2$ satisfies $\dot{e} = -e$, condition (iii) of Theorems 3 and 4 holds with the Lyapunov function

$$V(x, \hat{x}) = \frac{1}{2}(x_2 - \hat{x}_2)^2 = \frac{1}{2}e^2. \qquad (35)$$

To discretize (34) we first use the Euler approximation with $\delta = T$:

$$\chi(k+1) = \chi(k) + T[-\chi(k) - 2y(k) - y(k)^3], \qquad (36)$$

which coincides with our design (23)-(24) in Example 1. Thus, the simulation results in Figure 1 also illustrate Theorem 3. We next study the situation where T is fixed as in Theorem 4. Instead, we refine the discretization for (34) by dividing the sampling period into N steps of size $\delta = T/N$, and by applying an Euler approximation for each step. As $N \to \infty$, this approximation converges to the exact zero-order-hold equivalent of the continuous-time observer (34):

$$\chi(k+1) = \exp(-T)\chi(k) + (1 - \exp(-T))[-2y(k) - y^3(k)] \qquad (37)$$

which is computable in this example because the only nonlinearity in (34) is in the output-injection term $[-2y - y^3]$. Simulation results in Figure 3 show that the residual observer error is smaller for $N = 3$ in the middle plot, than for $N = 1$ ($\delta = T$) in the top plot. However, as predicted by Theorem 4, increasing N (that is, decreasing δ) does not reduce this residual error arbitrarily. Even with the exact zero-order-hold equivalent (37) in the bottom plot of Figure 3, we note that an observer error remains because the sampling period T is fixed.

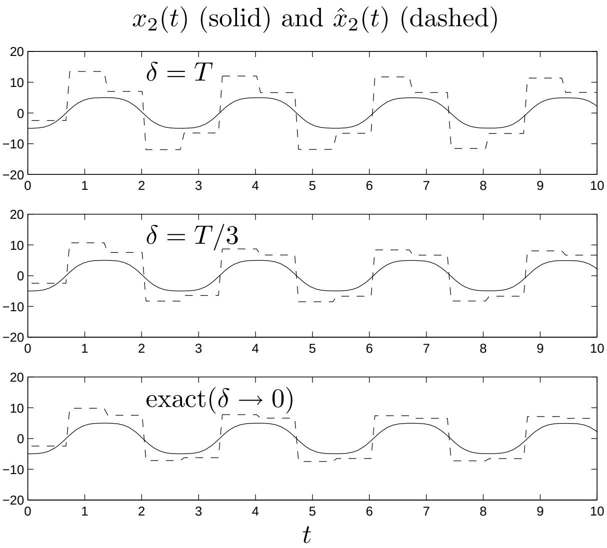

Fig. 3. Simulation results for $x_2(t)$ (solid) from the Duffing oscillator (20), and $\hat{x}_2(t)$ (dashed) from the Euler approximation of the continuous-time observer (36), when the sampling period is held constant at $T = 0.67$ sec. and the integration period δ is reduced.

5. CONCLUSIONS

We have specified conditions on the approximate model, continuous-time model, and the observer, guaranteeing that the observer that performs well on an approximate discrete-time model will also perform well on the exact discrete-time model. We have further discussed the effect of refining the approximate models with a modelling parameter δ, independently of the sampling period T. Although practical convergence is in principle achievable with a design based on approximate-models refined with smaller integration periods δ (Theorem 2), the complexity of such models grow significantly as δ is reduced and, thus, it may be infeasible to develop a family of observers for them. Because sampling periods cannot be arbitrarily reduced in applications, the question of how to reduce the residual observer error systematically for a prescribed T deserves further investigation.

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