SYSTEM THEORY FOR NUMERICAL ANALYSIS

Kenji Kashima*, Shinjiro Ashida* Yutaka Yamamoto*

Kyoto University, JAPAN.
{kashima,sashida,yy}@acs.i.kyoto-u.ac.jp

Abstract: System theory for numerical analysis has recently become a focus of research. In this paper we regard dynamics of Newton's method as a nonlinear feedback system and derive convergence conditions, based on the internal model principle and systems of Lur'e type. We then focus our attention on the analysis of the region of absolute stability of Runge-Kutta type methods. We derive a linear matrix inequality condition which characterizes a relationship between Runge-Kutta coefficients and the corresponding stability region. We also propose a new optimization procedure for designing a Runge-Kutta method based on this characterization. Copyright ©2005 IFAC

1. INTRODUCTION

There exist many iterative numerical schemes for solving linear or nonlinear equations, differential equations, having different characteristics suited for varied needs. They are mostly represented by difference equations, and hold an important position. In this connection, it is natural that system theory plays an important role, and numerical analysis has indeed been studied from this viewpoint, see [Bhaya et al. (2003); Kaszkurewicz et al. (1995); Schaerer et al. (2001); Wakasa et al. (2001)] and references therein. This paper starts by noting the role of the internal model principle in iterative methods for solving linear equations, and then aims at analyzing the stability of Newton's and Runge-Kutta methods by placing these in a system-theoretic framework.

We start by studying the linear equation $Ax = b$. We see that we can interpret iterative solution processes for this equation as tracking problems to the constant (step) input b [Schaerer et al. (2001); Wakasa et al. (2001)]. The objective here is clearly to track b in spite of minor computational or data errors arising in the process of computation.

The internal model principle plays a crucial role here. We then generalize this idea to nonlinear equations, especially to Newton's method. While the stability of Newton's method is considered in [Bhaya et al. (2003)], only the first order approximation of the given nonlinear function was taken into account. In contrast, we view Newton's iterative process as a nonlinear feedback system of Lur'e type [Khalil (1992)]. The internal model principle again plays an important role here. The present analysis enables us to relax the convergence region, and the result is compared with conventional analysis.

In the second half, we turn our attention to numerical integration methods of ordinary differential equations, particularly the analysis of the absolute stability region of Runge-Kutta type methods. The absolute stability at a point in the complex plane means that a corresponding linear test problem is stable. The absolute stability region governs the step size to guarantee accurate numerical solutions. While this is an important problem, it is also known to be difficult to describe relationships between this region and Runge-Kutta coefficients. Only for some special cases, algebraic
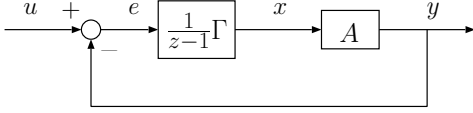
Fig. 1. Block diagram for iterative methods

Fig. 2. Block diagram for Newton's method with computational errors

conditions of the coefficients have been obtained [Scherer et al. (1989)]. We invoke a new generalized Kalman-Yakubovich-Popov (KYP) Lemma [Iwasaki et al. (2003)], to obtain a more general characterization of this region in terms of a linear matrix inequality (LMI). This allows us to *design* the coefficients of a Runge-Kutta type method by optimizing the region of absolute stability.

2. CONVERGENCE OF ITERATIVE PROCESSES

2.1 Iterative processes for linear equations

Let us start by reviewing iterative processes for solving linear equations [Schaerer et al. (2001); Wakasa et al. (2001)]. Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and consider the linear equation

$$Ax = b$$

for $x \in \mathbb{R}^n$. Suppose that A is nonsingular, and we wish to generate a sequence x_k that converges to the solution $A^{-1}b$. Decompose A as $A = D + E + F$, where D, E and F are diagonal, strictly lower rectangular and strictly upper rectangular matrices, respectively. Then the Jacobi, Gauss-Seidel (GS), and Successive Over-Relaxation (SOR) methods are given as follows [Quartteroni et al. (2000)]:

Jacobi: $x_{k+1} = -D^{-1}(E + F)x_k + D^{-1}b$,
GS : $x_{k+1} = -(D + E)^{-1}Fx_k + (D + E)^{-1}b$,
SOR : $x_{k+1} = (I + \omega D^{-1}E)^{-1}\{(1 - \omega)I - \omega D^{-1}F\}x_k$
$\qquad\qquad +\omega(D + \omega E)^{-1}b$,

where ω is relaxation parameter. Let us formulate these three methods as feedback systems. For example, we can rewrite the algorithm of the Jacobi method as $x_{k+1} = x_k + D^{-1}e_k$ with the error signal $e_k := b - Ax_k$. Similarly, the other two methods have similar feedback mechanisms

$$x_{k+1} = x_k + \Gamma e_k,$$

where Γ is given by $(D + E)^{-1}$ for GS and $\omega(D + \omega E)^{-1}$ for SOR. These are depicted in Fig. 1, with step input signal $u \equiv b$.

It is natural to require the following properties:

- For arbitrary b, the method should work, i.e., the output y_k should track arbitrary b in order that x_k converges to the exact solution.
- This must be satisfied robustly, i.e., in spite of some data errors.

The celebrated internal model principle tells us that this robust tracking property is satisfied if and only if the following two conditions hold:
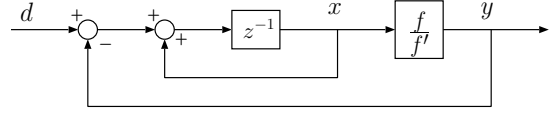
(i) The feedback system is internally stable, and

(ii) The loop transfer matrix contains the internal model $1/(z - 1)$.

In Fig. 1, the open-loop transfer matrix clearly contains $1/(z - 1)$. Thus x_k converges to $A^{-1}b$ if and only if condition (i) above is satisfied, i.e., all eigenvalues of $I - \Gamma A$ lie in the open unit disc. This is consistent with the conventional results [Quartteroni et al. (2000)]. This also tells us that why most, if not all, iterative processes assume the form $x_{k+1} = x_k +$ correction term: it is a result of the internal model principle. While the same equation is also put into a closed-loop feedback system in [Bhaya et al. (2003)], the integrator is considered to be a part of the plant and not the controller; thus the internal model principle and the resulting robust tracking property is not explicitly presented in [Bhaya et al. (2003)].

On the other hand, Bhaya et al. discussed choices of Γ which leads to better convergence properties and the conjugate gradient method as proportional-derivative control [Bhaya et al. (2003)]. Some other numerical schemes are also considered in [Kaszkurewicz et al. (1995); Schaerer et al. (2001)].

2.2 Newton's method and Lur'e system

We consider Newton's method for solving nonlinear equations of a single variable: $f(x) = 0$. We first suppose that f is continuously differentiable function and that $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Moreover we assume the existence of a solution, and denote it by x^*. Newton's method generates a sequence $\{x_k\}$ which converges to x^* with an adequate initial value x_0 [Quartteroni et al. (2000)]:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} + d_k, \quad k = 0, 1, 2, \ldots. \quad (1)$$

where $\{d_k\}$ represents computational errors. We can similarly describe this dynamics as a nonlinear feedback system as shown in Fig. 2. Here the convergence of x_k to x^* means that $y_k := f(x_k)/f'(x_k)$ converges to 0.

In [Bhaya et al. (2003)], stability conditions of this system have been shown by means of control Lyapunov functions. However only the first order approximation of f was considered there.

We now convert this system to a system of Lur'e type [Khalil (1992)] for stability analysis. In what follows, we denote f/f' by g for simplicity, and define $\tilde{g}(x) := g(x + x^*)$. Notice that $\tilde{g}(0) = 0$. Assume that there exist a, b such that

$$a \le \frac{\tilde{g}(x)}{x} \le b, \quad \text{for all } x \ne 0. \quad (2)$$

The introduction of $\tilde{g}$ changes the feedback system as shown in Fig. 3 with step disturbance $w \equiv -x^*$ because $y_k = g(x_k)$ is equal to $\tilde{g}(x_k - x^*)$. Thus the objective here is now modified to attenuate the step disturbance w in Fig. 3. It should be noted that we have introduced $\tilde{g}$ just for stability analysis and do not need $\tilde{g}$ to execute Newton's method.

Similarly to the case of linear equations in the previous subsection, the loop transfer function contains the step signal generator $1/(z-1)$, which is necessary to reject the unknown step disturbance w according to the internal model principle. To express this more precisely, we transform the system in Fig. 3 to an error system. With $\tilde{g}$ defined above, equation (1) becomes

$$e_{k+1} = e_k - \tilde{g}(e_k) + d_k, \quad (3)$$

where $e_k := x_k - x^*$ is the error signal. Equation (3) can be viewed as a system of Lur'e type as in Fig. 4, consisting of the linear system $1/(z-1)$ and nonlinear function $\tilde{g}$. In this error system, we have to make the error signal e_k converge to 0. The following theorem guarantees that x_k does not diverge due to the computational error d_k. Furthermore, when there exist no computational errors, numerical solutions converge to the exact solution.

Theorem 1. Consider the algorithm (1). Suppose that there exist constants a and b such that

$$0 < a \le \frac{f(x)}{(x - x^*)f'(x)} \le b < 2, \quad (4)$$

for all $x \ne x^*$. Then for any initial value x_0 and $d \in \ell^2$, the error e belongs to ℓ^2. Furthermore, if $d \equiv 0$, x_k converges to the exact solution x^*.

Proof We show only the first statement. We have (2) from the assumption (4). According to stability analysis for Lur'e system, the first statement holds if $1/(z + l - 1)$ is stable and

$$\left\| \frac{r}{z + l - 1} \right\|_\infty < 1,$$

where $l = (a + b)/2$ and $r = (b - a)/2$. It can be shown that this is equivalent to $0 < a \le b < 2$. ∎

2.3 Modification of the convergence theorem

Condition (4) may appear a bit restrictive in that the constants a, b are bounded by 2 and that it is
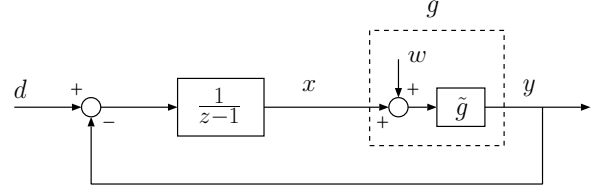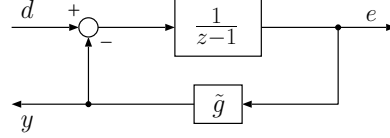

Fig. 3. Lur'e system for Newton's method

Fig. 4. Error system for Newton's method

assumed to hold globally. We show two corollaries of Theorem 1 to remedy this. The first corollary is concerned with the case that the condition (4) does not hold for $b < 2$. Even if $b \ge 2$ in (4), we can make x_k converge to x^* by introducing a suitable gain in the iteration process.

Corollary 1. Suppose that there exist a and b satisfying

$$0 < a \le \frac{f(x)}{(x - x^*)f'(x)} \le b, \quad \text{for all } x \ne x^*. \quad (5)$$

Consider the iterative process

$$x_{k+1} = x_k - L\frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots$$

with constant L such that $0 < L < 2/b$. Then x_k converges to the solution x^*.

The other corollary deals with local stability. As is well known, the convergence of Newton's method is guaranteed only locally [Quartteroni et al. (2000)]. This property corresponds to the fact that condition (4) holds only on a bounded set of $\mathbb{R}$. We now apply Theorem 1 to enlarge the domain of convergence.

Corollary 2. Suppose that the solution x^* is in $I := [\alpha, \beta]$ and that there exist a and b satisfying (4) for all $x \in I \backslash \{x^*\}$. If we set $x_0 \in I$ and

$$x_{k+1} = \begin{cases} \alpha, & \text{if } x_k < \alpha \\ x_k - \dfrac{f(x_k)}{f'(x_k)}, & \text{if } x_k \in I \\ \beta, & \text{if } x_k > \beta \end{cases} \quad k = 0, 1, 2, \ldots$$

then x_k converges to the solution x^*.

2.4 Comparison with conventional convergence condition

Here we discuss the difference between Theorem 2 and the convergence condition based on the contraction mapping principle. Define ϕ by

$$\phi(x) := x - \frac{f(x)}{f'(x)}.$$

Then Newton's method is described as $x_{k+1} = \phi(x_k)$. According to the contraction mapping principle, x_k converges to the exact solution x^* if the function ϕ is a contraction on a closed interval.

Theorem 2. Suppose that the solution x^* exists in a closed interval I and that the function f is twice continuously differentiable in I. If function ϕ is contractive in I, there exist a and b satisfying (4) for all $x \in I \backslash \{x^*\}$.

This theorem means that our result includes the conventional one as a special case. However the converse does not hold.

3. RUNGE-KUTTA TYPE METHODS

3.1 Absolute stability of Runge-Kutta type methods

Methods of the Runge-Kutta type are widely used for the numerical integration of the initial value problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0 \in \mathbb{R}^m \qquad (6)$$

with $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$. The algorithm with step size h is represented by

$$x_{n+1} = x_n + \sum_{i=1}^{s} b_i k_i,$$

where for $i = 1, 2, \ldots, s$

$$k_i = hf\left(t_n + c_i h, x_n + \sum_{j=1}^{s} a_{ij} k_j\right),$$

and $t_n = nh$. Then x_n is an approximation of the exact solution $x(t_n)$. We refer to the coefficients a_{ij} and b_i as the *Runge-Kutta coefficients*.

Let us consider the linear test problem

$$\dot{x} = \lambda x, \quad x(0) = x_0, \quad \text{Re}\,\lambda < 0. \qquad (7)$$

The numerical solution for (7) is given by ([Dekker et al. (1984)])

$$x_{n+1} = P(z)x_n, \ z = h\lambda,$$

where $P(z)$ is defined by

$$P(z) := 1 + b^T z(I - zA)^{-1}e, \qquad (8)$$

$$A := \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & \ddots & & a_{2s} \\ \vdots & & \ddots & \vdots \\ a_{s1} & \cdots & \cdots & a_{ss} \end{bmatrix}, \ b := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{bmatrix}, \ e := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Since the exact solution is clearly $x(t) = \exp(\lambda t)x_0$, numerical solution $\{x_n\}$ must decay. In view of this, the absolute stability is defined as follows [Dekker et al. (1984)].

Definition 1. A Runge-Kutta type method specified as above is *absolutely stable at $z \in \mathbb{C}$* if $|P(z)| < 1$. *The region of absolute stability* is defined as

$$\mathcal{R} := \{z \in \mathbb{C} \ : \ |P(z)| < 1\}.$$

The method is said to be *A-stable*, if $\mathcal{R}$ includes the open left half-plane.

In order to obtain the numerical solution for (7), we must choose so small h that $h\lambda$ belongs to $\mathcal{R}$ to guarantee a stable behavior of the numerical solution against various fluctuations. However we cannot take h very small, when we consider round-off errors or deal with so-called *stiff* systems. The region $\mathcal{R}$ thus characterizes how large the step size h can be, and hence it is desirable that we have a large region of absolute stability.

It is however not easy to describe the relationship between such regions and the Runge-Kutta coefficients. For this problem, Scherer [Scherer et al. (1989)] derived an algebraic condition on the coefficients of A-stable methods. It turned out that A-stable methods are inevitably implicit Runge-Kutta schemes that require complicated procedures of solving nonlinear equations. On the other hand, explicit Runge-Kutta methods, which cannot be A-stable, are widely used for their simplicity. In view of this, we here attempt to estimate the region of absolute stability quantitatively, for those methods which are not necessarily A-stable.

3.2 Problem formulation

Let us formulate the problem. Define the region

$$\Lambda_{\text{arc}}(r_1, r_2, \alpha) := \{z \in \mathbb{C} \ : \ r_1 \le |z| \le r_2, \ |\arg z - \pi| \le \alpha\}$$

where r_1, r_2 and α satisfy $0 < r_1 \le r_2$ and $0 \le \alpha \le \pi/2$ (Fig. 5). In particular, when $\alpha = 0$, this reduces to an interval in the real axis. To estimate the region of absolute stability, we consider the following:

Problem 1. For given $0 < r_1 \le r_2$, $0 \le \alpha \le \pi/2$ and the Runge-Kutta coefficients A and b, determine whether the corresponding region of absolute stability $\mathcal{R}$ includes the region $\Lambda_{\text{arc}}(r_1, r_2, \alpha)$.

To see the meaning of three parameters r_1, r_2 and α, consider the linear test problem (7) again. We here take α such that $|\arg \lambda - \pi| \le \alpha$, $r_2 := h|\lambda|$ and sufficiently small r_1. If Problem 1 is solvable and $\Lambda_{\text{arc}}(r_1, r_2, \alpha) \subset \mathcal{R}$ for these parameters, then we have $h\lambda \in \mathcal{R}$ and corresponding numerical solutions decay as desired. Furthermore, by a bisection algorithm on r_2, we can compute a (sub)maximal step size h.
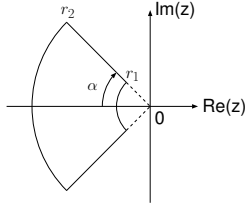
Fig. 5. Region $\Lambda_{\mathrm{arc}}(r_1, r_2, \alpha)$

3.3 LMI conditions of coefficients

Denote $K(z) := P(z^{-1})$ for $z \neq 0$, i.e.,

$$K(z) = 1 + b^T(zI - A)^{-1}e. \tag{9}$$

This can be viewed as a proper and rational transfer function. Clearly $|P(z)| < 1$ in $\Lambda_{\mathrm{arc}}(r_1, r_2, \alpha)$ if and only if $|K(z)| < 1$ in $\Lambda_{\mathrm{arc}}(1/r_2, 1/r_1, \alpha)$. The latter condition is the bounded-realness of the system. Since many results on relationships between bounded-realness and system matrices are known, it is natural to apply such results to characterization of absolute stability. In particular, a generalized KYP Lemma [Iwasaki et al. (2003)], recently obtained for the situations dealing with a finite frequency range, enables us to characterize system matrices so that the corresponding system be bounded-real on a class of curves in the complex plane. We can now give a solution to Problem 1 by using this lemma.

Theorem 3. Let r_1, r_2, α and the coefficients A and b be given. Suppose that $\Lambda_{\mathrm{arc}}(1/r_2, 1/r_1, \alpha)$ contains no eigenvalues of matrix A. Then the region of absolute stability $\mathcal{R}$ includes $\Lambda_{\mathrm{arc}}(r_1, r_2, \alpha)$ if and only if there exist Hermitian matrices P_i and Q_i $(i = 1, 2, 3)$ such that

$$Q_i > 0, \quad i = 1, 2, 3$$

$$F^T \begin{bmatrix} P_i & -Q_i \\ -Q_i & -\frac{1}{r_i{}^2}P_1 - \frac{2}{r_i}\cos\alpha Q_i \end{bmatrix} F + \Theta < 0, \ i = 1, 2$$

$$F^T \begin{bmatrix} -Q_3 & e^{-j\alpha}(jP_3 - r_cQ_3) \\ -e^{j\alpha}(jP_3 + r_cQ_3) & -\frac{1}{r_1r_2}Q_3 \end{bmatrix} F + \Theta < 0$$

where

$$F := \begin{bmatrix} A & e \\ I & 0 \end{bmatrix}, \ \Theta := \begin{bmatrix} bb^T & b \\ b^T & 0 \end{bmatrix}, \ r_c := \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Proof See Appendix. ∎

3.4 An application

Runge-Kutta methods of high order have some free parameters in the coefficients A, b and c. As an application of the result obtained, we propose a new procedure for designing the Runge-Kutta coefficients that maximize the region of absolute stability. Generally, the stability function $P(z)$ for

Table 1. Result of maximizing r_2

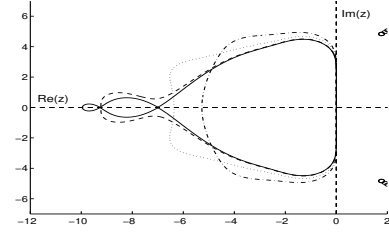| α	r_2
0	9.9575
$\pi/40$	9.1174
$\pi/6$	6.3504
$3\pi/8$	5.2513

Fig. 6. Regions of absolute stability: $\alpha = 0$ (solid), $\pi/40$ (dash), $\pi/6$ (dotted) and $3\pi/8$ (dash-dotted)

an s-stage explicit Runge-Kutta method of order p is represented as

$$P(z) = \sum_{i=0}^{p} \frac{z^i}{i!} + \sum_{i=p+1}^{s} \gamma_i z^i. \tag{10}$$

For a class of such methods, some of γ_i $(i = p + 1, \ldots, s)$ are free parameters and the coefficients A, b and c are determined by them. In what follows we suppose ideally that γ_i $(i = p+1, \ldots, s)$ are all free parameters. We denote

$$\boldsymbol{\gamma} := \begin{bmatrix} \gamma_s & \gamma_{s-1} & \cdots & \gamma_{p+1} \end{bmatrix},$$

and the corresponding region of absolute stability as $\mathcal{R}(\boldsymbol{\gamma})$. We attempt to find $\boldsymbol{\gamma}$ maximizing $\mathcal{R}(\boldsymbol{\gamma})$.

According to equation (10), $K(z) = P(z^{-1}) = C_k(zI - A_k)^{-1}B_k + 1$ with

$$A_k = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \ B_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ C_k = \begin{bmatrix} \boldsymbol{\gamma}^T \\ 1/p! \\ \vdots \\ 1/2! \\ 1 \end{bmatrix}^T.$$

Notice that $\boldsymbol{\gamma}$ appears only in C_k. By replacing A, e, b^T by A_k, B_k, C_k in Theorem 3, the pertinent matrix inequalities are characterized by $\boldsymbol{\gamma}$, P_i and Q_i $(i = 1, 2, 3)$. Furthermore, by using the Schur complement [Iwasaki et al. (2003)], they are all transformed to LMIs. Hence we can easily design $\boldsymbol{\gamma}$ which maximizes $\mathcal{R}(\boldsymbol{\gamma})$.

We illustrate an example of a 9-stage explicit Runge-Kutta method of order 7. For the class of the methods, γ_8 and γ_9 are free parameters [Tanaka et al. (1992)]. We design them by the procedure mentioned above. We take $r_1 = 0.1$ and $\alpha = 0$, $\pi/40$, $\pi/6$, $3\pi/8$, and maximize r_2. The result is shown in Table 1 and the corresponding regions are shown in Fig. 6. We see that the larger α becomes, the greater width the corresponding region assumes, as expected.

Finally we show the effectiveness of our procedure. We compute the Runge-Kutta coefficients corre-
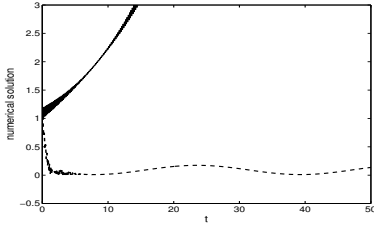
Fig. 7. Numerical solutions: $\alpha = 0$ (solid) and $\alpha = \pi/40$ (dash)

sponding to (sub)optimum γ_α for given α. Then by using these formulas, we obtain numerical solutions for the initial value problem

$$\dot{x} = \begin{bmatrix} -100 & 45.1 \\ -20 & -40 \end{bmatrix} x + \begin{bmatrix} 10\sin 0.2t \\ -1 \end{bmatrix}, \; x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (11)$$

We set the step size $h = 0.1$. The first component of the corresponding numerical solutions is shown in Fig. 7. This figure shows that the solution for the formula for $\alpha = \pi/40$ converges to the exact solution, while the solution for $\alpha = 0$ diverges. In this case, eigenvalues of the matrix in (11) are given by $\lambda = -70 \pm 1.4j$. In fact, $h\lambda$ belongs to $\mathcal{R}(\gamma_{\pi/40})$ but not to $\mathcal{R}(\gamma_0)$. This example illustrates the effectiveness of designing a new numerical scheme with stability issues taken into account.

4. CONCLUSION

We have derived stability conditions for Newton's method based on the stability analysis for systems of Lur'e type. This result includes as a special case conventional results based on the contraction mapping principle. In the second half, we have given an LMI condition to describe the relationship between the Runge-Kutta coefficients and the corresponding stability region, by invoking a generalized KYP lemma. By this characterization, we can design coefficients which optimize the region. It is natural to expect that a much wider variety of applications still remain to be investigated.

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Appendix A. PROOF OF THEOREM 3

First define Λ by

$$\Lambda(\Phi, \Psi) := \{ z \in \mathbb{C} \; : \; \sigma(z, \Phi) = 0, \; \sigma(z, \Psi) \geq 0 \},$$

$$\sigma(z, \Pi) := \begin{bmatrix} z \\ 1 \end{bmatrix}^* \Pi \begin{bmatrix} z \\ 1 \end{bmatrix}.$$

Note that $\Lambda(\Phi, \Psi)$ represents curves in the complex plane, when Φ and Ψ are chosen appropriately [Iwasaki et al. (2003)]. The following lemma, a special case of Theorem 3 in [Iwasaki et al. (2003)], gives system matrices such that corresponding transfer functions are bounded real on curves $\Lambda(\Phi, \Psi)$.

Lemma 1. Given Hermitian matrices $\Phi, \Psi \in \mathbb{C}^{2\times 2}$ and $K(z)$ by (9). Suppose that $\Lambda(\Phi, \Psi)$ represents bounded curves in the complex plane and that $K(z)$ is analytic on $\Lambda(\Phi, \Psi)$. Then $|K(z)| < 1$ for all $z \in \Lambda(\Phi, \Psi)$ if and only if there exist Hermitian matrices P and Q satisfying $Q > 0$ and

$$F^*(\Phi \otimes P + \Psi \otimes Q)F + \Theta < 0,$$

where $\otimes$ means the Kronecker product and F and Θ are defined in Theorem 3.

Now we are ready to prove Theorem 3.

Proof of Theorem 3 Since $K(z)$ is analytic on $\Lambda_{\mathrm{arc}}(1/r_2, 1/r_1, \alpha)$ by the assumption, $|K(z)| < 1$ on this domain if and only if this holds on its boundary, according to the maximum modulus theorem. This boundary consists of two arcs and two lines, as in Fig. 5. Here these can be represented in the form of $\Lambda(\Phi, \Psi)$ with adequately defined matrices Φ, Ψ. Hence by applying Lemma 1, we obtain Theorem 3. ∎