ON ROBUST EXPONENTIAL STABILITY OF
A CLASS OF ATTRACTOR NEURAL NETWORKS [*]

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Abstract: Robust Exponential stability of continuous-time attractor neural networks with delays is discussed. A new sufficient condition ensuring existence and uniqueness of periodic solution for a general class of interval dynamical systems are obtained. Discrete-time analogue of the continuous-time systems with periodic input is formulated and we study their dynamical characteristics. The robust exponential stability and periodicity of the continuous-time systems is preserved by the discrete-time analogue without any restriction imposed on the uniform discretization step-size. *Copyright © 2005 IFAC*

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1. INTRODUCTION

The properties of equilibrium points of neural systems play an important role in some practical problems, such as optimization solvers (Chen, 2000; Forti, 1995; Kennedy, 1988; Tank and Hopfield, 1986; Sudharsanan and Sundareshan, 1991), pattern recognition (Liu and Lu, 1997) and image compression (Venetianer and Roska, 1998). It is well known that an equilibrium point can be viewed as a special periodic solution of continuous-time neural systems with arbitrary period (Zhang, 2002; Sun and Feng, 2003, 2004;). In this sense the analysis of periodic solutions of neural systems may be considered to be more general sense than that of equilibrium points. In addition, the existence of periodic solutions of continuous-time neural networks is an interesting dynamic behavior. It has been found applications in learning theory (Townley, 2000), which is motivated by the fact that learning process usually requires repetition. On the other hand, the dynamical characteristics of neural networks may often be destroyed by its unavoidable uncertainty due to the existence of modeling errors, external disturbance and parameter fluctuation during the implementation on very-large-scale-integration (VLSI) chips. Thus, it is important to investigate the

periodicity and robustness of the network against such errors and fluctuation. In order to overcome this difficulty, (Liao, 1998; Arik, 2003; and Sun, 2003) have extended the model of delayed Hopfield neural networks to interval-delayed neural systems. We further extend the model of delayed Hopfield neural networks with constant input to interval dynamical systems with periodic input in (Sun, 2004).

Dynamical analysis of continuous-time attractor neural networks with delays have been the new world-wide focus (Van and Zou, 1998; Cao, 2000; Gopalsamy, 1994; Liao and Wang, 2003; Sun, Zhang et al, 2002; Mohamad, 2001). However, in implementing the continuous-time neural system for simulation or computational purposes, it is essential to formulate a discrete-time system which is an analogue of the continuous-time system. A method which is commonly used in the formulation of a discrete-time analogue is by discretizing the continuous-time system. Certainly, the discrete-time analogue when derived as a numerical approximation of continuous-time system is desired to preserve the dynamical characteristics of the continuous-time system. Once this is established, the discrete-time analogue can be used without loss of functional similarity to the continuous-time system and preserving any physical or biological reality that the continuous-time system has. Though there exist a lot of numerical schemes (such as Euler scheme, Runge-Kutta scheme) that can be used to obtain discrete-

time analogues of continuous-time neural networks. However, it is generally known that these numerical schemes can exhibit spurious equilibrium and spurious asymptotic behavior, which are not present in the continuous-time counterparts. The existence of spurious equilibrium and asymptotic behavior in a discrete-time analogue can happen if one holds the positive discretization step size fixed and let the original parameters (i.e. parameters of the continuous-time system) vary within the asymptotic parameter space of the continuous-time systems. As a consequence, one has to impose limitations either on the size of h or on the original parameters so as to avoid the existence of spurious equilibrium and spurious asymptotic behavior in the discrete-time analogues. With such limitations, the computational capability of the continuous-time network will not be achieved fully by the discrete-time analogues. Here, we refer to (Mohamad and Gopalsamy, 2003; Sun and Feng, 2005) and their references therein for related discussions on the importance of discrete-time counterparts.

In this paper, we will derive a easily checkable condition to guarantee robust exponential periodicity of continuous-time attractor neural networks with delays for pattern recognition. We will also study the robust exponential periodicity of discrete-time neural systems with delays. To the best knowledge of ours, this is the first time to investigate the robust exponential periodicity of discrete-time neural networks with delays.

2. PROBLEM STATEMENT

Consider the model of continuous-time neural network with delays described by the following functional differential equations

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau_j)) + I_i(t), \ t \geq 0 \qquad (1)$$

$$x_i(t) = \phi_i(t), \ i = 1,2,\cdots,n,$$

where $0 \leq \tau_j \leq \tau$ is the transmission time delay of the j th unit , $x_i(t)$ is the state vector of the i th unit at time t , a_{ij}, b_{ij} are constants. For simplicity, let D be an $n \times n$ constant diagonal matrix with diagonal elements $d_i > 0$, $i = 1,2,\ldots,n$, $A = (a_{ij})$ is $n \times n$ constant interconnection matrix, $f_j(x_j(t-\tau_j))$ denotes the activation function of the j th unit at time $t - \tau_j$, $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T \in R^n$ is an input periodic vector function with period ω , i.e., there exists a constant $\omega > 0$ such that $I_i(t+\omega) = I_i(t)$ $(i = 1,2,\cdots,n)$ for all $t \geq 0$, and $x = (x_1, x_2, \ldots, x_n)^T \in R^n$. Suppose further $f \in$GLC in R^n, i.e., for each $j \in \{1,2,\cdots,n\}$, $f_j : R \to R$ is globally Lipschitz continuous with Lipschitz constant

$M_j \geq 0$, that is, $\left| f_j(u) - f_j(v) \right| \leq M_j \left| u - v \right|$ for all $u, v \in R$. Moreover, we will use norms of vector x in R^n, which is defined as $\|x\| = \sum_{i=1}^{n} |x_i|$.

As a special case of neural system (1), the delayed neural networks with constant input vector $I = (I_1, I_2, \ldots, I_n)^T \in R^n$ have been studied widely by many researchers. This system is described by the following functional differential equations

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau_j)) + I_i, \ t \geq 0 \qquad (2)$$

$$x_i(t) = \phi_i(t), \ -\tau \leq t \leq 0, \ i = 1,2,\cdots,n,$$

Define $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $t \geq 0$. Let

$$\|x_t\| = \sup_{-\tau \leq \theta \leq 0} \sum_{i=1}^{n} \left| x_i(t+\theta) \right|,$$

Let $x^* = \left(x_1^*, x_2^*, \cdots, x_n^* \right)$ be the equilibrium point of system (2), i.e.,

$$d_i x_i^* = \sum_{j=1}^{n} a_{ij} f_j(x_i^*) + I_i \qquad (3)$$

In practice implementation of neural networks, the value of the constants d_i and connection weight coefficients a_{ij} depend on certain resistance and capacitance value which are subject to uncertainties. This may lead to some deviations in the value of d_i , and a_{ij} . The deviations and perturbations of the neuron charging time constants and the weights of interconnections are bounded in general. In the following, we correspondingly intervalize the above mentioned quantities by defining the set of matrices as follows similar to (Liao, 1998; Sun and Feng, 2003).

$$D_I = \{ D = \text{diag}(d_i)_{n \times n} : \underline{D} \leq D \leq \overline{D} \ ,$$
$$\text{i.e., } \underline{d}_i \leq d_i \leq \overline{d}_i, i = 1,2,\ldots,n \} \ ;$$
$$A_I = \{ A = (a_{ij})_{n \times n} : \underline{A} \leq A \leq \overline{A} \ ,$$
$$\text{i.e., } \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, i,j = 1,2,\ldots,n \} \ .$$

In the following, we formulate a discrete-time analogue of the continuous-time neural networks (1) by a semi-discretization technique. For detail, reader can refer to (Sun and Feng, 2004).

$$x_i(m+1) = x(m) e^{-d_i h}$$
$$+ \theta_i(h) \left\{ \sum_{j=1}^{n} a_{ij} f_j(x_j(m-\sigma_j)) + I_i(m) \right\} \qquad (4)$$

for $i = 1,2,\cdots,n$, $m \in Z_0^+$, where

$$\theta_i(h) = \frac{1 - e^{-d_i h}}{d_i}, \ i = 1,2,\cdots,n \ .$$

It can be verified that $\theta_i(h) > 0$ if $d_i > 0, h > 0$. System (4) is our discrete-time analogue of the continuous-time system (1). It is easily to verify that the discrete-time analogue (4) converges towards the continuous-time network (1) when $h \to 0^+$.

In order to study the discrete-time analogue (3), we assume that the system (3) is supplemented with an initial condition, a consequence of the discretization, of the form $x_i(l) = \varphi_i(l)$ for $l \in [-\sigma, 0]_z$ where $i = 1, 2, \cdots, n$, $\sigma = \max_{j=1,2,\cdots,n} \{\sigma_j\}$, and the external input function are periodic function with positive integer period ϖ, i.e.,

$$I_i(m + \varpi) = I_i(m), \; i = 1, 2, \cdots, n, \; m \in Z^+.$$

As a special case of neural system (4), the delayed discrete-time neural networks with constant input vector $I = (I_1, I_2, \ldots, I_n)^T \in R^n$ is described by the following functional differential equations

$$x_i(m+1) = x(m)e^{-d_i h}$$
$$+ \theta_i(h) \left\{ \sum_{j=1}^{n} a_{ij} f_j(x_j(m - \sigma_j)) + I_i \right\}. \quad (5)$$

Let $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ denote an equilibrium point of (5). We obtain from (5) that

$$x_i^* = x_i^* e^{-d_i h} + \theta_i(h) \left\{ \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + I_i \right\}.$$

from which one easily obtains the equilibrium point equation (3) of system (2). Obviously, the discrete-time analogue (5) preserves the equilibrium x^* of the continuous-time network (2).

3. MAIN RESULTS

In this section, we will present the results concerning the robust exponential periodicity of interval delayed neural networks of the form (1) and (3) for each

$$D = \text{diag}(d_i)_{n \times n} \in D_I, \quad A = (a_{ij})_{n \times n} \in A_I.$$

Theorem 1: Suppose $f \in$ GLC. If there exists a positive diagonal matrix $= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that the system parameter d_i, a_{ij} $(i, j = 1, 2, \cdots, n)$ satisfy the following conditions

$$\lambda_i \underline{d}_i - M_i \sum_{j=1}^{n} \lambda_j |a_{ji}^*| > 0, \text{ for } i = 1, \cdots, n, \quad (4)$$

where

$$|a_{ij}^*| = \max_{1 \le i, j \le n} \{|\underline{a}_{ij}|, |\overline{a}_{ij}|\},$$

then for every periodic input $I(t)$, the delayed neural system (1) is robustly exponentially periodic.

Proof. Let $C = C([-\tau, 0], R^n)$ be the Banach space of all continuous function from $[-\tau, 0]$ to R^n with the topology of uniform convergence. For any $\phi \in C$, let $\|\phi\| = \sup_{-\tau \le t \le 0} \sum_{i=1}^{n} |\phi_i(t)|$. Given any ϕ, $\varphi \in C$, let $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \cdots, x_n(t, \phi))^T$ and $x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \cdots, x_n(t, \varphi))^T$ be the solutions of (1) starting from ϕ and φ respectively. Define $x_t(\phi) = x(t + \theta, \phi)$, $\theta \in [-\tau, 0]$, then $x_t(\phi) \in C$ for all $t \ge 0$. Thus, it follows from (1) that

$$\dot{x}_i(t, \phi) - \dot{x}_i(t, \varphi) = -d_i \big(x_i(t, \phi) - x_i(t, \varphi) \big)$$
$$+ \sum_{j=1}^{n} a_{ij} \big(f_j(x_j(t - \tau_j, \phi)) - f_j(x_j(t - \tau_j, \varphi)) \big) \quad (5)$$

for $t \ge 0$, $i = 1, 2, \cdots, n$.

We consider function $F_i(\cdot)$ defined by

$$F_i(\varepsilon_i) = \lambda_i(\underline{d}_i - \varepsilon_i) - M_i e^{\varepsilon_i \tau} \sum_{j=1}^{n} \lambda_j |a_{ji}^*|$$

for $\varepsilon_i \in [0, \infty)$, $i = 1, \cdots, n$. \quad (6)

We have from (4) and (6) that $F_i(0) > 0$ for $i = 1, \cdots, n$ and hence by the continuity of $F_i(\cdot)$ on $[0, \infty)$, there exists a constant $\varepsilon > 0$ such that

$$F_i(\varepsilon) = \lambda_i(\underline{d}_i - \varepsilon) - M_i e^{\varepsilon \tau} \sum_{j=1}^{n} \lambda_j |a_{ji}^*| > 0,$$

for $i = 1, \cdots, n$ \quad (7)

We will use the following positive definite Lyapunov function candidate defined as follows:

$$V(t) = \sum_{i=1}^{n} \lambda_i \Big(|x_i(t, \phi) - x_i(t, \varphi)| e^{\varepsilon t}$$
$$+ \sum_{j=1}^{n} |a_{ij}| M_j \int_{t-\tau_j}^{t} |x_j(s, \phi) - x_j(s, \varphi)| \times e^{\varepsilon(s+\tau_j)} ds \Big). \quad (8)$$

Define the upper right derivative of $V(t)$ by

$$D^+ V(t) = \lim_{h \to 0^+} \sup [V(t+h) - V(t)]/h.$$

Computing the upper right derivative of $V(t)$ along (5) for $t \ge 0$, we have

$$D^+ V(t) = \sum_{i=1}^{n} \lambda_i \text{sgn}(x_i(t,\phi) - x_i(t,\varphi)) \Big[-(d_i x_i(t,\phi) - d_i x_i(t,\varphi))$$

$$+ \sum_{j=1}^{n} a_{ij} \big(f_j(x_j(t-\tau_j, \phi)) - f_j(x_j(t-\tau_j, \varphi)) \big) \Big] e^{\varepsilon t}$$
$$+ \sum_{i=1}^{n} \lambda_i |x_i(t,\phi) - x_i(t,\varphi)| \varepsilon e^{\varepsilon t}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |a_{ij}| M_j \Big(|x_j(t,\phi) - x_j(t,\varphi)| e^{\varepsilon(t+\tau_j)}$$
$$- |x_j(t-\tau_j, \phi) - x_j(t-\tau_j, \varphi)| e^{\varepsilon t} \Big)$$

$$\le \sum_{i=1}^{n} \lambda_i \Big[-d_i |x_i(t,\phi) - x_i(t,\varphi)|$$
$$+ \sum_{j=1}^{n} |a_{ij}| \big| f_j(x_j(t-\tau_j, \phi)) - f_j(x_j(t-\tau_j, \varphi)) \big| \Big] e^{\varepsilon t}$$
$$+ \sum_{i=1}^{n} \lambda_i |x_i(t,\phi) - x_i(t,\varphi)| \varepsilon e^{\varepsilon t}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |a_{ij}| M_j \Big(|x_j(t,\phi) - x_j(t,\varphi)| e^{\varepsilon(t+\tau_j)}$$
$$- |x_j(t-\tau_j, \phi) - x_j(t-\tau_j, \varphi)| e^{\varepsilon t} \Big)$$

$$\le \sum_{i=1}^{n} \lambda_i \Big[-d_i |x_i(t,\phi) - x_i(t,\varphi)| \Big] e^{\varepsilon t}$$
$$+ \sum_{i=1}^{n} \lambda_i |x_i(t,\phi) - x_i(t,\varphi)| \varepsilon e^{\varepsilon t}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i |a_{ij}| M_j \Big(|x_j(t,\phi) - x_j(t,\varphi)| e^{\varepsilon(t+\tau_j)} \Big)$$

$$\le -\sum_{i=1}^{n} \left\{ \lambda_i(\underline{d}_i - \varepsilon) - M_i e^{\varepsilon \tau} \sum_{j=1}^{n} \lambda_j |a_{ji}^*| \right\} |x_i(t,\phi) - x_i(t,\varphi)| e^{\varepsilon t}$$

$$\le 0.$$

Therefore,
$$V(t) \le V(0), \quad t \ge 0. \tag{9}$$
From (8), we obtain
$$e^{\varepsilon t}\left(\min_{1\le i\le n}\lambda_i\right)\sum_{i=1}^{n}\left|x_i(t,\phi)-x_i(t,\varphi)\right| \le V(t)$$
and
$$V(0) = \sum_{i=1}^{n}\lambda_i\Big[\left|x_i(0,\phi)-x_i(0,\varphi)\right|$$
$$+\sum_{j=1}^{n}\left|a_{ij}\right|M_j\int_{-\tau_j}^{0}\left|x_j(s,\phi)-x_j(s,\varphi)\right|e^{\varepsilon(s+\tau_j)}ds\Big]$$
$$\le\Big[\max_{1\le i\le n}\lambda_i+\overline{M}\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\lambda_i\max_{1\le j\le n}\left(\left|a_{ij}^{*}\right|\right)\Big]\left\|\phi-\varphi\right\|,$$
where $\overline{M}=\max_{1\le i\le n}\{M_i\}$ is constant. Therefore, from (9), we get
$$\sum_{i=1}^{n}\left|x_i(t,\phi)-x_i(t,\varphi)\right|\le\alpha\left\|\phi-\varphi\right\|\exp(-\varepsilon t) \tag{10}$$
for all $t\ge 0$, where
$$\alpha=\frac{\max_{1\le i\le n}\lambda_i+\overline{M}\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\lambda_i\max_{1\le j\le n}\left(\left|a_{ij}^{*}\right|\right)}{\min_{1\le i\le n}\lambda_i}\ge 1.$$
It follows from (10) that
$$\left\|x_t(\phi)-x_t(\varphi)\right\|\le\alpha\left\|\phi-\varphi\right\|\exp(-\varepsilon(t-\tau)) \tag{11}$$
We can choose a positive integer m such that
$$\alpha\exp(-\varepsilon(m\omega-\tau))\le\frac{1}{4}.$$
Define a Poincaré mapping $H:C\to C$ by $H\phi=x_\omega(\phi)$. Then from (11), we can derive that
$$\left\|H^m\phi-H^m\varphi\right\|\le\frac{1}{4}\left\|\phi-\varphi\right\|, \tag{12}$$
where $H^m\phi=x_{m\omega}(\phi)$. This implies that H^m is a contraction mapping. Therefore, there exists a unique fixed point $\phi^*\in C$ such that $H^m\phi^*=\phi^*$. So, $H^m(H\phi^*)=H(H^m\phi^*)=H\phi^*$. This shows that $H\phi^*\in C$ is also a fixed point of H^m, hence, $H\phi^*=\phi^*$, that is, $x_\omega(\phi^*)=\phi^*$. Let $x(t,\phi^*)$ be the solution of (1) through $(0,\phi^*)$. By using $I(t+\omega)=I(t)$ for $t\ge 0$, $x(t+\omega,\phi^*)$ is also a solution of (1). Note that $x_{t+\omega}(\phi^*)=x_t(x_\omega(\phi^*))=x_t(\phi^*)$ for $t\ge 0$, then $x(t+\omega,\phi^*)=x(t,\phi^*)$ for $t\ge 0$. This shows that $x(t,\phi^*)$ is a periodic solution of (1) with period ω. From (11), it is easy to see that all other solutions of (1) converge to this periodic solution exponentially as $t\to+\infty$.

Remark 1. When $I=(I_1,I_2,\cdots,I_n)$ is a constant vector, then for any constant $T\ge 0$ we have $I=I(t+T)=I(t)$ for $t\ge 0$. Thus by the results obtained, when the sufficient condition in Theorem 1 is satisfied, a unique periodic solution becomes a periodic solution with any positive constant as its period. So, the periodic solution reduced to a constant solution, that is, an equilibrium point. Furthermore, all other solutions globally

exponentially converge to this equilibrium point as $t\to+\infty$. The unique equilibrium point of the delayed neural system (2) is globally robustly exponentially stable. Obviously, the results obtained are consistent with the exponential stability results that were recently reported in (Cao, 2000; Sun, et al, 2002) and (Sun and Feng, 2003; Zhang, 2002).

Theorem 2: Suppose $f\in$ GLC and the positive step-size is fixed. If there exists a positive diagonal matrix $\Lambda=$diag$(\lambda_1,\lambda_2,\ldots,\lambda_n)$ such that the system parameter d_i, a_{ij} $(i,j=1,2,\cdots,n)$ satisfy the following conditions
$$\lambda_i\underline{d}_i-M_i\sum_{j=1}^{n}\lambda_j\left|a_{ji}^{*}\right|>0, \quad\text{for } i=1,\cdots,n,$$
where
$$\left|a_{ij}^{*}\right|=\max_{1\le i,j\le n}\left\{\left|\underline{a}_{ij}\right|,\left|\overline{a}_{ij}\right|\right\},$$
then for every periodic input $I(m)$, the delayed neural system (3) is robustly exponentially periodic.

Proof. Let $B=B\left([-k,0]_Z,\mathbb{R}^n\right)$ be the Banach space with norm $\|\phi\|=\max_{l\in[-k,0]_Z}\left(\sum_{i=1}^{n}\left|\phi_i(l)\right|\right)$ for any $\phi\in B$. Given any ϕ, $\psi\in B$, let $x(m,\phi)=(x_1(m,\phi),x_2(m,\phi),\cdots,x_n(m,\phi))^T$ and $x(m,\psi)=(x_1(m,\psi),x_2(m,\psi),\cdots,x_n(m,\psi))^T$ be the solutions of (16) starting from ϕ and ψ respectively. Define $x^m(\phi)(l)=x(m+l,\phi)$, $l\in[-\sigma,0]_Z$, then $x^m(\phi)(l)\in B$ for all $m\in Z^+$. Thus, it follows from (3) that
$$\left|x_i(m+1,\phi)-x_i(m+1,\psi)\right|\le\left|x_i(m,\phi)-x_i(m,\psi)\right|e^{-d_ih}$$
$$+\theta_i(h)\left\{\sum_{j=1}^{n}\left|a_{ij}M_j(x_j(m-\sigma_j,\phi)-x_j(m-\sigma_j,\psi))\right|\right\}$$
$$\text{for } i=1,2,\cdots,n. \tag{13}$$
We consider function $\widetilde{F}_i(\cdot)$ defined by
$$\widetilde{F}_i(r_i)=\lambda_i(1-r_ie^{-d_ih})-M_i\theta_i(h)\sum_{j=1}^{n}\lambda_j\left|a_{ji}^{*}\right|r_i^{\sigma_i+1}$$
$$\text{for } r_i\in[1,\infty), i=1,\cdots,n. \tag{14}$$
We have from (4) and (14) that
$$\widetilde{F}_i(1)\ge\theta_i(h)\left(\lambda_i\underline{d}_i-M_i\sum_{j=1}^{n}\lambda_j\left|a_{ji}^{*}\right|\right)>0 \quad\text{for}$$
$i=1,\cdots,n$ and hence by the continuity of $\widetilde{F}_i(\cdot)$ on $[1,\infty)$, there exists a constant $r>1$ such that
$$\widetilde{F}_i(r)=\lambda_i(1-re^{-d_ih})-M_i\theta_i(h)\sum_{j=1}^{n}\lambda_j\left|a_{ji}^{*}\right|r^{\sigma_i+1}\ge 0,$$
$$\text{for } i=1,\cdots,n \tag{15}$$
In the following, we consider nonnegative sequence $z_i(m)$ defined by
$$z_i(m)=\lambda_i^{-1}r^m\frac{\left|x_i(m,\phi)-x_i(m,\psi)\right|}{\theta_i(h)}$$
$$\text{for } i=1,\cdots,n, m\in[-\sigma,\infty)_Z. \tag{16}$$

By using (18) and (21) we obtain the following inequalities:

$$z_i(m+1) = \lambda_i^{-1} r^{m+1} \frac{|x_i(m+1,\phi) - x_i(m+1,\psi)|}{\phi_i(h)}$$

$$\leq re^{-d_i h} z_i(m) + \lambda_i^{-1} \sum_{j=1}^{n} |a_{ij}| \lambda_j M_j \theta_j(h) r^{\sigma_j+1} z_j(m-\sigma_j),$$

for $i = 1, \cdots, n$, $m \in [-\sigma, \infty)_Z$. (17)

Accordingly, we will use the following positive definite Lyapunov function candidate:

$$V(m) = \sum_{i=1}^{n} \left(z_i(m) + \lambda_j^{-1} \sum_{j=1}^{n} |a_{ij}| \lambda_j M_j \theta_j(h) r^{k_j+1} \sum_{l=m-\sigma_j}^{m-1} z_j(l) \right),$$

for $m \in Z_0^+$. (18)

Calculating the difference $\Delta V(m) = V(m+1) - V(m)$ along (22), we obtain

$$\Delta V(m) \leq \sum_{i=1}^{n} \Big\{ re^{-d_i h} z_i(m)$$

$$+ \lambda_i^{-1} r \sum_{j=1}^{n} |a_{ij}| \lambda_j M_j \theta_j(h) r^{\sigma_j+1} z_j(m-\sigma_j)$$

$$+ \lambda_i^{-1} \sum_{j=1}^{n} |a_{ij}| \lambda_j M_j \theta_j(h) r^{\sigma_j+1} \sum_{l=m+1-k_j}^{m} z_j(l) - z_i(m)$$

$$- \lambda_i^{-1} \sum_{j=1}^{n} |a_{ij}| M_j \lambda_j \theta_j(h) r^{\sigma_j+1} \sum_{l=m-k_j}^{m-1} z_j(l) \Big\}$$

$$= -\sum_{i=1}^{n} \Big\{ 1 - re^{-d_i h} - M_i \theta_i(h) \sum_{j=1}^{n} \lambda_i^{-1} \lambda_j |a_{ji}| r^{\sigma_i+1} \Big\} z_i(m)$$

$$= -\sum_{i=1}^{n} \lambda_i^{-1} \Big\{ \lambda_i(1 - re^{-d_i h}) - M_i \theta_i(h) \sum_{j=1}^{n} \lambda_j |a_{ji}| r^{\sigma_i+1} \Big\} z_i(m)$$

$$\leq -\sum_{i=1}^{n} \lambda_i^{-1} \Big\{ \lambda_i(1 - re^{-d_i h}) - M_i \theta_i(h) \sum_{j=1}^{n} \lambda_j |a_{ji}^*| r^{\sigma_i+1} \Big\} z_i(m) \quad (19)$$

for $m \in Z_0^+$. By using (20) in (24) we assert that

$$\Delta V(m) \leq 0 \quad \text{for} \quad m \in Z_0^+ \quad \text{which implies}$$

$V(m) \leq V(0)$ for $m \in Z^+$. It then follows from (19) that

$$\sum_{i=1}^{n} z_i(m) \leq \sum_{i=1}^{n} \Big\{ z_i(0)$$

$$+ \lambda_i^{-1} \sum_{j=1}^{n} |a_{ij}| \lambda_j M_j \theta_j(h) r^{\sigma_j+1} \sum_{l=-\sigma_j}^{-1} z_j(l) \Big\}$$

and by substituting (21) in the above we obtain

$$\sum_{i=1}^{n} \lambda_i^{-1} \frac{|x_i(m,\phi) - x_i(m,\psi)|}{\theta_i(h)} \leq \left(\frac{1}{r} \right)^m$$

$$\times \sum_{i=1}^{n} \lambda_i^{-1} \Big\{ 1 + M_i \theta_i(h) \lambda_i \sum_{j=1}^{n} |a_{ji}| r^{\sigma_i+1} \sigma_i \Big\}$$

$$\times \left(\sup_{l \in [-\sigma, 0]_Z} \frac{|\phi_i - \psi_i|}{\theta_i(h)} \right)$$

with the implication

$$\frac{\lambda_{\min}}{\theta_{\max}(h)} \sum_{i=1}^{n} |x_i(m,\phi) - x_i(m,\psi)| \leq \frac{\lambda_{\max}}{\theta_{\min}(h)} \left(\frac{1}{r} \right)^m$$

$$\times \sum_{i=1}^{n} \Big\{ 1 + M_i \theta_i(h) \lambda_j \sum_{j=1}^{n} |a_{ji}^*| r^{\sigma_j+1} \sigma_i \Big\} \times \left(\sup_{l \in [-\sigma, 0]_Z} |\phi_i - \psi_i| \right),$$

where

$$\lambda_{\min} = \min_{i=1,2,\cdots,n} \{\lambda_i\}, \quad \lambda_{\max} = \max_{i=1,2,\cdots,n} \{\lambda_i\},$$

$$\theta_{\min}(h) = \min_{i=1,2,\cdots,n} \{\theta_i(h)\},$$

$$\theta_{\max}(h) = \max_{i=1,2,\cdots,n} \{\theta_i(h)\}.$$

Therefore, we obtain

$$\sum_{i=1}^{n} |x_i^m(\phi) - x_i^m(\psi)| \leq \alpha \left(\frac{1}{r} \right)^{m-\sigma} \sum_{i=1}^{n} \left(\sup_{l \in [-\sigma, 0]_Z} |\phi_i - \psi_i| \right),$$

for $m \in [-\sigma, \infty)_Z$,

where

$$\alpha = \frac{\lambda_{\max} \theta_{\max}(h)}{\lambda_{\min} \theta_{\min}(h)} \max_{i=1,2,\cdots,n} \Big\{ 1 + M_i \theta_i(h) \sum_{j=1}^{n} |a_{ji}^*| r^{\sigma_i+1} \sigma_i \Big\}$$

$$\geq 1.$$

We can choose a positive integer M such that

$$\alpha \left(\frac{1}{r} \right)^{M\varpi - \sigma} \leq \frac{1}{4}. \quad (20)$$

Define a Poincaré mapping $H : B \to B$ by $H\phi = x_\varpi(\phi)$. Then from (20), we can derive that

$$\|H^M \phi - H^M \varphi\| \leq \frac{1}{4} \|\phi - \varphi\|, \quad (21)$$

where $H^M \phi = x_{m\varpi}(\phi)$. This implies that H^M is a contraction mapping. Therefore, there exists a unique fixed point $\phi^* \in B$ such that $H^M \phi^* = \phi^*$. So, $H^M(H\phi^*) = H(H^M \phi^*) = H\phi^*$. This shows that $H\phi^* \in B$ is also a fixed point of H^M, hence, $H\phi^* = \phi^*$, that is, $x_\varpi(\phi^*) = \phi^*$. Let $x(k, \phi^*)$ be the solution of (3) through $(0, \phi^*)$. By using $I(m+\varpi) = I(k)$ for $m \in Z^+$, $x(m+\varpi, \phi^*)$ is also a solution of (16). Note that $x_{m+\varpi}(\phi^*) = x_m(x_\varpi(\phi^*)) = x_m(\phi^*)$ for $m \in Z^+$, then $x(m+\varpi, \phi^*) = x(m, \phi^*)$ for $m \in Z^+$. This shows that $x(m, \phi^*)$ is a periodic solution of (3) with period ϖ. From (21), it is easy to see that all other solutions of (3) converge to this periodic solution exponentially as $m \to +\infty$.

Remark 2. When $I = (I_1, I_2, \cdots, I_n)$ is a constant vector, then for any constant $T \geq 0$ we have $I = I(m+T) = I(m)$ for $m \in Z^+$. Thus by the results obtained, when the sufficient condition in Theorem 2 is satisfied, a unique periodic solution becomes a periodic solution with any positive constants as its period. So, the periodic solution reduced to a constant solution, that is, an equilibrium point. Furthermore, all other solutions globally robustly exponentially converge to this equilibrium point as $m \to +\infty$. The unique equilibrium point of the discrete-time dynamical neural system (4) is globally exponentially stable. Obviously, the results obtained are consistent with the exponential stability results that were recently reported in (Mohamad and

Gopalsamy, 2003), (Sun and Feng, 2004) and (Mahamad, 2001).

4. CONCLUSIONS

In this paper, robust exponential stability and periodicity of continuous-time and discrete-time dynamical systems for pattern recognition have been proposed. Without assuming the boundedness and differentiability of the activation functions, the easily checked conditions ensuring the exponential periodicity of attractor neural networks are obtained. It is shown that the discrete-time analogue preserves the dynamical characteristics of the continuous-time neural systems.

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