CASCADE DISCRETE-CONTINUOUS STATE ESTIMATORS FOR A CLASS OF MONOTONE SYSTEMS

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Abstract: A cascade discrete-continuous state estimator is proposed for a class of monotone systems with both continuous and discrete state. The proposed estimator exploits the partial order preserved by the system dynamics in order to satisfy two properties. First, its computation complexity scales with the number of variables to be estimated instead of scaling with the size of the discrete state space. Second, a separation principle holds: the continuous state estimation error is bounded by a monotonically decreasing function of the discrete state estimation error, the latter one converging to zero. A multi-robot example is proposed.

Keywords: State estimation, hybrid systems, lattice theory, monotone systems, distributed systems.

1. INTRODUCTION

The number of systems of interest with "hybrid" dynamics has been increasing. Internet systems, biological systems, multi-agent systems, dynamic resource allocation systems and many others are all examples of such a hybrid behavior. The problem of estimating the state becomes relevant when asking to control these systems or to verify the correctness of their behavior, as is in the case of air-traffic control systems. Several of these systems have a partial order naturally associated with the space of discrete and continuous variables that is preserved by the dynamics. Dynamic resource allocation problems involving moving resources (agents) as in air-traffic controlled systems ((Tomlin *et al.*, 2001)) or weapon-target assignment problems, are examples where the tasks are usually associated with position in Euclidean space, where the cone partial order induces a partial order on the tasks. There is plenty of systems where partial order among events is naturally established by causal order relations, as for example in message-passing distributed systems ((Zeng *et al.*, 2004)). Most of these examples are also distributed, meaning that the size of the discrete state is so large as to render the estimation problem prohibitive if the partial order is not explicitly taken into account in the estimator design.

As pointed out also by (Bemporad *et al.*, 1999), one of the biggest issues in the estimator design for hybrid systems is complexity. In (DelVecchio and Murray, 2004), a low computation discrete state estimator is constructed, which exploits a partial order on the discrete variable space. The proposed estimator updates at each step the lower and upper bound of the set of discrete variable values compatible with the output sequence. The main contribution of this paper is to extend the results of (DelVecchio and Murray, 2004) to the case in which the continuous variables need to be estimated as well. In particular, an estimator in cascade form is constructed assuming that the discrete variables can be estimated independently of the continuous variables. A class of systems, the monotone systems, is considered for which the computation of the order relation between elements in the continuous variable space can be efficiently performed. However, it has been shown in (DelVecchio and Murray, 2005) that if the system is observable and independent discrete state observable, one can always find a partial order on the spaces of continuous and discrete variables for which the estimation approach here proposed is ap-

[1] Partially supported by AFOSR under grants F49620-01-1-0460 and FA9550-04-1-0169

plicable. The main advantage of a monotone structure is from a computational standpoint.

There is a wealth of research on hybrid observer design and discrete event observer design. In the purely discrete domain, there is the pioneering work of (Caines, 1991) who proposes an enumeration method for the estimation of the discrete state of a finite state machine. This method is also used in (Balluchi *et al.*, 2002) for the estimation of the discrete state. However, if the dimension of the discrete variables set is large, the estimation problem using this method becomes intractable. If the system has some order preserving properties with respect to a suitable partial order, the method proposed here generates an estimator whose computation scales with the number of variables to be estimated. The estimator of this paper is similar to the decoupled estimator design proposed by (Balluchi *et al.*, 2002), except that the continuous and the discrete state are estimated simultaneously in order to achieve a faster convergence of the continuous state estimate, and asymptotic convergence is achieved. As opposed to (Vidal *et al.*, 2002), which proposes to detect the discrete state change *a posteriori*, here the state of the system is tracked.

This paper is organized as follows. In Section 2, notions from partial order theory and observability related definitions are reviewed. In Section 3, the model is introduced. In Section 4, the problem is formulated, and a solution is proposed in Section 5. Section 6 presents a multi-robot example.

2. BASIC CONCEPTS

In this section, some basic definitions on deterministic transition systems and on partial order theory are reviewed (see (Davey and Priesteley, 2002) for details).

2.1 Partial Orders

A partial order is a set χ with a partial order relation "$\leq$", and it is denoted by the pair $(\chi, \leq)$. Define the *join* "$\vee$" and the *meet* "$\wedge$" of two elements x and w in χ as $x \vee w = \sup\{x, w\}$ and $x \wedge w = \inf\{x, w\}$, where by $\sup\{x, w\}$ is the smallest element in χ that is bigger than both x and w, and $\inf\{x, w\}$ is the biggest element in χ that is smaller than both x and w. Let $S \subseteq \chi$, its supremum is denoted $\bigvee S$ and its infimum $\bigwedge S$. If $x < w$ and there is no other element in between x and w, then $x \ll w$. Let $(\chi, \leq)$ be a partial order. If $x \wedge w \in \chi$ and $x \vee w \in \chi$ for any $x, w \in \chi$, then $(\chi, \leq)$ is a *lattice*. Let $(\chi, \leq)$ be a lattice and let $S \subseteq \chi$ be a non-empty subset of χ. Then $(S, \leq)$ is a *sublattice* of χ if $a, b \in S$ implies that $a \vee b \in S$ and $a \wedge b \in S$. If any sublattice of χ contains its least and greatest elements, then $(\chi, \leq)$ is called *complete*. Given a complete lattice $(\chi, \leq)$, this work is concerned with a special kind of a sublattice called an *interval sublattice* defined as follows. Any interval sublattice of $(\chi, \leq)$ is given by $[L, U] = \{w \in \chi : L \leq w \leq U\}$ for $L, U \in \chi$. That is, this special sublattice can be represented by only two elements. The cardinality of an interval sublattice $[L, U]$ is denoted $|[L, U]|$.

Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets. A map $f : P \to Q$ is (i) an *order preserving map* if $x \leq w \implies f(x) \leq f(w)$; (ii) an *order embedding* if $x \leq w \iff f(x) \leq f(w)$; (iii) an *order isomorphism* if it is order embedding and it maps P *onto* Q. A partial order induces a notion of distance between elements in the space. Define the distance function on a partial order in the following way. Let $(P, \leq)$ be a partial order. A distance d on $(P, \leq)$ is a function $d : P \times P \to \mathbb{R}$ such that the following properties are verified: (i) $d(x, y) \geq 0$ for any $x,\ y \in P$ and $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, y) = d(y, x)$; (iii) if $x \leq y \leq z$ then $d(x, y) \leq d(x, z)$.

Since this paper deals with a partial order on the space of the discrete variables and with a partial order on the space of the continuous variables, it is useful to introduce the Cartesian product of two partial orders. Let $(P_1, \leq)$ and $(P_2, \leq)$ be two partial orders. Their Cartesian product is given by $(P_1 \times P_2, \leq)$, where $P_1 \times P_2 = \{(x, y) \mid x \in P_1 \text{ and } y \in P_2\}$, and $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

2.2 Deterministic Transition Systems and Observability

The class of systems dealt with in this work are deterministic, infinite state systems with output. A *deterministic transition system* (DTS) is the tuple $\Sigma = (S, \mathcal{Y}, F, g)$, where S is a set of states with $s \in S$; $\mathcal{Y}$ is a set of outputs with $y \in \mathcal{Y}$; $F : S \to S$ is the state transition function; $g : S \to \mathcal{Y}$ is the output function. An execution of Σ is any sequence $\sigma = \{s(k)\}_{k \in \mathbb{N}}$ such that $s(0) \in S$ and $s(k+1) = F(s(k))$ for all $k \in \mathbb{N}$. The set of all executions of Σ is denoted $\mathcal{E}(\Sigma)$.

Definition 1. (Observability) The deterministic transition system $\Sigma = (S, \mathcal{Y}, F, g)$ is said to be *observable* if any two different executions $\sigma_1, \sigma_2 \in \mathcal{E}(\Sigma)$ are such that there is a $k > 0$ such that $g(\sigma_1(k)) \neq g(\sigma_2(k))$.

This class of systems is general. In the next section, the continuous state evolution and the discrete state evolution of the system are explicitly modeled, and the class of monotone DTSs is introduced.

3. THE MODEL

For a system $\Sigma = (S, \mathcal{Y}, F, g)$ suppose that $S = \mathcal{U} \times \mathcal{Z}$ with $\mathcal{U}$ a finite set, and $\mathcal{Z}$ a possibly infinite dense set; $F = (f, h)$, where $f : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$ and $h : \mathcal{U} \times \mathcal{Z} \to \mathcal{Z}$; $g : \mathcal{U} \times \mathcal{Z} \to \mathcal{Y}$ is the output map. These systems have the form

$$\alpha(k+1) = f(\alpha(k), y(k)) \quad (1)$$

$$z(k+1) = h(\alpha(k), z(k)) \quad (2)$$

$$y(k) = g(\alpha(k), z(k)),$$

and they are referred to as the tuple $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$. Note that the set $\mathcal{Y}$ in general can be both in continuous and discrete form. For such systems, an additional notion, called discrete state observability, is defined.

Definition 2. (Independent discrete state observability) The system $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$ is said to be *independent discrete state observable* if for any execution with output sequence $\{y(k)\}_{k\in\mathbb{N}}$, the following are verified

- (i) $\{\alpha \in \mathcal{U} \mid y(k) = g(\alpha, z(k))$ and $y(k+1) = g(f(\alpha, y(k)), h(\alpha, z(k)))\} := \mathcal{S}(k)$ does not depend on $z(k)$;
- (ii) if two executions $\sigma_1 = \{\alpha_1(k), z_1(k)\}_{k\in\mathbb{N}}$ and $\sigma_2 = \{\alpha_2(k), z_2(k)\}_{k\in\mathbb{N}}$ are such that if $\{\alpha_1(k)\}_{k\in\mathbb{N}} \neq \{\alpha_2(k)\}_{k\in\mathbb{N}}$, then there is $k > 0$ such that $\alpha_1(k) \in \mathcal{S}(k)$ and $\alpha_2(k) \notin \mathcal{S}(k)$.

A independent discrete state observable system admits a discrete state estimator that does not involve the continuous state estimate. This property will allow us to construct a cascade discrete-continuous state estimator.

Now, Σ is restricted to the case in which $\mathcal{Z}$ is partially ordered and the continuous dynamics of the system preserves the ordering. Monotone dynamical systems are usually defined on ordered Banach spaces. An *ordered Banach space* is a real Banach space $\mathcal{Z}$ with a nonempty closed subset K known as the positive cone with the following properties: (i) $\alpha K \subseteq K$ for any $\alpha \in \mathbb{R}_+$; (ii) $K + K \subseteq K$; (iii) the cone is pointed, i.e., $K \cap (-K) = \{\emptyset\}$. A partial ordering is then defined by $x \geq y$ for any $x, y \in \mathcal{Z}$ if and only if $x - y \in K$, with $x > y$ if and only if $x \geq y$ and $x \neq y$. The space and the partial order is denoted $(\mathcal{Z}, \leq)$ (for details see (Smith, 1995) and (Berman and Plemmons, 1994)). A monotone dynamical system on $\mathcal{Z}$ is one whose flow preserves the ordering on initial data. To extend this property to DTSs the notion of extended system is introduced.

Definition 3. (System extension) Consider the system $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$. Let $(\chi, \leq)$ be a lattice with $\mathcal{U} \subseteq \chi$. An extension of Σ on the lattice $(\chi \times \mathcal{Z}, \leq)$ is given by $\tilde{\Sigma} = (\chi \times \mathcal{Z}, \mathcal{Y}, (\tilde{f}, \tilde{h}), \tilde{g})$ such that $\tilde{f} : \chi \times \mathcal{Y} \to \chi$ and $\tilde{f}|_{\mathcal{U}\times\mathcal{Y}} = f$; $\tilde{h} : \chi \times \mathcal{Z} \to \mathcal{Z}$ with $\tilde{h}|_{\mathcal{U}\times\mathcal{Z}} = h$; $\tilde{g} : \chi \times \mathcal{Z} \to \mathcal{Y}$ and $\tilde{g}|_{\mathcal{U}\times\mathcal{Z}} = g$.

Definition 4. (Monotone deterministic transition systems) A system $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$, with $(\mathcal{Z}, \leq)$ an ordered Banach space, and $(\chi, \leq)$ a lattice with $\mathcal{U} \subseteq \chi$, is said to be *monotone* on the partial order $(\chi \times \mathcal{Z}, \leq)$ if there is an extension $\tilde{\Sigma} = (\chi \times \mathcal{Z}, \mathcal{Y}, (\tilde{f}, \tilde{h}), \tilde{g})$ on $(\chi \times \mathcal{Z}, \leq)$ with the property that $\tilde{h} : \chi \times \mathcal{Z} \to \mathcal{Z}$ is order preserving. The extension $\tilde{\Sigma}$ is termed the *monotone extension* of Σ on $(\chi \times \mathcal{Z}, \leq)$.

For a monotone system, the partial order $(\mathcal{Z}, \leq)$ can be used in the estimator design to bring the computational burden down, as the elements of $\mathcal{Z}$ are points, and their partial order relation can be computed efficiently using the definition of $(\mathcal{Z}, \leq)$.

4. PROBLEM STATEMENT

Given a monotone deterministic transition system $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$ and an output sequence $\{y(k)\}_{k\in\mathbb{N}}$, determine and track the current state $(\alpha(k), z(k))$. This is defined in the following problem.

Problem 5. (Cascade continuous-discrete state estimator) Given the monotone deterministic transition system $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$, find functions f_1, f_2, f_3, f_4, with $f_1 : \chi \times \mathcal{Y} \times \mathcal{Y} \to \chi$, $f_2 : \chi \times \mathcal{Y} \times \mathcal{Y} \to \chi$, $f_3 : \mathcal{Z} \times \chi \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{Z}$, $f_4 : \mathcal{Z} \times \chi \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{Z}$, with $\mathcal{U} \subseteq \chi$, $(\chi, \leq)$ a lattice, such that the update laws

$$\begin{aligned} L(k+1) &= f_1(L(k), y(k), y(k+1)) \\ U(k+1) &= f_2(U(k), y(k), y(k+1)) \\ z_L(k+1) &= f_3(z_L(k), L(k), y(k), y(k+1)) \\ z_U(k+1) &= f_4(z_U(k), U(k), y(k), y(k+1)) \end{aligned} \quad (3)$$

with $L(k), U(k) \in \chi$, $L(0) := \bigwedge \chi$, $U(0) := \bigvee \chi$, $z_L(k), z_U(k) \in \mathcal{Z}$, $z_L(0) = \bigwedge \mathcal{Z}$, and $z_U(0) = \bigvee \mathcal{Z}$, have the following properties

- (i) $L(k) \leq \alpha(k) \leq U(k)$ (correctness);
- (ii) $|[L(k+1), U(k+1)]| \leq |[L(k), U(k)]|$ (non-increasing error);
- (iii) There exists $k_0 > 0$ such that for any $k \geq k_0$, $[L(k), U(k)] \cap \mathcal{U} = \alpha(k)$ (convergence).
- (i') $z_L(k) \leq z(k) \leq z_U(k)$;
- (ii') there is a nonnegative function $V : \mathbb{N} \to \mathbb{R}$ such that $d(z_L(k), z_U(k)) \leq V(k)$, with $V(k+1) \leq V(k)$;
- (iii') There exists $k_0' \geq k_0$ such that for any $k \geq k_0'$, $d(z_{L'}(k), z_{U'}(k)) = 0$ where $L' = \bigwedge([L, U] \cap \mathcal{U})$ and $U' = \bigvee([L, U] \cap \mathcal{U})$, with $z_{L'}(k+1) = f_3(z_{L'}(k), L'(k), y(k), y(k+1))$, and $z_{U'}(k+1) = f_4(z_{U'}(k), U'(k), y(k), y(k+1))$, with $z_{L'}(0) = z_L(0)$ and $z_{U'}(0) = z_U(0)$,

for some distance function "d".

The update laws (3) are in cascade form as the variables L and U are not updated on the basis of the variables z_L and z_U. The lattice intervals $[L(k), U(k)]$ and $[z_L(k), z_U(k)]$ define the sets that contain the values of $\alpha(k)$ and $z(k)$ respectively. Properties (iii) and (iii') roughly establish that such sets shrink to $\alpha(k)$ and $z(k)$ respectively. The distance function "d" has been left unspecified for the moment, as its form depends on the particular partial order chosen $(\mathcal{Z}, \leq)$. The following section proposes a solution to the Problem 5.

5. MAIN RESULT

Given the monotone DTS $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$, a set of sufficient conditions that allow a solution to Problem 5 is given. First, some definitions involving the monotone extension $\tilde{\Sigma}$ are given.

Definition 6. (Interval compatibility) The pair $(\tilde{\Sigma}, (\chi, \leq))$ is said to be *interval compatible* if

(i) $\{w \in \chi \mid y(k+1) = \tilde{g}(\tilde{f}(w, y(k)), \tilde{h}(w, z(k))$ and $y(k) = \tilde{g}(w, z(k))\} = [l_w(k), u_w(k)]$;
(ii) the extension $\tilde{f} : \chi \times \mathcal{Z} \to \chi$ is such that $\tilde{f} : ([l_w(k), u_w(k)], y(k)) \to [\tilde{f}(l_w(k), y(k)), \tilde{f}(u_w(k), y(k))]$ is order isomorphic.

Item (i) establishes that the set of $w \in \chi$ compatible with the pair $(y(k), y(k+1))$ is a sublattice interval (see Figure 1). Note that $\mathcal{S}(k) \subseteq [l_w(k), u_w(k)]$. For the construction of a cascade estimator, it is interesting the case in which the partial order $(\mathcal{Z}, \leq)$ is related to $(\chi, \leq)$ by means of the system dynamics. Thus, a new notion of interval compatibility is introduced.

Definition 7. (Induced interval compatibility) The pair $(\tilde{\Sigma}, (\mathcal{Z}, \leq))$ is said to be *induced interval compatible* if

(i) for any $w \in \chi$, $\{z \in \mathcal{Z} \mid y(k+1) = \tilde{g}(\tilde{f}(w, y(k)), \tilde{h}(w, z))$ and $y(k) = \tilde{g}(w, z)\} = [l_z(k, w), u_z(k, w)]$, and if $w_1 \leq w_2$ then $l_z(k, w_1) \leq l_z(k, w_2)$ and $u_z(k, w_1) \leq u_z(k, w_2)$ (see Figure 1);
(ii) $\tilde{h} : \alpha \times [l_z(k, \alpha), u_z(k, \alpha)] \to [\tilde{h}(\alpha, l_z(k, \alpha)), \tilde{h}(\alpha, u_z(k, \alpha))]$ is order isomorphic for any $\alpha \in \mathcal{U}$;
(iii) $d(\tilde{h}(L, l_z(k, L)), \tilde{h}(U, u_z(k, U))) \leq \gamma(|[L, U]|)$, with $\gamma : \mathbb{N} \to \mathbb{R}$ a monotonic function of its argument.

Theorem 8. Consider the monotone DTS $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f, h), g)$. Assume that there is a lattice $(\chi, \leq)$ with $\mathcal{U} \subseteq \chi$ such that $\tilde{\Sigma}$ is a monotone extension of Σ with the properties that $(\tilde{\Sigma}, (\chi, \leq))$ and $(\tilde{\Sigma}, (\mathcal{Z}, \leq))$ are interval compatible and induced interval compatible respectively. A solution to Problem 5 is provided by

$$
\begin{aligned}
L(k+1) &= \tilde{f}(l_w(k) \vee L(k), y(k)) \\
U(k+1) &= \tilde{f}(u_w(k) \wedge U(k), y(k)) \qquad (4) \\
z_L(k+1) &= \tilde{h}(l_w(k) \vee L(k), z_L(k) \vee l_z(k, l_w(k) \vee L(k))) \\
z_U(k+1) &= \tilde{h}(u_w(k) \wedge U(k), z_U(k) \wedge u_z(k, u_w(k) \wedge U(k))).
\end{aligned}
$$

Proof. For the proof of (i)-(ii)-(iii) the reader is deferred to (DelVecchio and Murray, 2004). Define $U^* = u_w(k) \wedge U(k)$, $L^* = l_w(k) \vee L(k)$, $z_U^* = z_U(k) \wedge u_z(k, U^*)$, and $z_L^* = z_L(k) \vee l_z(k, L^*)$. The dependence of u_z and l_z on their arguments is omitted.

Proof of (i'). This is proved by induction on k. Since $z_L(0) = \bigwedge \mathcal{Z}$, and $z_U(0) = \bigvee \mathcal{Z}$, then at the first step $z_L(0) \leq z(k) \leq z_U(0)$ (base case). Assume that $z_L(k) \leq z(k) \leq z_U(k)$ (induction assumption), show that $z_L(k+1) \leq z(k+1) \leq z_U(k+1)$. Consider the third equation of (4). By the order preserving property of $\tilde{h}$, it follows that $\tilde{h}(L^*, z_L^*) \leq \tilde{h}(z(k), \alpha(k))$. Thus,

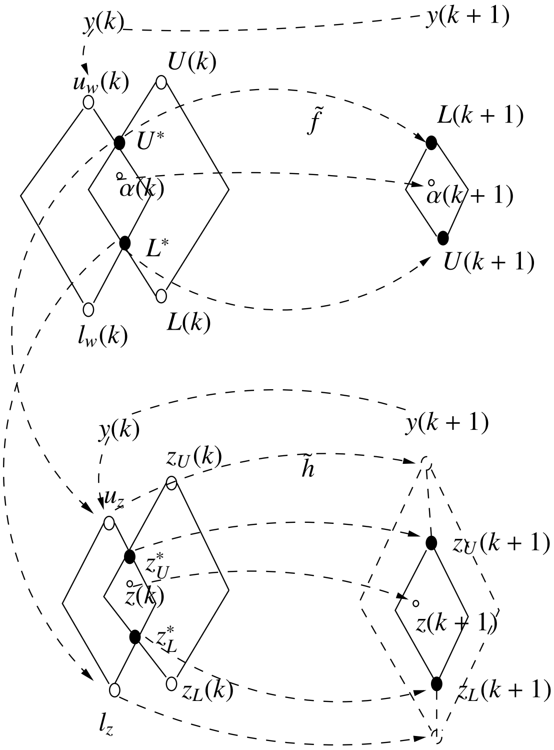

Fig. 1. Hasse diagrams representing the updates of the estimator in Theorem 8. $x < y$ if and only if there is a sequence of connected line segments moving upward from x to y.

$z_L(k+1) \leq z(k+1)$. Similar arguments can be used to prove that $z(k+1) \leq z_U(k+1)$ (see Figure 1).

Proof of (ii'). By the order preserving property of $\tilde{h}$, it follows that $\tilde{h}(L^*, z_L^*) \geq \tilde{h}(L^*, l_z)$, as $z_L^* \geq l_z$ (see the Figure 1). By similar reasonings, it is also true that $\tilde{h}(U^*, z_U^*) \geq \tilde{h}(U^*, u_z)$. The property (iii) of the distance function yields to $d(z_L(k+1), z_U(k+1)) \leq d(\tilde{h}(L^*, l_z), \tilde{h}(U^*, u_z))$. This along with (iii) of Definition 7, yields to $d(z_L(k+1), z_U(k+1)) \leq \gamma(|[L^*, U^*]|)$. Since $\tilde{f}$ is order isomorphic, it follows that $|[L^*, U^*]| = |[\tilde{f}(L^*, y), \tilde{f}(U^*, y)]|$. By the first two equations of (4), it follows that (ii') of Problem 5 is satisfied with $V(k) = \gamma(|[L(k), U(k)]|)$.

Proof of (iii'). The proof proceeds by contradiction. Assume that $d(z_{L'}(k+1), z_{U'}(k+1))$ is never zero. Then, there are at least two elements $z_1'', z_2'' \in [z_{L'}(k+1), z_{U'}(k+1)]$. Because of Property (ii) in Definition 7, and because (a) $z_{L'}(k+1) = \tilde{h}(\alpha(k), z_{L'}(k) \vee l_z(k, \alpha(k)))$, and (b) $z_{U'}(k+1) = \tilde{h}(\alpha(k), z_{U'}(k) \wedge u_z(k, \alpha(k)))$ for $k > k_0$, there are $z_1', z_2' \in [z_{L'}(k), z_{U'}(k)]$ such that $z_1'' = h(\alpha(k), z_1')$, $z_2'' = h(\alpha(k), z_2')$, and $z_1', z_2' \in [l_z(k, \alpha(k)), u_z(k, \alpha(k))]$ (see Figure 1). In analogous way, there are $z_1, z_2 \in [z_{L'}(k-1), z_{U'}(k-1)]$ such that $z_1' = h(\alpha(k-1), z_1)$, $z_2' = h(\alpha(k-1)z_2)$, and $z_1, z_2 \in [l_z(k-1, \alpha(k-1)), u_z(k-1, \alpha(k-1))]$. This implies that there are two executions of Σ, $\sigma_1 = \{\alpha(k), z_1(k)\}_{k \in \mathbb{N}}$ and $\sigma_2 = \{\alpha(k), z_2(k)\}_{k \in \mathbb{N}}$ that share the same output sequence $\{y(k)\}$ for all k. This contradicts the observability of Σ. □

Corollary 9. If in addition to the assumptions of Theorem 8, $\tilde{\Sigma}$ is observable and independent discrete state observable , then (iv) there exists $k'_0 > 0$ such that for any $k \geq k'_0$ $d(z_L(k), z_U(k)) = 0$; (v) there exist a $k_0 > 0$ such that for any $k > k_0$ $L(k) = U(k) = \alpha(k)$.

For the proof of (v), see (DelVecchio and Murray, 2004). The proof of (iv) can be carried out by contradiction in a way analogous to how (iii') of Theorem 8 was proved. In order to verify the properties of Definition 7, an algebraic check is given. For this purpose, define $\tilde{h}^k(w,z) := \tilde{h}(\tilde{h}^{k-1}(w,z), \tilde{f}^{k-1}(w, y(k-2)))$, and $\tilde{f}^k(w, y(k-1)) := \tilde{f}(\tilde{f}^{k-1}(w, y(k-2), y(k-1))$, with $\tilde{f}^0(w,y) := w$ and $\tilde{h}^0(w,z) := z$.

Proposition 10. Consider the monotone DTS $\Sigma = (\mathcal{U} \times \mathcal{Z}, \mathcal{Y}, (f,h), g)$. If its monotone extension $\tilde{\Sigma}$ is observable, there is $\bar{k} > 0$ such that $\{z \mid \tilde{g}(w_0, z) = y(0), ..., \tilde{g}(\tilde{h}^{\bar{k}-1}(w_0,z), \tilde{f}^{\bar{k}-1}(w_0, y(\bar{k}-2)) = y(\bar{k}-1)\} = \{z(0)\}$, where $y(k) = \tilde{g}(w(k), z(k))$, and $w_0 = w(0)$.

This proposition indicates that if the system $\tilde{\Sigma}$ is observable, the continuous state z can be expressed as a function of the output sequence and of the starting discrete state. Thus, there is a map that attaches to a discrete state, a value of the continuous state after some time given an output sequence: this map is defined to be the observability map.

Definition 11. (Observability map) Let the monotone extension $\tilde{\Sigma}$ of Σ be observable. Let $Y := \{y(k)\}_{k \in [1,\bar{k}]}$ be the output sequence up to the smallest step $\bar{k}$ such that the system of equations $\tilde{g}(z,w) = y(0)$,..., $\tilde{g}(\tilde{h}^{\bar{k}-1}(z,w), \tilde{f}^{\bar{k}-1}(w, y(\bar{k}-2))) = y(\bar{k}-1)$ has an unique solution for $z \in \mathcal{Z}$. Then, the *observability map*, denoted $O_Y : \chi \to \mathcal{Z}$, is the map that for a fixed Y attaches to w the unique z satisfying the above system. Also, $\tilde{\Sigma}$ is said to be observable in $\bar{k}$ steps if $\bar{k}$ is not dependent on z.

Here is an algebraic condition that guarantees that $\tilde{\Sigma}$ is induced interval compatible with $(\mathcal{Z}, \leq)$.

Proposition 12. If the monotone extension of Σ, $\tilde{\Sigma}$ is observable in two steps, and the observability map $O_y : \chi \to \mathcal{Z}$ is order preserving, then the pair $(\tilde{\Sigma}, (\mathcal{Z}, \leq))$ is induced interval compatible.

Proof. To prove (i) of Definition 7, let $Y = (y(k), y(k+1))$ be a pair of consecutive outputs in the output sequence $\{y(k)\}_{k \in \mathbb{N}}$ corresponding to an execution of $\tilde{\Sigma}$. By the observability in two steps hypothesis, it follows that $\{z \in \mathcal{Z} \mid y(k) = \tilde{g}(w,z),\ \ y(k+1) = \tilde{g}(\tilde{h}(w,z), \tilde{f}(w, y(k)))\} = \{z^*\}$, and thus $l_z(k,w) = z^* = u_z(k,w)$. Also, by the Definition 11, it follows that $z^* = O_Y(w)$. By the order preserving property of O_Y, it follows that $O_Y(w_1) \leq O_Y(w_2)$ if $w_1 \leq w_2$. Item (ii) of Definition 7 is clearly verified as $l_z(k,\alpha) = u_z(k,\alpha)$. Item (iii) can be proved in the following way. Let $\bar{d} := \max_{w_i \ll w_j} \|\tilde{h}(O_Y(w_i), w_i) - \tilde{h}(O_Y(w_j), w_j)\|$ for $w_i, w_j \in [L, U]$. Then, (iii) is verified with $\gamma(\|[L,U]\|) = \bar{d}\|[L,U]\|$. □

Remark 13. The basic assumption to have induced interval compatibility, is the order preserving property of the observability map. The two steps observability assumption can be abolished by relaxing item (i) of Definition 7 to consider a longer sequence of observations. This can be done with minor modifications.

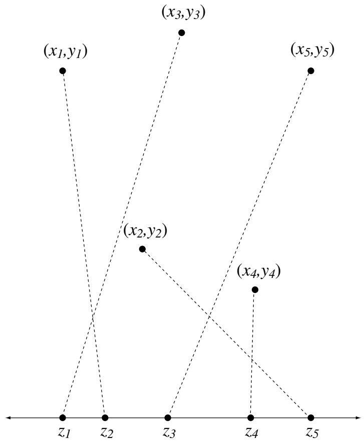

Fig. 2. An example state of the RoboFlag Drill for 5 robots. Here $\alpha = \{3, 1, 5, 4, 2\}$.

6. SIMULATION EXAMPLE

A version of the RoboFlag Drill system, already presented in (DelVecchio and Murray, 2004), is considered where the robots have partially measured second order dynamics. Briefly, there are two teams of N robots, say the attackers and the defenders, in which each defender is assigned to an attacker and moves toward it in order to intercept it before it passes over a defensive zone. There is an assignment protocol that establishes that two close defenders moving one toward the other will exchange their assignments. The dynamics of the defenders are different from our previous work. In this case in fact, they are second order dynamics in which the state is not entirely measured. Figure 2, represents an example with five robots per team. The attacker positions are denoted by (x_i, y_i) and their dynamics is given by if $y_i > \delta$ then $y'_i = y_i - \delta$. Let perm(N) denote the set of all possible permutations of N elements. For the defenders, let the assignment be denoted by $\alpha = (\alpha_1, ..., \alpha_N) \in$ perm(N), with α_i the assignment of defender i, $\mathcal{U} =$ perm(N), their state variable be denoted by $z = (z_{1,1}, z_{1,2}, ..., z_{N,1}, z_{N,2}) \in \mathcal{Z}$, with output $(z_{1,1}, ..., z_{N,1}) \in \mathcal{Y}$. We assume that the set $\mathcal{Z}$ is such that $z_{i,1} \in [x_i, x_{i+1}]$ and $z_{i,2} \in [x_i, x_{i+1}]$ for any i. The function $f : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$ that updates α is given by

$$\text{if } x_{\alpha_i} > z_{i,1} \text{ and } x_{\alpha_{i+1}} < z_{i+1,1} \text{ then } (\alpha'_i, \alpha'_{i+1}) = (\alpha_{i+1}, \alpha_i), \tag{5}$$

for any i. This updates state that whenever two close defenders have conflicting assignments, they swap them. The function $h : \mathcal{U} \times \mathcal{Z} \to \mathcal{Z}$ that updates the z variables is given by

$$\begin{aligned} z'_{i,1} &= (1-\beta)z_{i,1} - \beta z_{i,2} + 2\beta x_{\alpha_i} \\ z'_{i,2} &= (1-\lambda)z_{i,2} + \lambda x_{\alpha_i} \end{aligned} \tag{6}$$

for any i. These laws establish that each defender moves toward the attacker assigned to him. The fact that $z_{i,1} \in [x_i, x_{i+1}]$ and $z_{i,2} \in [x_i, x_{i+1}]$ for any i is guaranteed if β and λ are assumed sufficiently small.

It can be easily shown that the system is independent discrete state observable and interval compatible with $(\chi, \leq)$ defined in the following way. The set χ is the set of vectors in $\mathbb{N}^N$ with components less than N, and the order between any two vectors in χ is established component-wise. By construction perm$(N) \subset \chi$ (see (DelVecchio and Murray, 2004) for details). It can be verified that the system is observable in two steps. The system is monotone and the observability map is order preserving. To see this, consider the positive cone K in $\mathcal{Z}$ composed by all vectors $v = (v_{1,1}, v_{1,2}, ..., v_{N,1}, v_{N,2})$ such that $v_{i,2} \geq 0$, the system preserves this order as if $z^{(1)}_{i,2} < z^{(2)}_{i,2}$ and $w^{(1)}_i \leq w^{(2)}_i$ then $(1-\lambda)z^{(1)}_{i,2} + \lambda x_{w^{(1)}_i} \leq (1-\lambda)z^{(2)}_{i,2} + \lambda x_{w^{(2)}_i}$ because $x_{w^{(1)}_i} \leq x_{w^{(2)}_i}$ whenever $w^{(1)}_i \leq w^{(2)}_i$, and because $(1-\lambda) > 0$. The output map is readily seen to be order preserving in its argument $w = (w_1, ..., w_N) \in \chi$ as for any k, it follows that $z_{i,2}(k) = \frac{1}{\beta}\left((1-\beta)y_i(k) - y_i(k+1) + 2\beta x_{w_i(k)}\right)$.

The estimator in equations (4) has been implemented for system in equations (5) and (6). The discrete state estimator is identical to the one in (DelVecchio and Murray, 2004). For the continuous state estimator set $z_L = (z_{L,1}, ..., z_{L,N}) \in \mathbb{R}^N$ and $z_U = (z_{U,1}, ..., z_{U,N}) \in \mathbb{R}^N$, where $z_{L,i} \leq z_{i,2} \leq z_{U,i}$, that is $z_{L,i}$ and $z_{U,i}$ are respectively the lower and upper bound of the $z_{i,2}$. The first components $z_{i,1}$ are neglected as they are measured. Figure 3 illustrates the estimator performance. $W(k) = \sum_{i=1}^{N} |m_i(k)|$, where $|m_i(k)|$ is the cardinality of the sets $m_i(k)$ that are the sets of possible α_i for each component obtained from the sets $[L_i, U_i]$ by removing iteratively a singleton occurring at component i by all other components. When $[L(k), U(k)] \cap$ perm(N) has converged to α, then $m_i(k) = \alpha_i(k)$. The distance function for $z, x \in \mathbb{R}^N$ is defined $d(x, z) = \sum_{i=1}^{N} \text{abs}(z_i - x_i)$. The function $V(k)$ is defined as $V(k) = \frac{1}{2}\sum_{i=1}^{N}(x_{U_i(k)} - x_{L_i(k)})$, and it is always non increasing. Note that even if the discrete state has not converged yet, the continuous state estimation error after $k = 8$ is close to zero.

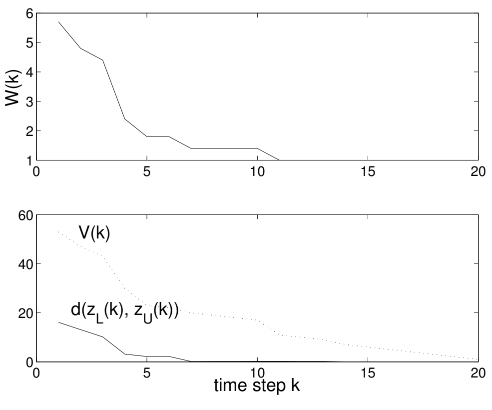

Fig. 3. Estimator performance with $N = 10$ agents.

7. CONCLUSIONS

In this paper, computational tractability of the discrete-continuous state estimation problem has been achieved by the use of a partial order on the space of continuous and discrete states. This was possible due to the order preserving properties of the system dynamics. An example showed how to apply the estimator in the case of a distributed system whose discrete state space is so large as to render prohibitive enumeration estimation methods. Future research will address the problem of relaxing the main assumption underlying this work, that is, that the discrete state dynamics does not depend on the continuous state.

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