STABILITY ANALYSIS OF SWITCHED TIME-DELAY SYSTEMS [1]

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Abstract: This paper addresses the asymptotic stability of switched time delay systems. Piecewise Lyapunov-Razumikhin functions are introduced for the switching candidate systems to investigate the stability in the presence of infinite number of switchings. We provide sufficient conditions in terms of the minimum dwell time to guarantee asymptotic stability under the assumptions that each switching candidate is delay-independently or delay-dependently stable. Conservatism analysis is also provided by comparing with the dwell time conditions for switched delay free systems.

Keywords: Asymptotic Stability, Switched Systems, Time-Delay, Dwell Time

1. INTRODUCTION

Switching control offers a new look into the design of complex control systems (e.g. nonlinear systems, parameter varying systems and uncertain systems) (Skafidas *et al.*, 1999; Morse, 1996; Hespanha *et al.*, 2003; Hespanha, 2004). Unlike the conventional adaptive control techniques that rely on continuous tuning, the switching control method updates the controller parameters in a discrete fashion based on the switching logic. The resulting closed-loop systems have hybrid behaviors (e.g. continuous dynamics, discrete time dynamics and jump phenomena, etc.). One of the most challenging issues in the area of hybrid systems is the stability analysis in the presence of control switching. We refer to (Hespanha *et al.*, 2003) for a general review on switching control methods.

In particular, we are interested in the stability analysis of switched time delay systems. In fact, time delay systems are ubiquitous in chemical processes, aerodynamics, and communication networks (Kharitonov, 1999; Dugard and Verriest, 1998). To further complicate the situation, the time delays are usually time varying and uncertain(Wu and Grigoriadis, 2001). It has been shown that robust $\mathcal{H}^\infty$ controllers can be designed for such infinite dimensional plants, where robustness can be guaranteed within some uncertainty bounds (Toker and Özbay, 1995). In order to incorporate larger operating range or better robustness, controller switching can be introduced, which results in switched closed-loop systems with time delays. For delay free systems, stability analysis and design methodology have been investigated recently in the framework of hybrid dynamical systems (Morse, 1996; Skafidas *et al.*, 1999; Yan and Özbay, 2003; Hochcerman-Frommer *et al.*, 1998; De Persis *et al.*, 2004; Hespanha, 2004). In particular, (Skafidas *et al.*, 1999) provided sufficient conditions on the stability of

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the switching control systems based on Filippov solutions to discontinuous differential equations and Lyapunov functionals; (Morse, 1996) proposed a dwell-time based switching control, where a sufficiently large dwell-time can guarantee the system stability. A more flexible result was obtained in (Hespanha and Morse, 1999), where the average dwell-time was introduced for switching control. In (Yan and Özbay, 2003) the results of (Hespanha and Morse, 1999) were extended to LPV systems. LaSalle's invariance principle was extended to a class of switched linear systems for stability analysis (Hespanha, 2004). Despite the variety and significance of the many results on hybrid system stability, stability of switched time delay systems has seldom been addressed due to the general difficulty of infinite dimensional systems (Hale and Verduyn Lunel, 1993).

Two important approaches in the stability analysis of time delay systems are (1) Lyapunov-Krasovskii method, and (2) Lyapunov-Razumikhin method . Various sufficient conditions with respect to the stability of time delay systems have been given using Riccati-type inequalities or LMIs (Kao and Lincoln, 2004; Kharitonov, 1999; Wu and Grigoriadis, 2001; Dugard and Verriest, 1998). In the presence of switching logic for time delay systems, stability can be guaranteed by introducing multiple Lyapunov functions properly. The main contribution of this paper is a collection of results on the stability of switched time delay systems using piecewise Lyapunov-Razumikhin functions. We provide sufficient stability conditions in terms of the dwell time of the switching signals for the delay independent case and the delay dependent case, respectively.

The paper is organized as follows. The problem definition is stated in Section 2. In Section 3, the main results on the stability of switched time delay systems are presented in terms of the dwell time of the switching signals. Conservatism analysis is provided by comparing with the dwell time conditions for switching delay free systems in Section 4, followed by concluding remarks in Section 5.

2. PROBLEM DEFINITION

Consider the following switched time delay systems:

$$\Sigma_t: \begin{cases} \dot{x} &= A_{q(t)}x(t) + \bar{A}_{q(t)}x(t-\tau_{q(t)}),\ t \geq 0 \\ x(t) &= \phi(t),\ \forall t \in [-\tau_{max}, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $q(t)$ is a piecewise switching signal taken values on the set $\mathcal{F} := \{1, 2, ..., l\}$, i.e. $q(t) = k_j$, $k_j \in \mathcal{F}$, for $\forall t \in [t_j, t_{j+1})$, where t_j, $j \in \mathbb{Z}^+ \cup \{0\}$, is the j^{th} switching time instant. We introduce the triplet $\Sigma_i := (A_i, \bar{A}_i, \tau_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^+$ to describe the i^{th} candidate system of (1). Thus for $\forall t \geq 0$, we have $\Sigma_t \in \mathcal{A} := \{\Sigma_i : i \in \mathcal{F}\}$, where $\mathcal{A}$ is a family of candidate systems of (1). In (1), $\phi(\cdot) : [-\tau_{max}, 0] \mapsto \mathbb{R}^n$ is a continuous and bounded vector-valued function, where $\tau_{max} = \max_{i\in\mathcal{F}}\{\tau_i\}$ is the maximal time delay of the candidate systems in $\mathcal{A}$.

We use $\|\cdot\|$ to denote the Euclidean norm of a vector in $\mathbb{R}^n$, and $|f|_{[t-r,t]}$ for the ∞-norm of f, i.e.

$$|f|_{[t-r,t]} := \sup_{t-r\leq\theta\leq t} \|f(\theta)\|,$$

where f is an element of the Banach space $C([t-r,t], \mathbb{R}^n)$.

Lemma 1. Suppose for a given triplet $\Sigma_i \in \mathcal{A}$, $i \in \mathcal{F}$, there exists symmetric and positive-definite $P_i \in \mathbb{R}^{n\times n}$, such that the following LMI with respect to P_i is satisfied for some $p_i > 1$ and $\alpha_i > 0$:

$$\begin{bmatrix} P_iA_i + A_i^TP_i + p_i\alpha_iP_i & P_i\bar{A}_i \\ \bar{A}_iP_i & -\alpha_iP_i \end{bmatrix} < 0. \tag{2}$$

Then Σ_i is asymptotically stable independent of delay (Dugard and Verriest, 1998; Kharitonov, 1999).

If all candidate systems of (1), $\Sigma_i \in \mathcal{A}$, are delay-independently asymptotically stable satisfying (2), we denote $\mathcal{A}$ by $\tilde{\mathcal{A}}$.

Lemma 2. Suppose for a given triplet $\Sigma_i \in \mathcal{A}$, $i \in \mathcal{F}$, there exists symmetric and positive-definite $P_i \in \mathbb{R}^{n\times n}$, and a scalar $p_i > 1$, such that

$$\begin{bmatrix} \tau_i^{-1}\Omega_i & P_i\bar{A}_iM_i \\ M_i^T\bar{A}_i^TP_i & -R_i \end{bmatrix} < 0 \tag{3}$$

where

$$\Omega_i = (A_i + \bar{A}_i)^TP_i + P_i(A_i + \bar{A}_i) + p_i(\alpha_i + \beta_i)P_i,$$
$$M_i = [A_i\ \bar{A}_i],$$
$$R_i = diag(\alpha_iP_i, \beta_iP_i),$$

and $\alpha_i > 0$, $\beta_i > 0$ are scalars. Then Σ_i is asymptotically stable dependent of delay (Dugard and Verriest, 1998; Kharitonov, 1999).

Similarly we denote $\mathcal{A}$ by $\tilde{\mathcal{A}}_d$ if all candidate systems of (1) are delay-dependently asymptotically stable satisfying (3).

In what follows, we will establish sufficient conditions to guarantee stability of switched system (1) for the delay independent case and the delay dependent case. Therefore, we will assume that $\mathcal{A} = \tilde{\mathcal{A}}$ and $\mathcal{A} = \tilde{\mathcal{A}}_d$ respectively in the corresponding sections in this paper. It is well known that switching between stable candidates may result in divergence of the switched system (Liberzon and Morse, 1999). An important method in stability analysis of switched systems is based on the construction of the common Lyapunov function (CLF), which allows for arbitrary switching. However, this method is too conservative from the perspective of controller design because it is usually difficult to find the CLF for all the candidate systems, particularly for time delay systems whose stability criteria are only sufficient and conservative. A recent paper (Zhai *et al.*, 2003) explored the CLF method for switched time delays systems with three very strong assumptions: (i) each candidate system has the same time delay τ; (ii) each candidate is assumed to be delay independently stable; (iii) The A-matrix is always symmetric and the $\bar{A}$-matrix is always in the form of δI. In the present paper, we consider an alternative method using piecewise Lyapunov-Razumikhin functions for a general class of systems (1) and obtain stability conditions in terms of the dwell time of the switching signal. This method can be used for the case with delay independent criterion (2) and the case with delay dependent criterion (3).

3. MAIN RESULTS ON DWELL TIME BASED SWITCHING

For a given positive constant τ_D, the switching signal set based on the dwell time τ_D is denoted by $S[\tau_D]$, where for any switching signal $q(t) \in S[\tau_D]$, the distance between any consecutive discontinuities of $q(t)$, $t_{j+1} - t_j,\ j \in \mathbb{Z}^+ \cup \{0\}$, is larger than τ_D (Hespanha and Morse, 1999; Morse, 1996). Sufficient condition on the minimum dwell time to guarantee the stable switching will be given using piecewise Lyapunov-Razumikhin functions. Note that the dwell time based switching is trajectory-independent (Hespanha, 2004).

Before presenting the main result of this paper, we recall the following the lemma (Hale and Verduyn Lunel, 1993) for general Retarded Functional Differential Equations (RFDE) in the form of

$$\dot{x}(t) = f(t, x_t) \tag{4}$$

with initial condition $\phi(\cdot) \in C([-r,0],\mathbb{R}^n)$, where x_t denotes the state variables x over the interval $[t-r,\ t]$, and $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$.

Lemma 3. (Hale and Verduyn Lunel, 1993) Suppose $u,v,w,p : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous,nondecreasing functions, $u(0) = v(0) = 0$, $u(s)$, $v(s)$, $w(s)$, $p(s)$ positive for $s > 0$, and v strictly increasing. If there is a continuous function $V : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ such that

$$u(\|x\|) \le V(t,x) \le v(\|x\|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \tag{5}$$

and

$$\dot{V}(t,x(t)) \le -w(\|x(t)\|), \tag{6}$$

if

$$V(t+\theta, x(t+\theta)) < p(V(t,x(t))) \ \ \forall\theta \in [-r,0],$$

then the solution $x = 0$ of the RFDE is uniformly asymptotically stable.

A particular case of (4) is a linear time delay system Σ_i, $i \in \mathcal{F}$, where we can construct the corresponding Lyapunov-Razumikhin function in the quadratic form

$$V_i(t,x) = x^T(t)P_i x(t),\ \ P_i = P_i^T > 0. \tag{7}$$

Apparently V_i can be bounded by

$$u_i(\|x\|) \le V_i(t,x) \le v_i(\|x\|),\ \forall x \in \mathbb{R}^n, \tag{8}$$

where

$$u_i(s) := \kappa_i s^2,\ \ v_i(s) := \bar{\kappa}_i s^2, \tag{9}$$

in which $\kappa_i := \sigma_{min}[P_i] > 0$ denotes the smallest singular value of P_i and $\bar{\kappa}_i := \sigma_{max}[P_i] > 0$ the largest singular value of P_i.

Proposition 4. For each time delay systems Σ_i with Lyapunov-Razumikhin function defined by (7) assume (6) is satisfied for some $w_i(s)$. Then we have

$$|x|_{[t_m-\tau_i,t_m]} \le \sqrt{\frac{\bar{\kappa}_i}{\kappa_i}}|x|_{[t_n-\tau_i,t_n]},\quad \forall t_m \ge t_n \ge 0. \tag{10}$$

Proof. Define

$$\bar{V}_i(t,x) := \sup_{-\tau_i\le\theta\le 0} V_i(t+\theta, x(t+\theta)) \tag{11}$$

for $t \ge 0$, we have

$$\kappa_i(|x|_{[t-\tau_i,t]})^2 \le \bar{V}_i(t,x) \le \bar{\kappa}_i(|x|_{[t-\tau_i,t]})^2,\ t\ge 0 \tag{12}$$

The definition of $\bar{V}_i(t,x)$ implies $\exists\theta_0 \in [-\tau_i, 0]$, such that $\bar{V}_i(t,x) = V(t+\theta_0, x(t+\theta_0))$. Introduce the upper right-hand derivative of $\bar{V}_i(t,x)$ as

$$\dot{\bar{V}}_i^+ = \limsup_{h\to 0^+}\frac{1}{h}[\bar{V}_i(t+h,x(t+h)) - \bar{V}_i(t,x(t))],$$

we have

(i). If $\theta_0 = 0$, i.e. $V_i(t+\theta, x(t+\theta)) \le V_i(t,x(t)) < p(V_i(t,x(t)))$, we have $\dot{V}_i(t,x) < 0$ by (6). Therefore $\dot{\bar{V}}_i^+ \le 0$.

(ii). If $-\tau_i < \theta_0 < 0$, we have $\bar{V}_i(t+h,x(t+h)) = \bar{V}_i(t,x)$ for $h > 0$ sufficiently small, which results in $\dot{\bar{V}}_i^+ = 0$.

(iii). If $\theta_0 = -\tau_i$, the continuity of $V_i(t,x)$ implies $\dot{\bar{V}}_i^+ \le 0$.

The above analysis shows that

$$\bar{V}_i(t_m) \le \bar{V}_i(t_n),\ \forall t_m \ge t_n \ge 0. \tag{13}$$

Recall (12), we have

$$\kappa_i(|x|_{[t_m-\tau_i,t_m]})^2 \le \bar{V}_i(t_m) \le \bar{V}_i(t_n) \le \bar{\kappa}_i(|x|_{[t_n-\tau_i,t_n]})^2, \tag{14}$$

for any $t_m \ge t_n \ge 0$. This implies (10) and proves Proposition 4. ■

Suppose all of the conditions of Lemma 3 are satisfied for general RFDE (4), we also have the follow result.

Lemma 5. (Hale and Verduyn Lunel, 1993) Suppose $|\phi|_{[t_0-r,t_0]} \le \delta_1, \bar{\delta}_1 > 0$, and $\bar{\delta}_2 > 0$ such that $v(\bar{\delta}_1) = u(\bar{\delta}_2)$. For all η satisfying $0 < \eta \le \bar{\delta}_2$, we have

$$V(t,x) \le u(\eta),\ \forall t \ge t_0 + T. \tag{15}$$

Here

$$T = \frac{Nv(\bar{\delta}_1)}{\gamma} \tag{16}$$

is defined by $\gamma = \inf_{v^{-1}(u(\eta))\le s\le\bar{\delta}_2} w(s)$ and $N = \lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil$, where $\lceil\cdot\rceil$ is the ceiling integer function and $a > 0$ satisfies $p(s) - s > a$ for $u(\eta) \le s \le v(\bar{\delta}_1)$.

3.1 The Case with Delay Independent Criterion

Consider the switched time delay systems Σ_t defined by (1) and assume each candidate system Σ_i, $i \in \mathcal{F}$ delay-independently asymptotically stable satisfying (2) (i.e. $\mathcal{A} = \tilde{\mathcal{A}}$). A sufficient condition on the minimum dwell time to guarantee the asymptotic stability can be derived using multiple piece-wise Lyapunov-Razumikhin functions.

Theorem 6. There exists a finite constant $\bar{T} > 0$, such that the switched time delay system (1) with $\Sigma_t \in \tilde{\mathcal{A}}$ is asymptotically stable for any switching rule $q(t) \in S[\tau_D]$, where $\tau_D > 0$ is defined by $\tau_D := \bar{T} + \tau_{max}$.

Proof. Consider an arbitrary switching interval $[t_j, t_{j+1})$ of the piecewise switching signal $q(t) \in S[\tau_D]$, where $q(t) = k_j$, $k_j \in \mathcal{F}$ for $\forall t \in [t_j, t_{j+1})$ and t_j is the j^{th} switching time instant for $j \in \mathbb{Z}^+ \cup \{0\}$ and $t_0 = 0$. The state variable $x_j(t)$ defined on this interval obeys:

$$\Sigma_{k_j} : \begin{cases} \dot{x}_j &= A_{k_j} x_j(t) + \bar{A}_{k_j} x_j(t - \tau_{k_j}),\ t \in [t_j,\, t_{j+1}) \\ x_j(t) &= \phi_j(t),\ \forall t \in [t_j - \tau_{k_j}, t_j]. \end{cases} \tag{17}$$

For the convenience of using "sup", we define $x_j(t_{j+1}) = \lim_{h\to 0^-} x_j(t_{j+1} + h) = x_{j+1}(t_{j+1})$ based on the fact that $x(t)$ is continuous for $t \geq 0$. Therefore $x_j(t)$ is now defined on a compact set $[t_j, t_{j+1}]$. The initial condition $\phi_j(t)$ of Σ_{k_j} is $\phi_j(t) = x_{j-1}(t), t \in [t_j - \tau_{k_j}, t_j]$ for $j \in \mathbb{Z}^+$, which is true because $\tau_D := \bar{T} + \tau_{max} > \tau_{max}$. And $\phi_0(t) = \phi(t), t \in [-\tau_{k_0}, 0]$.

Construct the Lyapunov-Razumikhin function

$$V_{k_j}(x_j, t) = x_j^T(t) P_{k_j} x_j(t),\ t \in [t_j, t_{j+1}] \tag{18}$$

for (17), we have

$$\kappa_{k_j} \|x_j\|^2 \leq V_{k_j}(t, x_j) \leq \bar{\kappa}_{k_j} \|x_j\|^2,\ \forall x_j \in \mathbb{R}^n. \tag{19}$$

A straightforward calculation gives the time derivative of $V_{k_j}(t, x_j(t))$ along the trajectory of (17)

$$\begin{aligned} \dot{V}_{k_j}(t, x_j) &= x_j^T (A_{k_j}^T P_{k_j} + P_{k_j} A_{k_j}) x_j \\ &\quad + 2 x_j^T(t) P_{k_j} \bar{A}_{k_j} x_j(t - \tau_{k_j}), \end{aligned} \tag{20}$$

where

$$\begin{aligned} &2 x_j^T(t) P_{k_j} \bar{A}_{k_j} x_j(t - \tau_{k_j}) \leq \alpha_{k_j} x_j^T(t - \tau_{k_j}) P_{k_j} x_j(t - \tau_{k_j}) \\ &\quad + \alpha_{k_j}^{-1} x_j^T(t) P_{k_j} \bar{A}_{k_j} P_{k_j}^{-1} \bar{A}_{k_j}^T P_{k_j} x_j(t),\ \forall \alpha_{k_j} > 0. \end{aligned}$$

Applying Razumikhin condition with $p(s) = p_{k_j} s$, $p_{k_j} > 1$, we obtain

$$x_j^T(t - \tau_{k_j}) P_{k_j} x_j(t - \tau_{k_j}) \leq p_{k_j} x_j^T(t) P_{k_j} x_j(t) \tag{21}$$

for

$$V_{k_j}(t + \theta, x_j(t + \theta)) < p_{k_j} V_{k_j}(t, x_j(t))\ \ \forall \theta \in [-\tau_{k_j}, 0].$$

Let

$$-S_{k_j} := A_{k_j}^T P_{k_j} + P_{k_j} A_{k_j} + p_{k_j} \alpha_{k_j} P_{k_j} + \alpha_{k_j}^{-1} P_{k_j} \bar{A}_{k_j} P_{k_j}^{-1} \bar{A}_{k_j}^T P_{k_j} \tag{22}$$

we have

$$\dot{V}_{k_j}(t, x_j) \leq -x_j^T(t) S_{k_j} x_j(t). \tag{23}$$

Because $\Sigma_t \in \tilde{\mathcal{A}}$, we have $S_{k_j} > 0$ from Lemma 1. Furthermore we can select $w(s) = w_{k_j} s^2$ in Lemma 3, such that (6) is satisfied, where $w_{k_j} := \sigma_{min}[S_{k_j}] > 0$.

Define

$$\lambda := \max_{i \in \mathcal{F}} \frac{\bar{\kappa}_i}{\kappa_i}, \tag{24}$$

$$\mu := \max_{i \in \mathcal{F}} \frac{\bar{\kappa}_i}{w_i}. \tag{25}$$

For some $0 < \beta < 1$ and $0 < \alpha < 1$, we choose

$$\bar{T} := \frac{\lambda \mu}{\alpha^2} \left\lceil \frac{\lambda - \alpha^2}{\alpha^2 \beta (\bar{p} - 1)} \right\rceil, \tag{26}$$

where $\bar{p} := \min_{i \in \mathcal{F}} \{p_i\} > 1$.

We claim that $\|x_j(t)\| \leq \alpha \delta_j$ for any $t \geq t_j + \bar{T}, t \in [t_j, t_{j+1}]$, where we assume $|\phi_j(t)|_{[t_j - \tau_{k_j}, t_j]} \leq \delta_j$.

To show this fact, we can choose $\bar{\delta}_1 = \delta_j$, $\bar{\delta}_2 = \bar{\delta}_1 \sqrt{\bar{\kappa}_{k_j} / \kappa_{k_j}} \geq \bar{\delta}_1$, and select $\eta = \alpha \bar{\delta}_1$ in Lemma 5. It is straightforward that $0 < \eta < \bar{\delta}_1 \leq \bar{\delta}_2$. Recall (15) and (16), we have

$$V_{k_j}(t, x_j) \leq \kappa_{k_j} \eta^2,\ \text{for}\ \ t \geq t_j + T, \tag{27}$$

where

$$\begin{aligned} T &= \frac{N v(\bar{\delta}_1)}{\gamma} \\ &= \frac{\lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil v(\bar{\delta}_1)}{\inf_{v^{-1}(u(\eta)) \leq s \leq \bar{\delta}_2} w(s)} \\ &= \frac{\bar{\kappa}_{k_j}^2 \lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil}{\alpha^2 w_{k_j} \kappa_{k_j}} \end{aligned} \tag{28}$$

Combining (19) and (27) yields

$$\|x_j(t)\| \leq \alpha \delta_j,\ \text{for}\ \ t \geq t_j + T. \tag{29}$$

Now choosing $a = \beta (p_{k_j} - 1) \kappa_{k_j} \eta^2$, we have

$$T = \frac{\bar{\kappa}_{k_j}^2 \left\lceil \frac{\frac{\bar{\kappa}_{k_j}}{\kappa_{k_j}} - \alpha^2}{\alpha^2 \beta (p_{k_j} - 1)} \right\rceil}{\alpha^2 w_{k_j} \kappa_{k_j}} \leq \bar{T} \tag{30}$$

Therefore

$$|x_j|_{[t_j + \bar{T}, t_{j+1}]} \leq \alpha \delta_j, \tag{31}$$

which is straightforward from (29) and (30).

Notice that $\phi_{j+1}(t) = x_j(t), t \in [t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]$ and recall the fact that $t_{j+1} - t_j > \tau_D = \bar{T} + \tau_{max} \geq \bar{T} + \tau_{k_{j+1}}$, we have

$$\begin{aligned} &|\phi_{j+1}|_{[t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]} = |x_j|_{[t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]} \\ &\leq |x_j|_{[t_j + \bar{T}, t_{j+1}]} \leq \alpha \delta_j := \delta_{j+1} \end{aligned} \tag{32}$$

and δ_0 is defined as $\delta_0 := |\phi|_{[-\tau_{max}, 0]} \geq |\phi|_{[-\tau_{k_0}, 0]}$. Therefore we obtain a convergent sequence $\{\delta_i\}, i = 0, 1, 2, \ldots$, where $\delta_i = \alpha^i \delta_0$.

Meanwhile, (10) implies

$$|x_j|_{[t-\tau_{k_j},t]} \le \sqrt{\frac{\bar{\kappa}_{k_j}}{\kappa_{k_j}}}|x|_{[t_j-\tau_{k_j},t_j]}, \quad \forall t \in [t_j, t_{j+1}]. \quad (33)$$

Hence

$$\begin{aligned}\sup_{t\in[t_j,t_{j+1}]} \|x_j(t)\| &\le \sup_{t\in[t_j,t_{j+1}]} |x_j|_{[t-\tau_{k_j},t]} \\ &\le \sqrt{\lambda}|x|_{[t_j-\tau_{k_j},t_j]} \\ &\le \sqrt{\lambda}\delta_j = \alpha^j\sqrt{\lambda}\delta_0, \qquad (34)\end{aligned}$$

which implies the asymptotic stability of the switched time delay system Σ_t with the switching signal $q(t) \in S_{[\tau_D]}$. This completes the proof. ■

3.2 The Case with Delay Dependent Criterion

In a similar fashion, we can investigate the stability of the switched time delay system Σ_t of (1) under the assumption that $\Sigma_t \in \bar{\mathcal{A}}_d$. Hence each candidate system Σ_i, $i \in \mathcal{F}$ is delay-dependently asymptotically stable satisfying (3).

Theorem 7. There exists a finite constant $\bar{T}_d > 0$, such that the switched time delay system (1) with $\Sigma_t \in \tilde{\mathcal{A}}_d$ is asymptotically stable for any switching rule $q(t) \in S[\tau_D^d]$, where $\tau_D^d > 0$ is defined by $\tau_D^d := \bar{T}_d + 2\tau_{max}$.

Proof. Similar to the proof of Theorem 6, we consider an arbitrary switching interval $[t_j, t_{j+1})$ of the piecewise switching signal $q(t) \in S[\tau_D^d]$, where the state variable $x_j(t)$ defined on this interval obeys (17). The first order model transformation (Hale and Verduyn Lunel, 1993) of (17) results in

$$\begin{aligned}&\dot{x}_j(t) = (A_{k_j} + \bar{A}_{k_j})x_j(t)\\ &-\bar{A}_{k_j}\int_{-\tau_{k_j}}^{0} [A_{k_j}x_j(t+\theta) + \bar{A}_{k_j}x(t+\theta-\tau_{k_j})]d\theta, \quad (35)\end{aligned}$$

where the initial condition $\psi_j(t)$ is defined as $\psi_j(t) = x_{j-1}(t), t \in [t_j - 2\tau_{k_j}, t_j]$ for $j \in \mathbb{Z}^+$, and $\psi_0(t)$ defined by

$$\psi_0(t) = \begin{cases} \phi(t), & t \in [-\tau_{max}, 0] \\ \phi(-\tau_{max}), & t \in [-2\tau_{max}, -\tau_{max}) \end{cases}$$

By using the Lyapunov-Razumikhin function (18), we obtain the time derivative of $V_{k_j}(t, x_j(t))$ along the trajectory of (35)

$$\begin{aligned}&\dot{V}_{k_j}(t,x_j) = x_j^T(t)[P_{k_j}(A_{k_j}+\bar{A}_{k_j}) + (A_{k_j}+\bar{A}_{k_j})^T P_{k_j}]x_j(t)\\ &-\int_{-\tau_{k_j}}^{0}[2x_j^T(t)P_{k_j}\bar{A}_{k_j}(A_{k_j}x_j(t+\theta)+\bar{A}_{k_j}x_j(t+\theta-\tau_{k_j})]d\theta.\end{aligned}$$

Assume $V_{k_j}(t+\theta, x_j(t+\theta)) < p(V_{k_j}(t, x_j(t)))$ for $\forall\theta \in [-2\tau_{k_j}, 0]$, where $p(s) = p_{k_j}s$, $p_{k_j} > 1$, we have (Dugard and Verriest, 1998; Kharitonov, 1999)

$$\dot{V}_{k_j}(t,x_j) \le -x_j^T(t)S_{k_j}^d x_j(t), \quad (36)$$

where

$$\begin{aligned}S_{k_j}^d := &-\{P_{k_j}(A_{k_j}+\bar{A}_{k_j}) + (A_{k_j}+\bar{A}_{k_j})^T P_{k_j}\\ &+\tau_{k_j}[\alpha_{k_j}^{-1}P_{k_j}\bar{A}_{k_j}A_{k_j}P_{k_j}^{-1}\bar{A}_{k_j}^T A_{k_j}^T P_{k_j}\\ &+\beta_i^{-1}P_{k_j}(\bar{A}_{k_j})^2 P_{k_j}^{-1}(\bar{A}_{k_j}^T)^2 P_{k_j}\\ &+p_{k_j}(\alpha_{k_j}+\beta_{k_j})P_{k_j}]\}. \qquad (37)\end{aligned}$$

Because $\Sigma_t \in \tilde{\mathcal{A}}_d$, we have $S_{k_j}^d > 0$ from Lemma 2. Therefore we can select $w(s) = w_{k_j}^d s^2$ in Lemma 3, such that (6) holds, where $w_{k_j}^d := \sigma_{min}[S_{k_j}^d] > 0$. We choose

$$\bar{T}_d := \frac{\lambda\mu_d}{\alpha^2}\lceil\frac{\lambda-\alpha^2}{\alpha^2\beta(\bar{p}-1)}\rceil, \quad (38)$$

where

$$\mu_d := \max_{i\in\mathcal{F}}\frac{\bar{\kappa}_i}{w_i^d} \quad (39)$$

and the other parameters are the same as those defined in the proof of Theorem 6.

We can obviously apply analogues of Theorem 6 to obtain the following inequality:

$$\sup_{t\in[t_j,t_{j+1}]}\|x_j(t)\| \le \sqrt{\lambda}\delta_j^d, \quad (40)$$

where $|\psi_j(t)|_{[t_j-2\tau_{k_j},t_j]} \le \delta_j^d$, and $\delta_{j+1}^d = \alpha\delta_j^d$. Note that δ_0^d can be selected as

$$\delta_0^d := |\psi|_{[-2\tau_{max},0]} = |\phi|_{[-\tau_{max},0]} = \delta_0.$$

It is clear that $|\psi|_{[-2\tau_{k_0},0]} \le \delta_0^d$, which further implies $\delta_j^d = \delta_j$, $j \in \mathbb{Z}^+ \cup \{0\}$. The upper bound of the state variable $x(t)$ of the switched time delay systems Σ_t is bounded by a decreasing sequence $\{\delta_i\}, i = 0,1,2,\ldots$ converging to zero, which implies the asymptotic stability and proves this theorem. ■

The dwell time based stability analysis proposed in this paper is general in the sense that it can be used for other stability results based on Razumikhin theorems as long as the correspondingly Lyapunov functions are in quadratic forms. Particularly, Theorem 7 can be extended easily to the case where Σ_t has time-varying time delays and parameter uncertainties, which has important applications such as TCP congestion control of computer networks (Kelly, 2001).

4. CONSERVATISM ANALYSIS

The dwell time based stability results has been obtained for linear delay free switched systems (Morse, 1996). It is interesting to compare the conservatism of the results presented in this paper with those for delay free systems.

In fact, one extreme case of the switched system Σ_t is $\tau_i = 0$ and $\bar{A}_i = 0$ for $i \in \mathcal{A}$, which corresponds to the delay free scenario. For each candidate system $\dot{x} = A_i x$, a sufficient and necessary condition to guarantee asymptotic stability is $\exists P_i = P_i^T > 0$, such that $Q_i :=$

$-(A_i^T P_i + P_i A_i) > 0$. Correspondingly a dwell time based stability for such switched delay free system is $q(t) \in S_{[\tilde{\tau}_D]}$, where

$$\tilde{\tau}_D = \tilde{\mu} \ln \lambda, \tag{41}$$

where λ is defined by (25) and

$$\mu := \max_{i \in \mathcal{F}} \frac{\overline{\kappa}_i}{\tilde{w}_i}, \tag{42}$$

where $\tilde{w}_i := \sigma_{min}[Q_i] > 0$.

On the other hand in our case, for $\tau_i = 0$ and $\overline{A}_i = 0$, we observe that

$$\lim_{\alpha_i \to 0^+} S_i = \lim_{\alpha_i, \beta_i \to 0^+} S_i^d = Q_i, \quad i \in \mathcal{F} \tag{43}$$

from (22) and (37), which indicates $\mu = \mu_d = \tilde{\mu}$ by (25), (39), and (42). Accordingly we can select $p_i > 1$, $i \in \mathcal{F}$ sufficiently large such that $\lceil \frac{\lambda - \alpha^2}{\alpha^2 \beta(\overline{p}-1)} \rceil = 1$ in (26) and (38), and obtain

$$\tau_D = \overline{T} = \frac{\lambda \mu}{\alpha^2} = \frac{\lambda \mu_d}{\alpha^2} = \overline{T}_d = \tau_D^d. \tag{44}$$

Therefore

$$\tau_D = \tau_D^d = \frac{\lambda \tilde{\mu}}{\alpha^2} > \lambda \tilde{\mu} > \tilde{\mu} \ln \lambda = \tilde{\tau}_D. \tag{45}$$

The dwell times derived in this paper for switched time delay systems are more conservative than that for switched delay free systems, although they are tight results based on Razumikhin theorems. It is due to the fact that asymptotic stability for linear delay free systems is equivalent to exponential stability. For time delay systems, the sufficient stability conditions based on Lyapunov-Razumikhin theorem do not guarantee exponential stability, however. As a matter of fact, the exponential estimates for time delay systems require additional assumptions besides asymptotic stability (Kharitonov and Hinrichsen, 2004).

5. CONCLUDING REMARKS

We provided stability analysis for switched linear systems with time delays, where each candidate system is assumed to be delay-independently or delay-dependently asymptotically stable. We showed the existence of a dwell time of the switching signal, such that the switched time delay system is asymptotically stable independent of the trajectory. The results are compared with the dwell time conditions for switched delay free systems. An extension of this work is to consider switching control methods for time varying time delay systems.

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