SUB OPTIMAL CONTROL OF SWITCHED NONLINEAR SYSTEMS UNDER LOCATION AND SWITCHING CONSTRAINTS [1]

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Abstract: This paper considers an optimal control problem for switched nonlinear systems. The objective is to minimize an associated cost functional, by finding an appropriate continuous control input and location switching strategy. We propose an extension of an algorithm based on strong variations to handle constraints on both locations and switching instants. Numerical experiments testify the viability and the tractability of such a scheme.

Keywords: Switched systems, Minimum principle, Optimal algorithms, Suboptimal control.

1. INTRODUCTION

Switched systems are one of the most important classes of mixed continuous discrete dynamical systems known as *hybrid systems*. This class includes applications from power systems, automotive control, consumer electronics, petrochemical industry and networked control systems. The theoretical foundations for such systems are now enough mature as evidenced by the published literature. The lack is the availability of general computational schemes, mainly computational optimal control methods. Although a great effort has been dedicated to this subject in the last decade, to the best knowledge of the authors no algorithm can handle all the features of hybrid systems. However, some promising design methodologies appear for the switched systems class. These include the two stages optimization approach initiated independently in (Cassandras *et al.*, 2001) and (Xu and Antsaklis, 2002). At the first stage the aim is to find an optimal continuous control and the switching instants while the second allows the variation of the sequences of active locations and the number of switches. Several works have been done in this direction (Xu and Antsaklis, 2003) and (Gokbayrak and Cassandras, 2000), see also (Egerstedt *et al.*, 2003) where a simple gradient formula is derived to update the switching instants.

With the advent of new versions of the Minimum principle due in part to (Sussmann, 1999; Piccoli, 1999) and recently (Riedinger *et al.*, 2003), a computational framework is set up in (Shaikh and Caines, 2003*b*) for fixed switching schedules. Some guidelines are also given to tackle the not *à* priori known switching schedule using a combinatorial search technique. This is further extended in (Shaikh and Caines, 2003*a*) where extended partitions of the time-state space parallel to those in (Bemporad *et al.*, 2002) are used.

Another important methodology is the one based on the Mixed Logical Dynamical (*MLD*) modelling framework initiated in (Bemporad and

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Morari, 1999). This framework handles complex systems where constrained dynamics and logics are interacting. Such systems include a large class of hybrid systems where the continuous dynamics can be expressed or at least finely approximated by Piece Wise Linear functions. A mixed integer predictive controller is developed to stabilize MLD systems on desired reference trajectories. The resulting on line optimizations are solved through mixed integer quadratic programming.

The problem is also investigated in the context of singular arcs in (Bengea and DeCarlo, 2003) without any assumption neither on the number of switches nor on the sequence of active locations. The authors express some sufficient conditions for the existence of an optimal solution by making use of an embedding technique similar to the one used in (Alamir and Attia, 2004) where the problem of optimal control of switched systems is solved using a strong variation approach. This alleviates the combinatorics associated with hierarchical algorithms like those developed in e.g, (Gokbayrak and Cassandras, 2000; Xu and Antsaklis, 2002).

We propose here an extension of the algorithm developed in (Alamir and Attia, 2004) that deals with the case of locations constraints, this is actually the case in many systems for instance, in a geared car, the switch from gear 1 to 4 is not allowed. Moreover, in order to further limit the number of switches, constraints are introduced on the switching candidates instants. The algorithm complexity is shown to be linear in the number of locations which makes it very appealing from a computational view point.

This paper is organized as follows : section 2 presents basic definitions of switched systems. Section 3 is devoted to the algorithm formulation and complexity studies. In Section 4 some validating simulations are reported. Finally, some conclusions and future orientations are given in section 5.

2. BASIC DEFINITIONS

Definition 1. **Switched system**

A switched system is a tuple $\mathcal{S} = (\mathcal{D}, \mathcal{F})$ where

- $\mathcal{D} = (\mathcal{Q}, \mathcal{E})$ is a directed graph representing the discrete structure of the system. The node or location set $\mathcal{Q} = \{1, 2, \ldots, Q\}$ is the set of indices for the configurations. The directed edge set $\mathcal{E}$ is a subset of the cartesian product $\mathcal{Q} \times \mathcal{Q}$ which contains all valid controlled transitions represented by the elements (events) of the type $(q_1,\ q_2)$ meaning that a switching from location q_1 to location q_2 is allowed.
- $\mathcal{F} = \{f_p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, p \in \mathcal{Q}\}$ is a set of vector fields, so that the system wherein location p is described by the following dynamics $\dot{x} = f_p(x, u)$ where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathcal{U}_p \subset \mathbb{R}^m$ is the control input, $\mathcal{U}_p$ is some set of admissible controls (generally a compact set). The vector fields in $\mathcal{F}$ are twice continuously differentiable w.r.t the states x.

For a switched system $\mathcal{S}$, the control inputs consist of a continuous input u and a switching strategy q.

Definition 2. For a switched system $\mathcal{S}$, an admissible switching strategy or profile $q(\cdot)$ with $K \in \mathbb{N}$ switches is a piecewise constant function defined for all $t \in [t_0,\ T)$ as

$$q(t) = \begin{cases} q_0 & t_0 \le t < t_1 \\ q_1 & t_1 \le t < t_2 \\ \vdots & \vdots \\ q_K & t_K \le t < t_{K+1} \end{cases} \tag{1}$$

where $t_0 < t_1 < t_2 \ldots < t_K < t_{K+1}$ are the switching instants with $t_{K+1} = T$, and $(q_k,\ q_{k+1}) \in \mathcal{E}$ for all $k \in \mathcal{K} := \{0, 1, 2, \ldots, K\}$. Define also $\Sigma_{[t_0,\ T]} \triangleq \{q(\cdot)\ in\ [t_0,\ T)\}$.

For a switched system to be well posed, pathological phenomenon like *Zeno*, i.e, accumulation of location switchings at finite time, have to be excluded. In this paper, time discretization is used in conjunction with the following constraints defined by one of the sets Σ_1 and Σ_2

$$\Sigma_1 = \{q(\cdot) \in \Sigma_{[t_0,T]} | \forall k \in \mathcal{K}, \quad \exists j \in \mathbb{Z}^+ : t_{k+1} - t_k = j \times dt_{min} > 0\} \tag{2}$$

$$\Sigma_2 = \{q(\cdot) \in \Sigma_{[t_0,T]} | \forall k \in \mathcal{K} \quad : t_{k+1} - t_k \ge dt_{min}\} \tag{3}$$

where

$$dt_{min} = N \times h \tag{4}$$

with $N \in \mathbb{Z}^+$ and h is a decision sampling period.

Remark 1. The sets defined in (2) and (3) allow only a finite number of switches in any finite time interval. For (2) the decision to stay further in a location or to leave it is made possible every dt_{min} unlike (3) where some location dwell time is specified.

Definition 3. For a switched system $\mathcal{S}$, an admissible control set $\bar{\mathcal{U}}_q$ associated to an admissible switching profile $q(\cdot)$ is defined as

$$\bar{\mathcal{U}}_q = \left\{u(\cdot) | \forall t \in [t_0, T), u(t) \in \mathcal{U}_{q(t)}\right\} \tag{5}$$

Problem 1. Consider a switched system $\mathcal{S} = (\mathcal{D}, \mathcal{F})$. Given a fixed time interval $[t_0,\ T]$, find a switching sequence $q(\cdot) \in \Sigma_1$ (or $q(\cdot) \in \Sigma_2$) and an associated control input $u \in \bar{\mathcal{U}}_q$ such that the cost functional

$$J = \int_{t_0}^{T} L(x(t), u(t), q(t))dt \quad (6)$$

is minimized under $q(t_0) = q_0$ and $x(t_0) = x_0$.

Let us also define the following family of indexed sets $\{\mathcal{A}_p\}_{p\in\mathcal{Q}}$ as

$$\mathcal{A}_p = \{s \in \mathcal{Q} : (p,\ s) \in \mathcal{E}\} \quad (7)$$

a set $\mathcal{A}_p$ represents a one step ahead *reachable* locations from the actual one p.

3. ALGORITHM FORMULATION

If no constraints are present neither on locations (meaning that $\mathcal{E} = \mathcal{Q} \times \mathcal{Q}$) nor on the switching instants, the Pontryagin Minimum principle (Pontryagin *et al.*, 1962) still holds for switched systems (with controlled switches).
This can be seen by taking binary valued variables $\alpha_p \in \{0, 1\}$ each one corresponding to a location (where well posedness properties impose $\sum_{p=1}^{Q} \alpha_p = 1$ at each instant), as a part of the control vector and writing down the dynamics of the switched system as a unique system $\sum_{p=1}^{Q} \alpha_p f_p(x, u)$. This embedding makes possible the use of algorithms for partially non convex control sets to solve optimal control problems, see (Alamir and Attia, 2004) where this is exploited.

Now if one considers constraints on locations ($\mathcal{E} \subset \mathcal{Q} \times \mathcal{Q}$). The α_p's time evolution will involve a memory map, i.e, the set of admissible controls cannot be defined independently from information about the corresponding switching strategy, leading to a violation of the hypotheses upon which the Minimum principle is built. Until one considers a fixed switching profile, see, e.g (Sussmann, 1999; Piccoli, 1999) where these arguments are explored. Here the Minimum principle is used only to orient the search for an optimal solution, and thus the trajectories obtained are by essence suboptimal. Let us define the Hamiltonian function associated to a switching profile $q(\cdot)$ as

$$H_q(x, u, \lambda) = L_q(x, u) + \lambda' f_q(x, u) \quad (8)$$

and the Hamiltonian system as

$$\dot{x} = \frac{\partial H_q}{\partial \lambda}, \quad \dot{\lambda} = -\frac{\partial H_q}{\partial x} \quad (9)$$

The pointwise minimization in the Minimum principle (Pontryagin *et al.*, 1962) and for the case of full accessibility of the locations is equivalent to activating the location with the least value of the Hamiltonian at each instant. This is exploited in the following algorithm to extract a suboptimal solution to the problem 1 under consideration.

Let N_d be a positive integer ($N_d > 1$), $h = \frac{T-t_0}{N_d-1}$ be a sampling period and define the sampling instants $t_{i+1} = t_i + h$. Define also piecewise constant controls as $u(t_i + \tau) = \bar{u}(i)$ and $q(t_i + \tau) = \bar{q}(i)$ for all $i \in \{1, 2, \dots, N_d\}$ and $\tau \in [0, h)$. The finite dimensional approximations $\bar{x}$ and $\bar{\lambda}$ are defined by integrating (9) using e.g, a second order Runge-Kutta method, this is shortly written as (with $\bar{x}(1) = x_0$, $\bar{\lambda}(N_d) = 0$ and $\bar{q}(1) = q_0$)

$$\bar{x}(i+1) = RK_2^F(\bar{x}(i), \bar{u}(i), \bar{q}(i)) \quad (10)$$

$$\bar{\lambda}(i-1) = RK_2^B(\bar{\lambda}(i), \bar{x}(i), \bar{u}(i), \bar{x}(i-1), \bar{u}(i-1), \bar{q}(i)) \quad (11)$$

where the superscripts F and B stand respectively for forward and backward. The following approximation of the performance index is used

$$\bar{J} = \bar{J}(\bar{u}, \bar{q}) = h \sum_{i=1}^{N_d-1} L(\bar{x}(i), \bar{u}(i), \bar{q}(i)) \quad (12)$$

Before we proceed further, let us define the following discrete metric $d_d : \mathcal{Q} \times \mathcal{Q} \to \{0, 1\}$ by

$$d_d(q_1, q_2) = \begin{cases} 0 \ if \ q_1 = q_2 \\ 1 \ if \ q_1 \neq q_2 \end{cases} \quad (13)$$

Consider now the following algorithm pseudo code.

ALGORITHM PSEUDO CODE

Step 0 : Fix some small ϵ_u, ϵ_J, some integer l_{max} and the reals $d\mu$, $\gamma > 1$, μ^0, choose some initial guess $\bar{q}^0 \in \Sigma_1$ (or $\bar{q}^0 \in \Sigma_2$) and $\bar{u}^0 \in \bar{\mathcal{U}}_{\bar{q}^0}$ under the initial condition q_0
Step 1 : Compute $\bar{x}^0$ solution of (10) with $\bar{u} = \bar{u}^0$, $\bar{q} = \bar{q}^0$ and let $l := 1$,
Step 2 : Compute $\bar{\lambda}^{l-1}$ solution of (11) with $\bar{u}^{l-1}$, $\bar{x}^{l-1}$ and $\bar{q} = \bar{q}^{l-1}$
Step 3 :

for $i = 1$ **to** N_d **do begin**

If ($i \times h$ is a candidate instant) **then**

- Compute $(\bar{q}^l(i), \bar{u}^l(i))$ and $\bar{x}^l(i)$ such that $\bar{x}^l(i)$ is solution of (10) with $\bar{u}(i) = \bar{u}^l(i)$ and $\bar{q}(i) = \bar{q}^l(i)$ such that

$$(\bar{q}^l(i), \bar{u}^l(i)) := \arg \min_{s\in\mathcal{A}_{\bar{q}^l(i-1)}} \Bigg\{ \min_{u\in\mathcal{U}_s} \Big[H_s(\bar{x}^l(i), u, \bar{\lambda}^{l-1}(i)) + \mu^{l-1}\|u - \bar{u}^{l-1}(i)\|^2 + \mu^{l-1} d_d(\bar{q}^{l-1}(i), s) \Big] \Bigg\} \quad (14)$$

else

- Compute $\bar{u}^l(i)$ and $\bar{x}^l(i)$ such that $\bar{x}^l(i)$ is solution of (10), with $u(i) = \bar{u}^l(i)$ and $\bar{q}(i) = \bar{q}^l(i)$ such that

$$\bar{q}^l(i) := \bar{q}^l(i-1), \quad (\bar{q}^l(0) = q_0) \tag{15}$$

$$\bar{u}^l(i) := \arg \min_{u \in \mathcal{U}_{\bar{q}^l(i)}} \left[H_{\bar{q}^l(i)}(\bar{x}^l(i), u, \bar{\lambda}^{l-1}(i)) + \mu^{l-1} \| u - \bar{u}^{l-1}(i) \|^2 \right] \tag{16}$$

end if
end

Step 4 : **If** $\left(\bar{J}(\bar{u}^l, \bar{q}^l) > \bar{J}(\bar{u}^{l-1}, \bar{q}^{l-1}) - \epsilon_J\right)$ and $\left(\|\bar{u}^l - \bar{u}^{l-1}\| + \max(d(\bar{q}^l, \bar{q}^{l-1})) > \epsilon_u\right)$ **then** let $\mu^{l-1} = \max(\mu^{l-1} + d\mu, \gamma \times \mu^{l-1}))$, and return to **Step 3**,
Step 5 : $\mu^{l-1} = \max(0, \min(\mu^{l-1} - d\mu, \mu^{l-1}/\gamma))$
Step 6 : **If** $(l > l_{max})$ **Then** Stop **else** let $l := l + 1$ and return to **Step 2**.

✓ The algorithm is essentially composed of two parts, the second part of the if bloc in step 3 permits only the update of the continuous input for a frozen location so that at this stage we solve a conventional optimal control problem (a succession of nonlinear programming problems), while the first uses a unified updating scheme for both the continuous input and the switching profile, this is in fact possible only for time instants that satisfy the constraints (2) or (3) leading to a limitation of the number of switching which is in general imposed by the user or by the application under consideration. More details on the internal structure of the algorithm can be found in (Alamir and Attia, 2004).

✓ Like any numerical scheme, the algorithm parameters are chosen on a trial-error basis. For a large number of problems, the following values have shown good convergence properties $\mu_0 = 10$, $\gamma = 1.5$ and $d\mu = 0.5$.

3.1 Algorithm complexity

Once the method of minimizing the penalized Hamiltonian has been chosen (the optimization stage of step 3), the worst case complexity (because $card(\mathcal{A}_p)$ is location dependent and $card(\mathcal{A}_p) \le card(\mathcal{Q})$) is given by

$$\Big[(\eta - 1)\, card(\mathcal{Q}) + \eta(N-1) + 1 \Big] \times \mathcal{C}_{NLP}(m) + 2 \times \mathcal{C}_{RK}(n) \tag{17}$$

where $\eta = \frac{N_d}{N}$, $N = \frac{dt_{min}}{h}$ defined in (4), $\mathcal{C}_{NLP}(m)$ denotes the complexity of the nonlinear programming algorithm and $\mathcal{C}_{RK}(n)$ the complexity of the Runge-Kutta method. Two extreme cases can be distinguished namely $\eta = 1$ and $\eta = N_d$. The first case corresponds to the one where only a location is fired and thus corresponds to a complexity of solving a conventional optimal control problem, this can also seen by putting $card(\mathcal{Q}) = 1$ in (17), while the second corresponds to the case reported in (Alamir and Attia, 2004) for a frozen initial location. This upper bounds holds both for Σ_1 and Σ_2 switching types strategies. There are no exponential terms in n, Q or N_d.

4. COMPUTATIONAL RESULTS

The following example is taken from (Xu and Antsaklis, 2002). The system dynamics are

$$\text{Location 1}: \begin{cases} \dot{x}_1 = -x_1 + 2x_1 u \\ \dot{x}_2 = x_2 + x_2 u \end{cases} \tag{18}$$

$$\text{Location 2}: \begin{cases} \dot{x}_1 = x_1 - 3x_1 u \\ \dot{x}_2 = 2x_2 - 2x_2 u \end{cases} \tag{19}$$

$$\text{Location 3}: \begin{cases} \dot{x}_1 = 2x_1 + x_1 u \\ \dot{x}_2 = -x_2 + 3x_2 u \end{cases} \tag{20}$$

with the following cost functional

$$J = \frac{1}{2}(x_1(T) - 2)^2 + \frac{1}{2}(x_2(T) - 2)^2 + \frac{1}{2} \int_0^T \left[(x_1(\tau) - 2)^2 + (x_2(\tau) - 2)^2 + u^2(\tau) \right] d\tau \tag{21}$$

$x(0) = (1,1)'$, $T = 3$ *sec* and $|u| \le 2$. The first simulations concern the case where an initial condition q_0 is imposed on the location. The algorithm parameters ;when not specified elsewhere; are the same in all the simulations and are taken as $\mu^0 = 10$, $u^0(\cdot) = -1$, $d\mu = 0.5$ and $\gamma = 1.5$. In table 1 are indicated the performance indices achieved for different initial locations.

$\mathbf{q_0}$	$\bar{\mathbf{J}}$	**Execution times**
1	0.3541	2.63 sec
2	0.6212	3.37 sec
3	0.3873	4.04 sec

Table 1. Performance index achieved for different initial locations

In figure 2 is shown the convergence history for the case $q_0 = 1$, roughly speaking the key point is that if the algorithm internal variable μ^l converges to 0 then one can show that the continuous control profile at the last iteration satisfies the Minimum principle (Alamir and Attia, 2004) (for a final fixed switching strategy). Consider now the case where the need is to identify critical locations. This can be the case when the designer has to choose whether to include components into a system or not, or the case where we have a finite set of control input and we have to choose the components of this vector that contributes the most to a performance index, e.g, networked

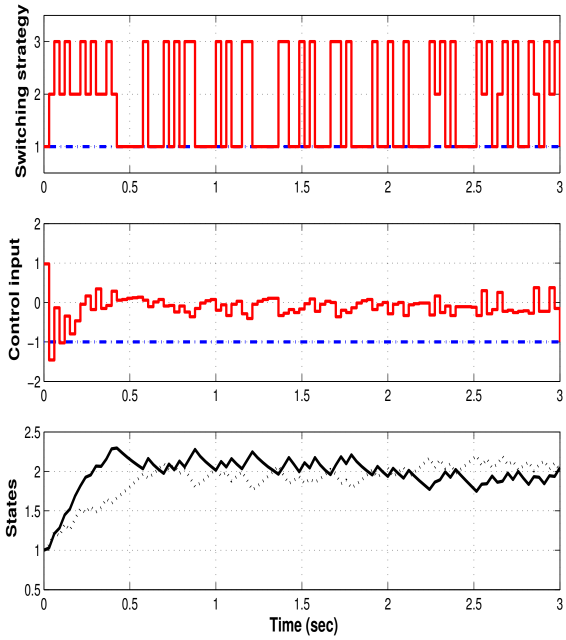

Fig. 1. Switching strategy, continuous input and the states, Initial condition : Location 1

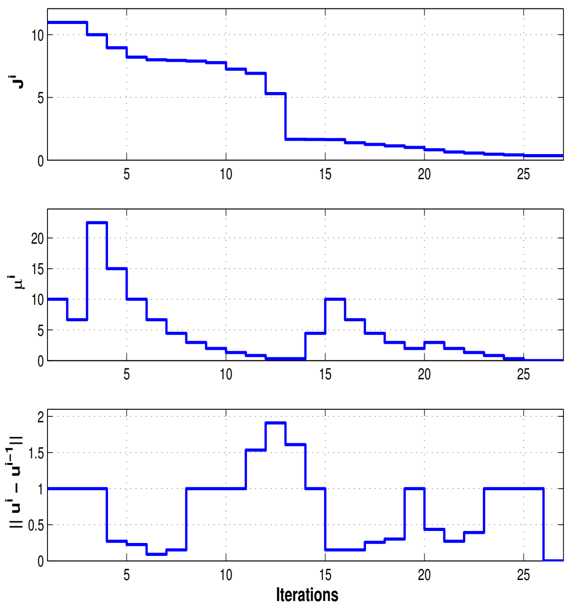

Fig. 2. Performance index, μ^i and $\|u^i - u^{i-1}\|$, Initial condition : Location 1

control systems.
As evidenced in figure 1 the jump from location 1 to location 3 dominates the switching behavior. In the next simulation we ban this switch by properly adjusting the set $\mathcal{E}$ and taking μ^0 and u^0 as $\mu^0 = 10000$ and $u^0(\cdot) = 2$, the results are shown in figure 3. Here we see clearly that in order to maintain acceptable performance more control effort u is used, the performance index achieved is $J_{1-3} = 0.8152$ in 98 iterations is still acceptable.

Next we investigate the case where we simply eliminate locations. The results are given in table 2. Recall that the three last problems are conven-

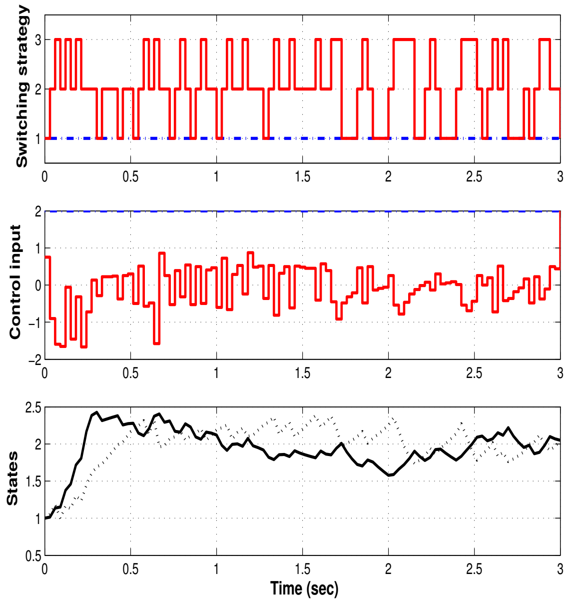

Fig. 3. Switching strategy, continuous input and the states, with jump 1-3 prohibited

Eliminated location	$\mathbf{u^0}(\cdot)$	μ^0	$\bar{\mathbf{J}}$	l
none	-0.5	10	0.2366	26
1	-1.5	10	2.6140	16
2	2.0	100	0.5676	35
3	1.0	10	7.69	17
$1 \wedge 2$	-2.0	10	12.45	14
$1 \wedge 3$	0.0	10	7.539	23
$2 \wedge 3$	1.0	200	8.049	26

Table 2. Identification of critical locations

tional optimal ones since only a location is fired along the time window. The problem corresponding to the first row is solved in (Xu and Antsaklis, 2002) by allowing only two switches (because of the combinatorics involved), the performance index achieved is more than 15 times the one reported here. From table 2, one concludes that the location 3 is somehow the most critical node in the system, this stems from the fact that the performance index varies from 0.3541 to 7.69, a ratio of approximately 22, see figure 4. Finally, the number of switches can be implicitly bounded by constraining the system to switch every $N \times h$ with N properly chosen. Suppose that the number of allowed switches is $N_{switching}$ so that N can be computed as $N = \frac{N_d}{N_{switching}}$. Figure 5 gives some indications on the evolution of the performance index versus the number of switches. For the example under study, the performance remains almost constant when the number of switches exceeds 25.

5. CONCLUSIONS

In this paper, an extension of an algorithm based on strong variations is developed. Simulations results are reported showing the efficiency of such

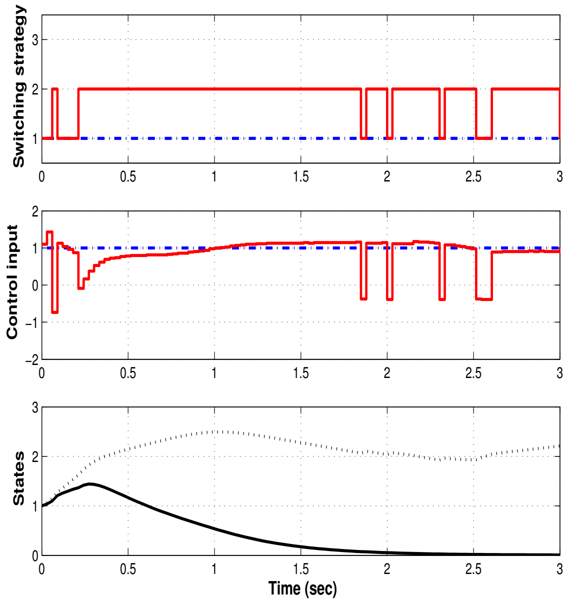

Fig. 4. Switching strategy, continuous input and the states, with location 3 eliminated

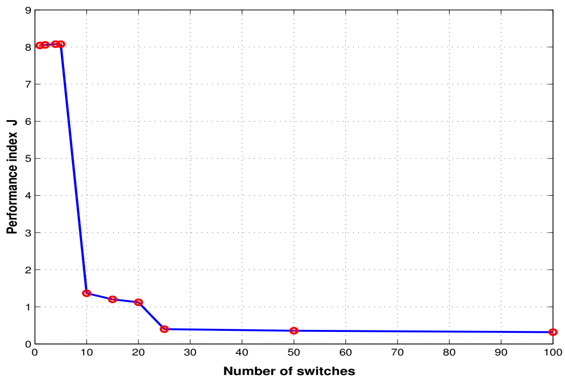

Fig. 5. Performance index versus the number of switches

algorithms to solve control problems for switched systems. The complexity is shown to be reasonable and state dimension independent except for the integration scheme.

Future work concerns the comparison of this approach to other dynamic programming based methods, where the aim will be to measure the performance loss against the complexity and the computational burden associated to a dynamic programming approach.

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