A CLOSED-FORM OPTIMAL CONTROL FOR LINEAR SYSTEMS WITH MULTIPLE STATE DELAYS

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Abstract This paper presents the optimal regulator for a linear system with multiple state delays and a quadratic criterion. The optimal regulator equations are obtained using the maximum principle. Performance of the obtained optimal regulator is verified in the illustrative example against the best linear regulator available for linear systems without delays. Simulation graphs demonstrating better performance of the obtained optimal regulator with respect to the criterion value are included.

Keywords: Optimal control, linear time-delay system

1. INTRODUCTION

Although the optimal control (regulator) problem for linear system states was solved in 1960s (see (Kwakernaak and Sivan, 1972; Fleming and Rishel, 1975)), the optimal control problem for linear systems with delays is still open, depending on the delay type, specific system equations, criterion, etc. Various linear-quadratic optimal control problems have been studied in (Eller *et al.*, 1969; Alekal *et al.*, 1971; Delfour, 1986; Uchida *et al.*, 1988). A detailed comment on the up-to-date state of the control theory for time-delay systems is given in (Basin *et al.*, 2003; Basin *et al.*, 2004). Comprehensive reviews of theory and algorithms for time delay systems can be found in (Malek-Zavarei and Jashmidi, 1987; Kolmanovskii and Shaikhet, 1996; Kolmanovskii and Myshkis, 1999; Dion *et al.*, 1999; Mahmoud, 2000; Niculescu, 2001; Boukas and Liu, 2002; Gu and Niculescu, 2003; Richard, 2003).

This paper concentrates on the solution of the optimal control problem for a linear system with multiple state delays and a quadratic criterion. Using the maximum principle (Pontryagin *et al.*, 1962; Kharatashvili, 1967), the solution to the stated optimal control problem is obtained in a closed form, i.e., it is represented as a linear feedback control law, whose gain matrix satisfies an ordinary differential (quasi-Riccati) equation, which does not contain time-advanced arguments and does not depend on the state variables. The obtained optimal regulator makes an advance with respect to general optimality results for time delay systems (such as given in (Alekal *et al.*, 1971; Malek-Zavarei and Jashmidi, 1987; Kolmanovskii and Shaikhet, 1996; Kolmanovskii and Myshkis, 1999)), since it is realizable using only two equations: a delay-differential equation for the state and an ordinary differential one for the gain matrix. Taking into account that the state space of a delayed system is infinite-dimensional (Malek-Zavarei and Jashmidi, 1987), this seems to be a significant advantage. It should be also noted that the optimal control is indeed obtained as a linear current-time feedback control, although in some papers (see, for example, (Alekal *et al.*, 1971)) it was derived as an integral of the system state over the delay interval. A comment on this result is given in Remark at the end of Section 3.

Finally, performance of the obtained optimal control for a linear system with multiple state delays and a quadratic criterion is verified in the illustrative example against the best linear regulator available for linear systems without delays. The simulation results show a definitive (about three times) difference in the values

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of the cost function in favor of the obtained optimal regulator.

The paper is organized as follows. Section 2 states the optimal control problem for a linear system with multiple state delays. The solution to the optimal control problem is given in Section 3. The proof of the obtained results, based on the maximum principle (Pontryagin *et al.*, 1962; Kharatashvili, 1967), is given in Appendix. Section 4 presents an example illustrating the efficiency of control provided by the obtained optimal regulator for linear systems with multiple state delays against the best linear regulator available for systems without delays. Simulation graphs demonstrating better performance of the obtained optimal regulator with respect to the cost function value are included.

2. OPTIMAL CONTROL PROBLEM STATEMENT

Consider a linear system with multiple time delays in the state

$$\dot{x}(t)=\sum_{i=1}^{p}a_i(t)x(t-h_i)+B(t)u(t), \qquad (1)$$

with the initial condition $x(s)=\phi(s)$, $s\in[t_0-h,t_0]$, $h=\max(h_1,\ldots,h_p)$, $h_1,\ldots,h_p>0$ are positive time delays. Here, $x(\cdot)\in C([t_0-h,t_0];R^n)$ is the system state, $u(t)\in R^m$ is the control variable, and $\phi(s)$ is a continuous function given in the interval $[t_0-h,t_0]$. The matrix coefficients $a_i(t)$ and $B(t)$ are considered bounded measurable functions of time. Note that the state of a delayed system (1) is infinite-dimensional (Malek-Zavarei and Jashmidi, 1987). Existence of the unique forward solution of the equation (1) is thus assured by the Carathéodory theorem (see, for example, (Filippov, 1988)), and existence of the unique backward solution follows from analyticity of the right-hand side functional with respect to the system state (see (Hale and Verduyn-Lunel, 1993; Richard, 2003))

The quadratic cost function to be minimized is defined as follows:

$$J=\frac{1}{2}[x(T)]^T\psi[x(T)]+$$

$$\frac{1}{2}\int_{t_0}^{T}u^T(s)R(s)u(s)ds+\frac{1}{2}\int_{t_0}^{T}x^T(s)L(s)x(s)ds, \qquad (2)$$

where R is positive and ψ, L are nonnegative definite symmetric matrices, and $T>t_0$ is a certain time moment.

The optimal control problem is to find the control $u^*(t)$, $t\in[t_0,T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t\in[t_0,T]$, generated upon substituting $u^*(t)$ into the state equation (1). The solution to the stated optimal control problem is given in the next section and then proved using the maximum principle (Pontryagin *et al.*, 1962; Kharatashvili, 1967) in Appendix.

3. OPTIMAL CONTROL PROBLEM SOLUTION

The solution to the optimal control problem for the linear system with state delay (1) and the quadratic criterion (2) is given as follows. The optimal control law is given by

$$u^*(t)=(R(t))^{-1}B^T(t)Q(t)x(t), \qquad (3)$$

where the matrix function $Q(t)$ satisfies the matrix equation

$$\dot{Q}(t)=L(t)-Q(t)(\sum_{i=1}^{p}N_i(t)a_i(t))- \qquad (4)$$

$$(\sum_{i=1}^{p}a_i^T(t)N_i^T(t))Q(t)-Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T)=-\psi$. The auxiliary matrices $N_i(t)$ are defined as $N_i(t)=(\partial x(t-h_i)/\partial x(t))$, whose value is equal to zero, $N_i(t)=0$, if $t\in[t_0,t_0+h_i)$, and are determined as $N_i(t)=\Phi^{-1}(t,t-h_i)=\Phi(t-h_i,t)=$ $\exp(-\int_{t-h_i}^{t}B(s)R^{-1}(s)B^T(s)Q(s)ds)$, if $t\geq t_0+h_i$, where $\Phi(t,\tau)$ satisfies the matrix equation

$$\frac{d\Phi(t,\tau)}{dt}=B(t)R^{-1}(t)B^T(t)Q(t)\Phi(t,\tau),$$

with the initial condition $\Phi(t,t)=I$, and I is the identity matrix.

Upon substituting the optimal control (3) into the state equation (1), the optimally controlled state equation is obtained

$$\dot{x}(t)=\sum_{i=1}^{p}a_i(t)x(t-h_i)+B(t)R^{-1}(t)B^T(t)Q(t)x(t), \qquad (5)$$

with the initial condition $x(s)=\phi(s)$, $s\in[t_0-h,t_0]$.

It should be noted that the obtained optimal regulator makes an advance with respect to general optimality results for time delay systems (such as given in (Alekal *et al.*, 1971; Malek-Zavarei and Jashmidi, 1987; Kolmanovskii and Shaikhet, 1996; Kolmanovskii and Myshkis, 1999)), since (a) the optimal control law is given explicitly and not as a solution of a system of integro-differential or PDE equations, and (b) the quasi-Riccati equation for the gain matrix does not contain any time advanced arguments and does not depend on the state variables and, therefore, leads to a conventional two points boundary-valued problem generated in the optimal control problems with quadratic criterion and finite horizon (see, for example, (Kwakernaak and Sivan, 1972)). Thus, the obtained optimal regulator is realizable using two delay-differential equations.

Remark. In some papers (see, for example, (Alekal *et al.*, 1971)), the optimal control $u^*(t)$ is obtained as an integral of the previous values of $x(t)$ over the interval $[t-h,t]$. However, since the backward solution of the equation (1) exists and is unique, as mentioned in Section 2, any previous value $x(\tau)$, $\tau\in[t-h,t]$ can be uniquely represented as a function of the current value $x(t)$ (as well as of any delayed value $x(t-r), r > 0$). Thus, the optimal control $u^*(t)$ can be obtained as a function of $x(t)$ in the form (3). The current value $x(t)$ is selected to form the closed-loop control (3), first, because the transversality condition induced by the cost function (3) can readily be satisfied (see Appendix) and, second, due to practical applicability of the current-time control in real technical problems.

4. EXAMPLE

This section presents an example of designing the optimal regulator for a system (1) with a criterion (2), using the scheme (3)–(5), and comparing it to the regulator where the matrix Q is selected as in the optimal linear regulator for a system without delays.

Let us start with a scalar linear system

$$\dot{x}(t) = x(t-0.1) + 10x(t-0.25) + u(t), \qquad (6)$$

with the initial conditions $x(s) = 1$ for $s \in [-0.25, 0]$. The control problem is to find the control $u(t)$, $t \in [0, T]$, $T = 0.5$, that minimizes the criterion

$$J = \frac{1}{2}[\int_0^T u^2(t)dt + \int_0^T x^2(t)dt]. \qquad (7)$$

In other words, the control problem is to minimize the overall energy of the state x using the minimal overall energy of control u.

Let us first construct the regulator where the control law and the matrix $Q(t)$ are calculated in the same manner as for the optimal linear regulator for a linear system without delays, that is $u(t) = R^{-1}(t)B^T(t)Q(t)x(t)$ (see (Kwakernaak and Sivan, 1972) for reference). Since $B(t) = 1$ in (6) and $R(t) = 1$ in (7), the optimal control is actually equal to

$$u(t) = Q(t)x(t), \qquad (8)$$

where $Q(t)$ satisfies the Riccati equation

$$\dot{Q}(t) = -(a_1(t) + a_2(t))^T Q(t) - Q(t)(a_1(t) + a_2(t)) + L(t) - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = -\psi$. Since $a_1(t) = 1$, $a_2(t) = 10$, $a_1(t) + a_2(t) = 11$, $B(t) = 1$ in (6), and $L(t) = 1$, $\psi = 0$ in (7), the last equation turns to

$$\dot{Q}(t) = 1 - 22Q(t) - Q^2(t), \quad Q(0.5) = 0. \qquad (9)$$

Upon substituting the control (8) into (6), the controlled system takes the form

$$\dot{x}(t) = x(t-0.1) + 10x(t-0.25) + Q(t)x(t). \qquad (10)$$

The results of applying the regulator (8)–(10) to the system (6) are shown in Fig. 1, which presents the graphs of the criterion (7) $J(t)$ and the control (8) $u(t)$ in the interval $[0, T]$. The value of criterion (7) at the final moment $T = 0.5$ is $J(0.5) = 19.58$.

Let us now apply the optimal regulator (3)–(5) for linear systems with multiple state delays to the system (6). The control law (3) takes the same form as (8)

$$u^*(t) = Q^*(t)x(t), \qquad (11)$$

where $Q^*(t)$ satisfies the equation

$$\dot{Q}^*(t) = 1 - 2Q^*(t)N_1(t) - 20Q^*(t)N_2(t) - Q^{*2}(t), \qquad (12)$$
$$Q^*(0.5) = 0,$$

where $N_1(t) = 0$ for $t \in [0, 0.1)$ and $N_1(t) = \exp(-\int_{t-0.1}^t Q^*(s)ds)$ for $t \in [0.1, 0.5]$; $N_2(t) = 0$ for $t \in [0, 0.25)$ and $N_2(t) = \exp(-\int_{t-0.25}^t Q^*(s)ds)$ for $t \in [0.25, 0.5]$.

Note that the obtained equation (12) does not contain any advanced arguments and, therefore, can be solved using simple numerical methods, such as "shooting." This method consists in varying initial conditions of (12), taking into account monotonicity of the solution of (12) with respect to initial conditions, until the terminal condition is satisfied. In this example, the equation (12) has been solved with the approximating terminal condition $Q^*(0.5) = 0.05$, in order to reduce the computation time.

Upon substituting the control (11) into (6), the optimally controlled system takes the same form as (10)

$$\dot{x}(t) = x(t-0.1) + 10x(t-0.25) + Q^*(t)x(t). \qquad (13)$$

The results of applying the regulator (11)–(13) to the system (6) are shown in Fig. 2, which presents the graphs of the criterion (7) $J(t)$ and the optimal control (11) $u^*(t)$ in the interval $[0, T]$. The value of the criterion (7) at the final moment $T = 0.5$ is $J(0.5) = 5.83$. There is a definitive improvement (about three times) in the values of the cost function in comparison to the preceding case, due to the optimality of the regulator (3)–(5) for linear systems with multiple state delays.

5. APPENDIX

Proof of the optimal control problem solution. Define the Hamiltonian function (Pontryagin *et al.*, 1962; Kharatashvili, 1967) for the optimal control problem (1),(2) as

$$H(x, u, q, t) = \frac{1}{2}(u^T R(t)u + x^T L(t)x) + q^T[(\sum_{i=1}^p a_i(t)x_i) + B(t)u], \qquad (14)$$

where $x_i(x) = x(t - h_i)$, $i = 1, \ldots, p$. Applying the maximum principle condition $\partial H/\partial u = 0$ to this specific Hamiltonian function (14) yields

$$\partial H/\partial u = 0 \Rightarrow R(t)u(t) + B^T(t)q(t) = 0,$$

and the optimal control law is obtained as

$$u^*(t) = -R^{-1}(t)B^T(t)q(t).$$

Taking linearity and causality of the problem into account, the co-state $q(t)$ is sought as a linear function in $x(t)$ to readily satisfy the transversality condition (Pontryagin *et al.*, 1962; Kharatashvili, 1967) induced by the cost function (2):

$$q(t) = -Q(t)x(t), \tag{15}$$

where $Q(t)$ is a square symmetric matrix of dimension n. This yields the complete form of the optimal control

$$u^*(t) = R^{-1}(t)B^T(t)Q(t)x(t). \tag{16}$$

Note that the transversality condition (Pontryagin *et al.*, 1962; Kharatashvili, 1967) for $q(T)$ implies that $q(T) = \partial J/\partial x(T) = \psi x(T)$ and, therefore, $Q(T) = -\psi$.

Using the co-state equation $dq(t)/dt = -\partial H/\partial x$ and denoting $(\partial x_i(t)/\partial x) = N_i(t)$ yields

$$-dq(t)/dt = L(t)x(t) + (\sum_{i=1}^{p} a_i^T(t)N_i^T(t))q(t), \tag{17}$$

and substituting (15) into (17), we obtain

$$\dot{Q}(t)x(t) + Q(t)d(x(t))/dt = L(t)x(t) - \tag{18}$$

$$(\sum_{i=1}^{p} a_i^T(t)N_i^T(t))Q(t)x(t).$$

Substituting the expression for $\dot{x}(t)$ from the state equation (1) into (18) yields

$$\dot{Q}(t)x(t) + Q(t)(\sum_{i=1}^{p} a_i(t)x(t-h_i)) + Q(t)B(t)u(t) = \tag{19}$$

$$L(t)x(t) - (\sum_{i=1}^{p} a_i^T(t)N_i^T(t))Q(t)x(t).$$

In view of linearity of the problem, differentiating the last expression in x does not imply loss of generality. Upon substituting the optimal control law (16) into (19), taking into account that $(\partial x(t-h_i)/\partial x(t)) = N_i(t)$, and differentiating the equation (19) in x, it is transformed into the quasi-Riccati equation

$$\dot{Q}(t) = L(t) - Q(t)(\sum_{i=1}^{p} N_i(t)a_i(t)) - \tag{20}$$

$$(\sum_{i=1}^{p} a_i^T(t)N_i^T(t))Q(t) - Q(t)B(t)R^{-1}(t)B^T(t)Q(t).$$

with the terminal condition $Q(T) = -\psi$.

Let us now obtain the values of $N_i(t)$, $i = 1,\ldots,p$. By definition, $N_i(t) = (\partial x(t-h_i)/\partial x(t))$. Substituting the optimal control law (16) into the equation (1) gives

$$\dot{x}(t) = \sum_{i=1}^{p} a_i(t)x(t-h_i) + B(t)R^{-1}(t)B^T(t)Q(t)x(t), \tag{21}$$

with the initial condition $x(s) = \phi(s)$, $s \in [t_0 - h, t_0]$. Integrating (21) yields

$$x(t_0+h_i) = x(t_0) + \int_{t_0}^{t_0+h_i} (\sum_{k=1}^{p} a_k(s)x(s-h_k))ds + \tag{22}$$

$$\int_{t_0}^{t_0+h_i} B(s)R^{-1}(s)B^T(s)Q(s)x(s)ds.$$

Analysis of the formula (22) shows that $x(t)$ does not explicitly depend on $x(t-h_i)$, if $t \in [t_0, t_0+h_i)$. Therefore, $N_i(t) = 0$ for $t \in [t_0, t_0+h_i)$. On the other hand, if $t \geq t_0 + h_i$, the following Cauchy formula is valid for the solution $x(t)$ of the equation (21)

$$x(t) = \Phi(t, t-h_i)x(t-h_i) + \tag{23}$$

$$\int_{t-h_i}^{t} \Phi(t,s)(\sum_{k=1}^{p} a_k(s)x(s-h_k))ds,$$

where $\Phi(t,\tau)$ satisfies the matrix equation

$$\frac{d\Phi(t,\tau)}{dt} = B(t)R^{-1}(t)B^T(t)Q(t)\Phi(t,\tau),$$

with the initial condition $\Phi(t,t) = I$, and I is the identity matrix. The expression (23) immediately implies that $N_i(t) = \Phi^{-1}(t, t-h_i) = \Phi(t-h_i, t) = \exp\left(-\int_{t-h_i}^{t} B(s)R^{-1}(s)B^T(s)Q(s)ds\right)$ for $t \geq t_0 + h_i$. The optimal control problem solution is proved.

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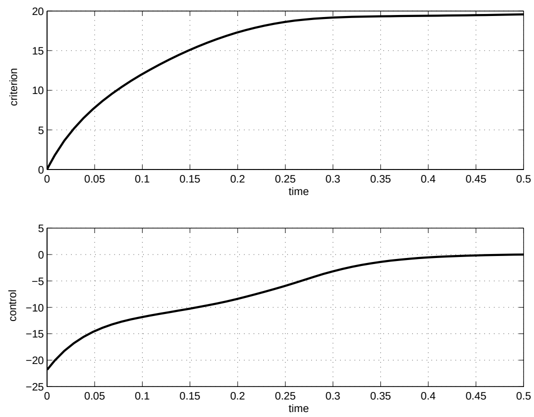

Figure 1. Best linear regulator available for linear systems without delays. Graphs of the criterion (7) $J(t)$ and the control (8) $u(t)$ in the interval $[0, 0.5]$.

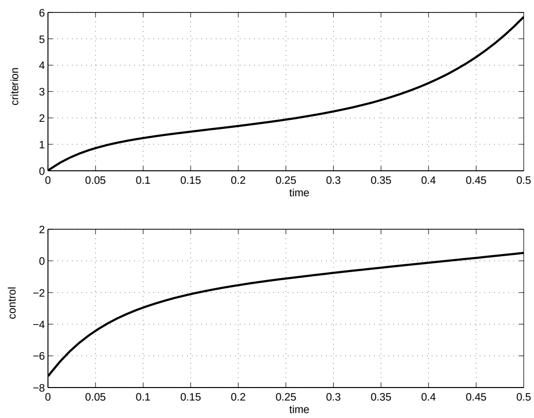

Figure 2. Optimal regulator obtained for linear systems with multiple state delays. Graphs of the criterion (7) $J(t)$ and the optimal control (11) $u^*(t)$ in the interval $[0, 0.5]$.