# NONLINEAR SAMPLED–DATA CONTROLLER REDESIGN VIA LYAPUNOV FUNCTIONS [1]

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Abstract: We provide results for redesign of Lyapunov function based continuous time controllers for sampled-data implementation, using a particular form of the redesigned controller and the Taylor expansion of the sampled-data Lyapunov difference. We develop two types of redesigned controllers that (i) make the lower order terms (in $T$) in the series expansion of the Lyapunov difference with the redesigned controller more negative and (ii) make the terms in the Taylor expansions of the Lyapunov difference for the sampled-data system with the redesigned controller behave as close as possible to the respective values of the continuous-time system with the original controller. Simulation studies illustrate the performance of our controllers. 

Keywords: sampled–data control, redesign, Lyapunov function, Taylor expansion

## 1. INTRODUCTION

One of the most popular methods to design sampled-data controllers is the design of a controller based on the continuous-time plant model, followed by a discretization of the controller (Chen and Francis, 1995; Franklin *et al.*, 1997; Laila *et al.*, 2002). This method, often called emulation, is very attractive since the controller design is carried out in two relatively simple steps. The first (design) step is done in continuous-time, completely ignoring sampling, which is easier than the design that takes sampling into account. The second step involves the discretization of the controller and there are many methods that can be used for this purpose. Classical discretization methods, such as the Euler, Tustin or matched pole-zero discretization are attractive for their simplicity but they may not perform well in practice since the required sampling rate may exceed the hardware limitations even for linear systems (Katz, 1981; Anderson, 1993). For linear systems this has lead to a range of advanced controller discretization techniques based on optimization ideas that compute "the best discretization" of the continuous-time controller, see (Anderson, 1993) and (Chen and Francis, 1995).

Also for a large class of nonlinear sampled-data systems emulation preserves a range of important properties, see (Laila *et al.*, 2002), if the discretized controller is consistent with the continuous-time controller and the sampling period is small enough. While optimization based approaches could be probably carried out for nonlinear systems, these approaches inevitably re-

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quire solutions of partial differential equations of Hamilton-Jacobi type that are very hard to solve.

In this paper we present Lyapunov based redesign techniques of continuous-time controllers for sampled-data implementation. We assume that an appropriate continuous-time controller $u_0(x)$ has been designed together with an appropriate Lyapunov function $V(\cdot)$ for the closed-loop continuous-time system. Then, we presuppose the following structure of the redesigned controller

$$u_{dt}(x) = u_0(x) + \sum_{i=1}^{N} T^i u_i(x) \ ,$$

where $T$ is the sampling period and $u_i(x)$ are the terms that need to be designed. This structure yields a particularly useful structure of the Taylor expansion of the first difference for $V(\cdot)$ along solutions of the sampled-data system with the redesigned controller which we use for the systematic computation of the correction terms $u_i$.

This structure was obtained in several papers as an outcome of the design procedure, see, e.g., (Arapostathis *et al.*, 1989; Nešić and Teel, 2001). Here, however, we *impose* this structure of the controller, similar to the approach in (Laila and Nešić, 2003) that used the Euler scheme and $u_{dt} = u_{ct}(x) + Tu_1$. Motivated by the promising results in this reference our goal is to develop a systematic methodology for controller redesign.

Our method is very flexible and allows for several redesign objectives, two of them being addressed in this paper. The first is to make the lower order terms in the Taylor expansions more negative by choosing $u_i$. This often leads to the correction terms of the form "$-L_gV$" useful in robustification of continuous-time controllers (see, e.g., (Corless, 1993; Sepulchre *et al.*, 1997)). The second objective is to make the first terms of the expansion of the first difference for $V(\cdot)$ along solutions of the sampled-data system with the redesigned controller as close as possible to the respective value for the "ideal" response of the continuous-time system.

The paper is organized as follows. In Section 2 we present the notation, the main assumption and pose the problem we consider. Section 3 contains the main result on the Taylor expansion which is used in Section 4 to show two distinct ways to redesign continuous-time controllers. Simulation results are given in Section 5 and conclusions are presented in the last section.

## 2. PRELIMINARIES

As usual, a function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is also unbounded. The notation $|\cdot|$ always denotes the Euclidean 2-norm. We will say that a function $G(T,x)$ is of order $T^p$ and we write $G(T,x) = O(T^p)$ if, whenever $G$ is defined, we can write $G(T,x) = T^p\tilde{G}(T,x)$ and there exists $\gamma \in \mathcal{K}_\infty$ such that for each $\Delta > 0$ there exists $T^* > 0$ such that $|x| \leq \Delta$ and $T \in (0,T^*)$ implies $\left|\tilde{G}(T,x)\right| \leq \gamma(|x|)$.

Consider the system

$$\dot{x} = g_0(x) + g_1(x)u \ , \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are respectively the state and the control input of the system. We will assume that all functions are sufficiently smooth. For simplicity, we concentrate on single input systems but the results can be extended to the multiple input case $u \in \mathbb{R}^m, m \in \mathbb{N}$.

For several classes of systems (1), there exist nowadays systematic methods to design a continuous-time control law of the form

$$u = u_0(x) \ , \tag{2}$$

and a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \ \leq \ V(x) \leq \alpha_2(|x|) \tag{3}$$

$$\frac{\partial V}{\partial x}[g_0(x) + g_1(x)u_0(x))] \leq -\alpha_3(|x|) \, \forall x \in \mathbb{R}^n . \tag{4}$$

Examples are methods like backstepping (Krstić *et al.*, 1995; Freeman and Kokotović, 1996), forwarding (Sepulchre *et al.*, 1997) or Sontag's formula (Sontag, 1989).

In most cases the controller (2) is implemented digitally using a sampler and zero order hold. Since (2) is static, it is often proposed to simply implement it as follows (see (Laila *et al.*, 2002)):

$$u(t) = u_0(x(k)) \quad \forall t \in [kT, (k+1)T), \, k \in \mathbb{N} \ . \tag{5}$$

It was shown, for instance, in (Laila *et al.*, 2002) that this digital controller will recover performance of the continuous-time system in a semiglobal practical sense ($T$ is the parameter that needs to be chosen sufficiently small). However, (5) typically requires very small sampling periods $T$ to work well and, hence, may not produce the desired behaviour for a fixed given $T$. The purpose of this paper is to systematically redesign the controller $u_0(\cdot)$ so that the redesigned sampled-data controller performs better than (5) in an appropriate sense.

In order make this "appropriate sense" more precise, consider the solution $y(t)$ of the scalar differential equation [2]

$$\dot{y} = -\alpha_3 \circ \alpha_2^{-1}(y) \qquad y(0) = y_0 \ . \tag{6}$$

[2] Without loss of generality we need to assume here that $\alpha_3 \circ \alpha_2^{-1}(\cdot)$ is a locally Lipschitz function (see footnote in (Khalil, 1996, pg. 153)).

Proposition 4.4 in (Khalil, 1996) states that the function $\sigma(y_0, t) := y(t)$ is of class $\mathcal{KL}$. Then with

$$\beta(s,t) := \alpha_1^{-1}\left(\sigma(\alpha_2(s), t)\right) \ . \tag{7}$$

we obtain that solutions of the closed loop system (1), (2) satisfy:

$$|x(t,x_0)| \leq \beta(|x_0|, t) \qquad \forall x_0 \in \mathbb{R}^n,\ t \geq 0 \ , \tag{8}$$

Based on these considerations we can now state our main assumption.

*Assumption 1.* Suppose that a continuous static state feedback controller (2) has been designed for the system (1) so that the following holds:

(1) There exists a Lyapunov function $V(\cdot)$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ satisfying (3) and (4).
(2) The function $\beta \in \mathcal{KL}$ defined in (7) satisfies all performance specifications in terms of overshoot and speed of convergence.
(3) The controller (2) is to be implemented digitally using a sampler and zero order hold, that is for a given sampling period $T > 0$ we measure $x(k) := x(kT), k \in \mathbb{N}$ and $u(t) = u(k) = const., t \in [kT, (k+1)T), k \in \mathbb{N}$.

*Remark 2.* At a first glance either item (i) and (ii) in Assumption 1 may seem enough. However, in our approach we will use both (i), since we need the Lyapunov function $V(\cdot)$ in order to carry out the redesign and (ii), because the objectives we use in redesign rely on the function $\beta$. For instance, in Subsection 4.1 the objective is to make the Lyapunov difference for the sampled-data system as close as possible to the Lyapunov difference along the continuous-time system. Hence, for our controller redesign objective to be plausible we need to assume that item (ii) of Assumption 1 holds. In other words, the bound (ii) on the continuous-time closed-loop response obtained from the Lyapunov function is regarded as "ideal" or a "reference" stability bound.

The exact discrete-time model of the system with the zero order hold assumption is obtained by integrating (1) starting from $x_0 = x(k)$ with the control $u(t) = u(k), t \in [kT, (k+1)T)$:

$$\begin{aligned} x(k+1) &= x(k) + \int_{kT}^{(k+1)T} g_0(x(s)) + g_1(x(s))u(k)ds \\ &=: F_T^e(x(k), u(k)). \end{aligned} \tag{9}$$

## 3. TAYLOR EXPANSION

In this section we propose a particular structure for the redesigned controller. This structure of the controller yields an interesting structure of the series expansion of the Lyapunov difference along the solutions of closed loop system with the redesigned controller and will allow us to redesign the controller in a systematic manner. We propose to modify the continuous-time controller as follows:

$$u_{dt}(x) := \sum_{j=0}^{M} u_j T^j \ , \tag{10}$$

where $u_0(x)$ comes from Assumption 1 and $u_j = u_j(x), j = 1, 2, \ldots, M$ are corrections that we want to determine. Note that for fixed $M$ and $T \to 0$ we obtain $u_{dt} \to u_0$. The number $M$ of correction terms needed for a suitable performance depends on the choice of $u_i$, the chosen sampling rate and the plant dynamics, see Remarks 7, 8.

The idea is to use the Lyapunov function $V$ as a control Lyapunov function for the discrete-time model (9) of the sampled-data system with the modified controller (10) where we treat $u_i, i = 1, 2, \ldots, M$ as new controls which are determined from the Lyapunov difference

$$\frac{V(F_T^e(x, u_{dt}(x))) - V(x)}{T}. \tag{11}$$

Since in general it is not possible to compute $F_T^e(x,u)$ in (9) exactly we need to work with an approximation of (11). For this purpose we use the following Taylor expansion of (11) that is particularly suitable for controller redesign.

*Theorem 3.* Consider system (1) and controller (10) and suppose that Assumption 1 holds. Then, for sufficiently small $T$, there exist functions $p_i(x, u_0, \ldots, u_{i-1})$ such that we can write:

$$\begin{aligned} &\frac{V(F_T^e(x, u_{dt})) - V(x)}{T} \\ &= \ L_{g_0}V + L_{g_1}V \cdot u_0 \\ &\quad + \ \sum_{s=1}^{M} T^s [L_{g_1}V \cdot u_s + p_s(x, u_0, \ldots, u_{s-1})] \\ &\quad + \ G(T, x, u_0, u_1, \ldots, u_M) \ , \end{aligned} \tag{12}$$

where $G(T, x, u_0, u_1, \ldots, u_M) = O(T^{M+1})$. □

The proof follows from a careful examination of the terms in the Fliess series expansion (Isidori, 2002, formula (3.7)) [3] of $V(F_T^e(x, u_{dt})$ in $T$, using our controller structure. Details can be found in (Nešić and Grüne, 2005).

The functions $p_s$ can be obtained by straightforward computations where computer algebra systems like, e.g., MAPLE can be efficiently used. For instance, for $s = 1, 2$ we obtain

[3] Since $u_{dt} \equiv const$ on $[0, T]$ the Fliess expansion coincides with the usual Taylor expansion along solutions of ODEs.

$$p_1 = \frac{1}{2!}\Big(L_{g_0}L_{g_0}V + (L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V)u_0 \qquad (13)$$
$$+L_{g_1}L_{g_1}Vu_0^2\Big) .$$
$$p_2 = \frac{1}{2!}\Big(u_1(L_{g_0}L_{g_1}V + L_{g_1}L_{g_0}V + 2_{g_1}L_{g_1}Vu_0)\Big)$$
$$+\frac{1}{3!}\Big(L_{g_0}L_{g_0}L_{g_0}V$$
$$+(L_{g_0}L_{g_0}L_{g_1}V \qquad (14)$$
$$+L_{g_0}L_{g_1}L_{g_0}V + L_{g_1}L_{g_0}L_{g_0}V)u_0$$
$$+(L_{g_0}L_{g_1}L_{g_1}V$$
$$+L_{g_1}L_{g_0}L_{g_1}V + L_{g_1}L_{g_1}L_{g_0}V)u_0^2$$
$$+L_{g_1}L_{g_1}L_{g_1}Vu_0^3\Big) .$$

## 4. REDESIGN TECHNIQUES

In this section we propose controller redesign procedures that are based on (12). There is a lot of flexibility in this procedure and in general one needs to deal with systems on a case-by-case basis. Here we consider two different goals for controller redesign in Subsections 4.1 and 4.2.

### *4.1 High gain controller redesign*

Our first case is reminiscent of the Lyapunov controller redesign of continuous-time systems for robustification of the system (see (Corless, 1993; Khalil, 1996)), providing more negativity to the Lyapunov difference. This typically yields high gain controllers that may have the well known "$-L_gV$" structure which was used, for example, in (Sepulchre *et al.*, 1997).

Observe that the terms in the series expansion have the following form:

$$O\left(T^0\right): \quad L_{g_1}V \cdot u_0 + L_{g_0}V \qquad (15)$$
$$O\left(T^1\right): \quad L_{g_1}V \cdot u_1 + p_1(x,u_0) \qquad (16)$$
$$O\left(T^2\right): \quad L_{g_1}V \cdot u_2 + p_2(x,u_0,u_1) \qquad (17)$$
$$O\left(T^3\right): \quad L_{g_1}V \cdot u_3 + p_3(x,u_0,u_1,u_2) \qquad (18)$$
$$\vdots \quad \vdots$$

This special triangular structure allows us to use a recursive redesign. Assuming that $u_0$ is designed based on the continuous-time plant model (1), at each step $s \in \{1,\ldots,M\}$ we design $u_s$ to make the terms of order $O(T^s)$ more negative. For this purpose we can use $p_s(x,u_0,\ldots,u_{s-1})$ since at stage $s$ all previous $u_i, i = 0,1,2,\ldots,s-1$ have already been designed.

For the actual design of $u_s$ we now discuss some possible choices. It is obvious from (12) that any function $u_j$ with

$$u_j = u_j(x) \ \text{ such that } \begin{cases} u_j \le 0 & \text{if } L_{g_1}V \ge 0 \\ u_j \ge 0 & \text{if } L_{g_1}V \le 0 \end{cases}$$

will achieve more decrease of $V(\cdot)$. For example, one such choice is

$$u_j(x) = -\gamma_j(V(x)) \cdot (L_{g_1}V(x)) \ , \qquad (19)$$

where $\gamma_j \in \mathcal{K}$ is a design parameter that can be determined using the $p_s(x,u_0,\ldots,u_{s-1})$ functions from (12), e.g., chosing $\gamma_j$ such that the sign indefinite functions $p_s(x,u_0,\ldots,u_{s-1})$ are dominated by the negative term $u_s(x)L_{g_1}V(x)$. The following theorem shows that this can be accomplished up to higher order terms.

*Theorem 4.* Consider the system (1) and suppose that Assumption 1 holds. For any $j \in \{0,1,2,\ldots,M\}$ denote $u^j(x) := \sum_{i=0}^{j} T^i u_i(x)$. Then, suppose that whenever $F_T^e$ is well defined, we have for some $j \in \{0,1,2,\ldots,M\}$ that the following holds:

$$\frac{V(F_T^e(x,u^j(x))) - V(x)}{T} \le -\alpha_3(|x|) + G_1(T,x) \ , \qquad (20)$$

and $G_1(T,x) = O(T^p)$ for some $p \in \mathbb{N}$. Suppose now that the controller $u^{j+1}(x)$ is implemented, where $u_{j+1}(x) := -\gamma_{j+1}(V(x)) \cdot L_gV(x)$. Then, whenever $F_T^e$ is well defined, we have that:

$$\frac{V(F_T^e(x,u^{j+1}(x))) - V(x)}{T} \qquad (21)$$
$$\le -\alpha_3(|x|) - T^{j+1}\gamma_{j+1}(V(x))\left(\frac{\partial V}{\partial x}g(x)\right)^2$$
$$+G_1(T,x) + G_2(T,x) \ ,$$

where $G_1(T,x)$ is the same as in (20) and $G_2(T,x) = O(T^{j+2})$. □

The proof follows directly from Theorem 3.

*Remark 5.* Whenever $L_{g_1}V(x) \neq 0$ we can in principle dominate the terms $p_s(x,u_0,\ldots,u_{s-1})$ by increasing the gain of $u_s$. However, due to saturation arbitrary increase in gain is not feasible. If we know an explicit bound on the control signals, such as $|u_j| \le \gamma(|x|)$, then the control that produces most decrease of $V(\cdot)$ under this constraint is

$$u_j(x) = \begin{cases} -\gamma(|x|) & \text{if } L_{g_1}V(x) \ge 0 \\ \gamma(|x|) & \text{if } L_{g_1}V(x) \le 0 \end{cases} .$$

We will use such a controller in the jet engine example presented below.

*Remark 6.* It is well known (see (Sepulchre *et al.*, 1997)) that the control laws of the form (19) robustify the controller to several classes of uncertainties and lead to improved stability margins. Our results show that adding the $-L_{g_1}V$ terms of the form (19) robustifies the controller also with respect to sampling.

*Remark 7.* The approach indicated above needs the sampling period $T$ to be sufficiently small so that terms of order $O(T^{M+1})$ are negligible. Since the $O(T^{M+1})$ terms depend in general on $u_0, u_1, \ldots, u_M$, larger magnitudes of $u_i$ will in general make these terms less negligible, cf. (13) and (14). Nevertheless, we will show in our example that a judicious choice of $u_i$ and $T$ does produce controllers that perform better than (5).

*Remark 8.* We again emphasize that the procedure we described above is very flexible and we only outlined some of the main guiding principles. However, even the simplest choice of redesigned controller of the form $u_{dt}(x) = u_{ct}(x) - TL_{g_1}V(x)$ improves the transients of the sampled-data system. If this is not significant then exploiting the structure of $p_s$ terms becomes important.

*Remark 9.* Often the redesign procedure is more important for states away from the origin, because near the origin the simple controller (5) either works well or can be replaced by a linear controller. This simplifies the difficult task of finding a control Lyapunov function satisfying Assumption 1, because we can restrict ourselves to the "interesting" region of the state space. This is the situation in the example in Section 5, below.

### *4.2 Model reference based controller redesign*

In this subsection, the goal of the controller redesign procedure is to make the sampled data Lyapunov difference $V(F_T^e(x, u_{dt}(x))) - V(x)$ as close as possible to the continuous time Lyapunov difference $V(\phi(T,x)) - V(x)$, where $\phi(T,x)$ is the solution of the continuous time closed loop system (1), (2) at time $t = T$ and initialized at $x(0) = x$. This makes sense in situations when we want the bound on our sampled-data response to be as close as possible to the "ideal" bound on the response generated by the solution of the continuous-time closed-loop system (1), (2). We use the following notation:

$$\Delta V_{dt}(T,x,u) := V(F_T^e(x,u)) - V(x)$$
$$\Delta V_{ct}(T,x) := V(\phi(T,x)) - V(x) \ .$$

The main result of this subsection is:

*Theorem 10.* If Assumption 1 holds then we have

$$\Delta V_{ct}(T,x) - \Delta V_{dt}(T,x,u_0(x)) = O(T^2) \ . \quad (22)$$

Defining $u_{dt}(x) = u_0(x) + Tu_1(x)$ with

$$u_1(x) = \frac{1}{2}\frac{\partial u_0(x)}{\partial x}[g_0(x) + g_1(x)u_0(x)] \quad (23)$$

we have

$$\Delta V_{ct}(T,x) - \Delta V_{dt}(T,x,u_{dt}(x)) = O(T^3) \ . \quad (24)$$

□

The proof follows from Theorem 3 by comparing $\Delta V_{dt}(T,x,u_0 + Tu_1)$ with the Taylor expansion of $\Delta V_{ct}(T,x)$ in $T = 0$.

Observe that in contrast to the control law from the previous section this controller does not depend on the Lyapunov function.

*Remark 11.* It may be tempting to repeat this procedure iteratively for $N \geq 2$ in order to obtain $O(T^{N+2})$ in (24). However, the computations in (Nešić and Grüne, 2005) show that even though some higher order terms in (24) above can be canceled for all $N \geq 2$, we can not in general make (24) smaller than $O(T^3)$.

## 5. AN EXAMPLE

Consider the following simplified Moore-Greitzer model of a jet engine taken from (Krstić *et al.*, 1995, Section 2.4.3)

$$\dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3, \quad \dot{x}_2 = -u \ .$$

The control law $u_0(x) = -k_1x_1 + k_2x_2$ and the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{c_0}{8}x_1^4 + \frac{1}{2}(x_2 - c_0x_1)^2$, have been derived in (Krstić *et al.*, 1995, pg. 72), where $k_1$, $k_2$, $c_0$, $c_1$ and $c_2$ are design parameters. Using $k_1 = 7$, $k_2 = 5$, $c_0 = 2$, $c_1 = \frac{7}{8}$ and $c_2 = \frac{3}{7}$ we obtain $u_0(x) = -7x_1 + 5x_2$ and

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}(x_2 - 2x_1)^2, \quad (25)$$

and the closed loop system becomes

$$\dot{x}_1 = -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3, \quad \dot{x}_2 = 7x_1 - 5x_2,$$

which has a very nice response. However, the Lyapunov function (25) does not satisfy our Assumption 1 because it does not capture this nice response. Indeed, while the trajectories converge very quickly with no overshoot, the Lyapunov function (25) has level sets that are elongated very much along the $x_2$ axis and, hence, the function $\beta \in \mathcal{KL}$ from (7) allows for very large overshoots. Motivated by simulations we try to use the Lyapunov function

$$V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \ . \quad (26)$$

Direct calculations show that $\dot{V}_1$ is strictly negative on the set $S := S_1 \cup (S_1^C \cap S_2)$ with $S_1 := \{x \in \mathbb{R}^2 : x_1 \notin [-4, +1], \ x_2 \in \mathbb{R}\}$ and $S_2 := \{x \in \mathbb{R}^2 : 2x_1^2 - 6x_1x_2 + 5x_2^2 > 18.1\}$. Hence, $V_1$ is a Lyapunov function on the above set and, moreover, it satisfies our Assumption 1 since it shows that trajectories are converging without any overshoot.

We use $V_1$ as a control Lyapunov function for redesign of the controller on the set $S$. Based on Theorem 4 and Remark 5 and noting that $L_{g_1}V_1 = -x_2$, we implemented the controller

$$u_{dt}^{Lf}(x) = \begin{cases} u_0(x) + Tu_1^{Lf}(x) & \text{if } x \in S \\ u_0(x) & \text{otherwise} \end{cases}$$

with

$$u_1^{Lf}(x) = \begin{cases} x_1^2 + x_2^2 & \text{if } L_{g_1}V_1 = -x_2 < 0 \\ -(x_1^2 + x_2^2) & \text{otherwise} \end{cases}.$$

The chosen gain $\gamma(|x|) = |x|^2$ here was tuned such that the redesigned controller yields a significant improvement in the response in the state space region $[-25, 25]^2$ with sampling rate $T = 0.1$.

For this example the model reference controller from Theorem 10 reads $u_1^{mr}(x) = \frac{35}{2}x_1 + \frac{21}{4}x_1^2 + \frac{7}{4}x_1^3 - 9x_2$. As for $u_{dt}^{Lf}$ we used a saturation with $\gamma(|x|) = |x|^2$, which allows for a "fair" comparison between the two controllers $u_{dt}^{Lf}$ and $u_{dt}^{mr}$.

Figure 1 shows the trajectories (top), sampled control values (bottom left) and the Lyapunov function $V_1(x)$ (bottom right) for initial value $x_0 = [22, 21]$ and sampling rate $T = 0.1$. The curves show the continuous time system (unmarked), the sampled continuous time controller $u_{dt} = u_0$ (circles), the controller $u_{dt}^{Lf}$ (squares) and the controller $u_{dt}^{mr}$ (crosses).

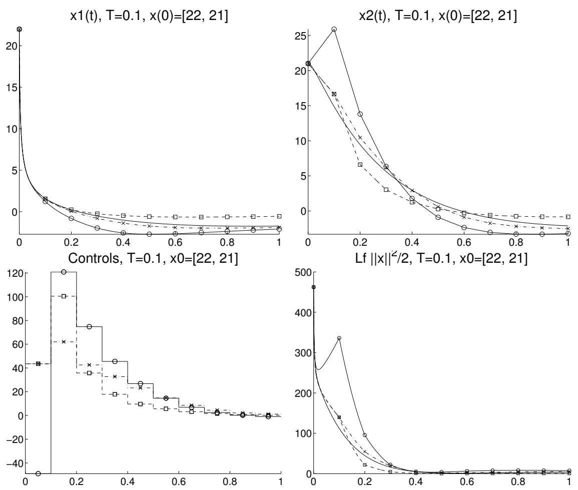


Fig. 1. Solutions for different controllers

As expected, the trajectory corresponding to $u_{dt}^{Lf}$ tends to 0 faster while $u_{dt}^{mr}$ keeps the trajectory closer to the continuous time one. Both redesigned controllers avoid the overshoot in the $x_2$–component visible in the sampled continuous time controller.

## 6. CONCLUSIONS

We have presented a method for a systematic redesign of continuous-time controllers for digital implementation. This method is very flexible and we illustrated its usefulness through an example. Many variations of this method are possible and the main directions for further improvement are including dynamical and observer based controllers and relaxing some of the assumptions that we use at the moment.

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