OUTPUT FEEDBACK STABILIZATION CONTROL DESIGN FOR THE DISTURBANCE ATTENUATION OF A CLASS OF MIMO NONLINEAR SYSTEMS

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Abstract: In this paper, an output feedback discontinuous controller is proposed for a class of nonlinear systems to attenuate the disturbance influence on the system performance. Our main goal is to globally stabilize the nonlinear system in the presence of unknown structural system uncertainties and external disturbances. Our approach consists of constructing a high gain nonlinear observer to reconstruct the unavailable states, and then design discontinuous controller using a backstepping like design procedure to ensure stability of the nonlinear system. The design parameters of the observer and the controller are determined in a complementary manner to ensure stability. As a result the whole system can be stabilized while internal stability of the system states is also ensured. Finally, an example is presented to show the effectiveness of the proposed scheme.

Keywords: Output Feedback, Disturbance Attenuation, Discontinuous Control, Input-to-State Stability, High Gain Observer, Nonlinear Systems, Uncertainties

1. INTRODUCTION

Output feedback stabilization has been the subject of constant research over the past several decades. Despite these efforts, robust stabilization of general nonlinear systems remains an open problem (Khalil, 1996). For linear systems or nonlinear systems that can be linearized near the equilibrium point, the well known H_∞ control method offers a systematic approach in which the influence of system uncertainties can be directly incorporated into the design. However, for nonlinear systems, the solution of the nonlinear H_∞ control problem has proven to be very difficult. Indeed, the synthesis of H_∞ optimal controller requires solving the Hamilton Jacobi-Isaacs (HJI) equation (James and Baras, 1995), which is either very difficult or in most cases impossible to solve. Solving the HJI equation can be avoided by the inverse optimal design proposed as in (Krstić and Li, 1998). However, in this approach, a prescribed performance level cannot be guaranteed.

Several researchers have proposed various approaches for the disturbance attenuation problems of nonlinear systems with different forms and assumptions. In (Isidori and Lin, 1998), the nonlin-

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ear functions in the system are linearized near the equilibrium $\mathbf{x} = 0$. The control law can then be obtained by utilizing the solution of the Riccati Algebraic Equation for the linearized plant to get a global solution of the Hamilton Jacobi Isaacs partial differential equation arising in the nonlinear H_∞ control. Though the control target is obtained, only a single input single output (SISO) system with specific formulation is considered. In (Marino and Tomei, 1999), a SISO nonlinear system with unknown parameters is considered and it is transformable to be a minimum phase system where the nonlinearity depends on the output only. It is the same situation as in (Ezal *et al.*, 2001) where the nonlinearity also solely depends on the measured output and the disturbance attenuation with local optimality can be realized according to the linearized plant. From these references, we note that it would be interesting to consider the disturbance attenuation task for a more general nonlinear system.

Many kinds of observers have been proposed for linear and nonlinear systems when the state variables are not available. When there are uncertainties in the system, high gain observer owns the advantage to acquire the states information while neglecting the influence of disturbance (Farza *et al.*, 2004), Due to the high gain in the feedback form, the observer is effective in ensuring the convergence of the estimation error such that the real states can be obtained for the controller design.

In this paper, we present a more general system plant with null space dynamics and range space dynamics together. Only partial states can be measured and one subsystem in the null space dynamics possesses a certain property with respect to the Lyapunov stability theory. Our approach has two main objectives: (i) to globally stabilize the nonlinear system, in the input-to-state (ISS) sense, in the existence of the structural unknown system uncertainties and external disturbance by an output feedback discontinuous controller, and (ii) to attenuate the disturbance influence on the system performance to a prescribed level. The attribute of this approach is that: (a) we can construct a high gain nonlinear observer to observe the states and only partial state estimation of the nonlinear system is necessary; (b) a resulting discontinuous controller can be designed according to the backstepping like design procedure to ensure the stability of the nonlinear system; (c) the design parameters in the observer and the controller are related and should be determined together to ensure stability. Hence the whole system can be stabilized while internal stability of the system states is also ensured. Usually, a discontinuous term is used to handle the matched $L_\infty[0,\infty)$ type system disturbance where the upper-bound knowledge is available (Utkin, 1992) (Xu *et al.*, 2003). However in this paper, a discontinuous term is used to ensure convergence of the observer since the switching surface is defined to be a combination of the estimated states while there are no uncertain term in the observer dynamics. In the example, it is shown that the stabilization can be achieved under the proposed controller while the high nonlinear system is originally not stable without control efforts.

Notations: $\mathcal{R}^n$ denotes an n-dimension real vector space; $\|\cdot\|$ is the Euclidean norm and induced matrix norm; $\lambda_i(A)$ denotes the i-th eigenvalue of the matrix A; $Re(\cdot)$ denotes the real part with respect to its argument; $D_{\mathbf{x}}f = \frac{\partial f(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}}$ and $D_{\mathbf{y}}f = \frac{\partial f(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}}$ are row vectors.

2. PROBLEM FORMULATION

A general nonlinear system with control input $\mathbf{u}(\cdot)$ and uncertainties $\mathbf{d}(\cdot)$ can be written as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \mathbf{d}). \tag{1}$$

Without any additional assumption, it is very difficult to design a suitable control law to stabilize this nonlinear system. Hence, in this paper we study a class of nonlinear systems in a cascade form and only output information is available. More explicitly, we consider a system of the form

$$\begin{cases} \dot{\mathbf{z}} = \mathbf{f}_0(t, \mathbf{z}, \mathbf{y}) + G_0(t, \mathbf{z}, \mathbf{y})\mathbf{d}(t) \\ \dot{\mathbf{x}}_1 = A\mathbf{x}_2 + \mathbf{f}_1(t, \mathbf{z}, \mathbf{x}_1) + G_1(t, \mathbf{z}, \mathbf{x}_1)\mathbf{d}(t) \\ \dot{\mathbf{x}}_2 = B\mathbf{u} + \mathbf{f}_2(t, \mathbf{z}, \mathbf{x}) + G_2(t, \mathbf{z}, \mathbf{x})\mathbf{d}(t) \\ \mathbf{y} = \mathbf{x}_1 := C\mathbf{x} \end{cases} \tag{2}$$

where $\mathbf{z} \in \mathcal{R}^p$, $\mathbf{x}_1 \in \mathcal{R}^n$ and $\mathbf{x}_2 \in \mathcal{R}^m$ are the states, $\mathbf{u} \in \mathcal{R}^m$ denotes the control input, $\mathbf{d} \in \mathcal{R}^l$ is the external disturbance. $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{m\times m}$ and $C \in \mathcal{R}^{n\times(n+m)}$. The mappings $\mathbf{f}_0 \in \mathcal{R}^p$ and $G_0 \in \mathcal{R}^{p\times l}$ are known and smooth with respect to $\mathbf{z}$, $\mathbf{y}$ and continuous with respect to time t. $\mathbf{f}_1 \in \mathcal{R}^n$, $G_1 \in \mathcal{R}^{n\times l}$, $\mathbf{f}_2 \in \mathcal{R}^m$ and $G_2 \in \mathcal{R}^{m\times l}$ are unknown functions. The relation $m \leq n$ holds for the system.

This system maintains the popular triangular structure used in the backstepping approaches (Khalil, 1996) (Isidori and Lin, 1998), but extended with additional structure given by the uncertainty terms and disturbances. In addition, only the output $\mathbf{y} = \mathbf{x}_1$, but not the state, is assumed to be available. The control objective of this paper is to stabilize the system, which is originally not ISS stable in the existence of the external disturbance input when $\mathbf{u} = 0$, to be ISS stable with respect to the external disturbance by using the measurable output information. The system in (2) satisfies the following assumptions.

Assumption 1. $\|\mathbf{f}_1\|^2 \leq c_{11}\|\mathbf{x}_1\|^2 + c_{12}\|\mathbf{z}\|^2$, $\|\mathbf{f}_2\|^2 \leq c_{21}\|\mathbf{x}\|^2 + c_{22}\|\mathbf{z}\|^2$, where c_{ij} $(i = 1, 2, j = 1, 2)$ are known positive constants. $\|G_i(\cdot)\| \leq \beta_i$, where β_i are positive constants.

Assumption 2. The function $\mathbf{f}_2$ has the following property,

$$\|\mathbf{f}_2(t, \mathbf{z}, \mathbf{x}) - \hat{\mathbf{f}}_2(t, \mathbf{x}_1, \hat{\mathbf{x}}_2)\|^2 \leq \alpha_1\|\mathbf{e}\|^2 + \alpha_2\|\mathbf{z}\|^2, \tag{3}$$

where $\hat{\mathbf{f}}_2(t, \mathbf{x}_1, \hat{\mathbf{x}}_2)$ is the estimate of the function $\mathbf{f}_2(t, \mathbf{z}, \mathbf{x})$, $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$, $\alpha_1 > 0$ and $\alpha_2 > 0$ are known positive constants.

We further make the following assumption on the null space $\mathbf{z}$-dynamics according to the definition of the ISS Lyapunov functions (Sontag, 1989) (Marquez, 2003). In the $\mathbf{z}$-dynamics, $\mathbf{y}$ and $\mathbf{u}$ can be treated as two external inputs.

Assumption 3. There is a nonempty set of ISS *Lyapunov* Functions, $\mathcal{V}$, such that, for any choice of $\mathcal{C}^1$ function $V_0(t, \mathbf{z}) \in \mathcal{V} : \mathcal{R}^n \times \mathcal{R}^+ \to \mathcal{R}^+$,

$$\begin{aligned}
&\gamma_1(\|\mathbf{z}\|) \leq V_0(t, \mathbf{z}) \leq \gamma_2(\|\mathbf{z}\|),\\
&D_t V_0 + (D_{\mathbf{z}} V_0)\left[\mathbf{f}_0(t, \mathbf{y}, \mathbf{z}) + G_0(t, \mathbf{y}, \mathbf{z})\mathbf{d}(t)\right]\\
\leq\ & -\gamma_3\|\mathbf{z}\|^2 + \gamma_4\|\mathbf{y}\|^2 + \gamma_5\|\mathbf{d}\|^2,
\end{aligned}\tag{4}$$

where $\gamma_1(\cdot), \gamma_2(\cdot) : \mathcal{R}^+ \to \mathcal{R}^+$ are class $\mathcal{K}_\infty$ functions, γ_3, γ_4 and γ_5 are positive constants.

3. HIGH GAIN OBSERVER DESIGN

In this section, a nonlinear observer with high gain is proposed for the partial system dynamics to estimate the state $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]^T$. Based on the structure of the system plant, the observer is constructed as the follows

$$\begin{cases} \dot{\hat{\mathbf{x}}}_1 = A\hat{\mathbf{x}}_2 + kK_1(\mathbf{x}_1 - \hat{\mathbf{x}}_1) \\ \dot{\hat{\mathbf{x}}}_2 = B\mathbf{u} + \hat{\mathbf{f}}_2(t, \mathbf{x}_1, \hat{\mathbf{x}}_2) + kK_2(\mathbf{x}_1 - \hat{\mathbf{x}}_1). \end{cases}\tag{5}$$

Define $\mathbf{e}_i = \mathbf{x}_i - \hat{\mathbf{x}}_i$, then the error dynamics is

$$\begin{bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} -kK_1 & A \\ -kK_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1 + G_1\mathbf{d} \\ \mathbf{f}_2 - \hat{\mathbf{f}}_2 + G_2\mathbf{d} \end{bmatrix}\tag{6}$$

In order to extract the design variable k in the presentation of the state space equation (6), we define $\mathbf{e} = [\mathbf{e}_1, \mathbf{e}_2/k]^T$. Then

$$\dot{\mathbf{e}} = kE_a\mathbf{e} + \mathbf{f} + G\mathbf{d}\tag{7}$$

where $\mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 - \hat{\mathbf{f}}_2/k \end{bmatrix}$, $G = \begin{bmatrix} G_1 \\ G_2/k \end{bmatrix}$, $k > 1$ is a positive constant, K_1 and K_2 are designed gains such that the matrix $E_a = \begin{bmatrix} -K_1 & A \\ -K_2 & \mathbf{0} \end{bmatrix}$ is Hurwitz, i.e, $\exists P > 0$ such that $PE_a + E_a^T P = -Q$.

According to Assumption 1 and Assumption 2, the terms $\mathbf{f}$ and $G\mathbf{d}$ in (7) could be upper-bounded as

$$\begin{aligned}
\|\mathbf{f}\|^2 &\leq \|\mathbf{f}_1\|^2 + \frac{\|\mathbf{f}_2 - \hat{\mathbf{f}}_2\|}{k^2}\\
&\leq c_{11}\|\mathbf{x}_1\|^2 + (c_{12} + \frac{\alpha_2}{k^2})\|\mathbf{z}\|^2 + \frac{\alpha_1}{k^2}\|\mathbf{e}\|^2\\
\|G\mathbf{d}\| &\leq max\{\beta_i\}(1 + \frac{1}{k})\|\mathbf{d}\|\\
&\leq 2max\{\beta_i\}\|\mathbf{d}\| \triangleq \beta\|\mathbf{d}\|, \quad i = 1, 2,
\end{aligned}$$

where $\beta \triangleq 2max\{\beta_i\}$.

Remark 1. Note that the term $(\mathbf{f}+G\mathbf{d})$ is bounded by the states, hence we cannot ensure a bounded state estimation error by this observer design. Fortunately, we have one degree of freedom to increase the value of k. Furthermore, as shown in the following section, we can design a nonlinear discontinuous controller and also can increase k such that the stability of the whole controlled system can be finally ensured.

Remark 2. Assumption 2 implies that the function $\mathbf{f}_2(\cdot)$ cannot be highly nonlinear with respect to the argument $\mathbf{x}_2$.

4. CONTROLLER DESIGN AND STABILITY ANALYSIS

Since the system in (2) has a cascade form, we can apply a backstepping like method to design the controller as summarized in Theorem 1.

Theorem 1. Under the control law designed as

$$\mathbf{u} = \mathbf{u}_c + \mathbf{u}_s, \tag{8}$$

$$\begin{aligned}\mathbf{u}_c = -\Gamma^{-1}\Big[&D_t\boldsymbol{\sigma} + (D_{\hat{\mathbf{x}}_1}\boldsymbol{\sigma})(A\hat{\mathbf{x}}_2 + kK_1\mathbf{e}_1)\\ &+(D_{\hat{\mathbf{x}}_2}\boldsymbol{\sigma})(\hat{\mathbf{f}}_2(t, \mathbf{x}_1, \hat{\mathbf{x}}_2) + kK_2\mathbf{e}_1)\Big],\end{aligned}\tag{9}$$

$$\mathbf{u}_s = -k_s\frac{\Gamma^T\boldsymbol{\sigma}}{\|\Gamma^T\boldsymbol{\sigma}\|}, \tag{10}$$

where $\boldsymbol{\sigma} = \hat{\mathbf{x}}_2 + \frac{A^T}{r(t, \hat{\mathbf{x}})}\hat{\mathbf{x}}_1 \in \mathcal{R}^m$, $\Gamma = (D_{\hat{\mathbf{x}}_2}\boldsymbol{\sigma})B \in \mathcal{R}^{m\times m}$, the designed function $r(t, \hat{\mathbf{x}}) > 0$ is a positive scalar function and $k_s > 0$ is a positive constant, the system is globally ISS stable with respect to the external disturbance input.

Proof: The proof can be seperated into the following three steps.

Step 1: Construct a Lyapunov function $V_1(t,\mathbf{z},\mathbf{e}) = V_0(t,\mathbf{z}) + \mathbf{e}^T P\mathbf{e}$ where V_0 satisfies (4), and select γ_6 such that $\frac{c_{12}k^2+\alpha_2}{2\gamma_6 k^2} < \gamma_3$, then the derivative of the Lyapunov function $V_1(\cdot)$ becomes

$$\begin{aligned}
&\dot{V}_1(t,\mathbf{z},\mathbf{e}) = \dot{V}_0(t,\mathbf{z}) + k\mathbf{e}^T(PE_a + E_a^T P)\mathbf{e} \\
&+2\mathbf{e}^T P\mathbf{f} + 2\mathbf{e}^T PG\mathbf{d} \\
&\le -\gamma_3\|\mathbf{z}\|^2 + \gamma_4\|\mathbf{y}\|^2 + \gamma_5\|\mathbf{d}\|^2 - k\|Q\|\|\mathbf{e}\|^2 \\
&+2\gamma_6\|P\|^2\|\mathbf{e}\|^2 + \frac{1}{2\gamma_6}\left[\left(c_{12} + \frac{\alpha_2}{k^2}\right)\|\mathbf{z}\|^2 + c_{11}\|\mathbf{x}_1\|^2\right. \\
&\left.+\frac{\alpha_1}{k^2}\|\mathbf{e}\|^2\right] + 2\|P\|^2\|\mathbf{e}\|^2 + \frac{\beta^2}{2}\|\mathbf{d}\|^2 \\
&= -\left[\gamma_3 - \frac{(c_{12}k^2+\alpha_2)}{2\gamma_6 k^2}\right]\|\mathbf{z}\|^2 + \left(\gamma_4 + \frac{c_{11}}{2\gamma_6}\right)\|\mathbf{y}\|^2 \\
&-\left(k\|Q\| - 2\gamma_6\|P\|^2 - 2\|P\|^2 - \frac{\alpha_1}{2\gamma_6 k^2}\right)\|\mathbf{e}\|^2 \\
&+\left(\frac{\beta^2}{2} + \gamma_5\right)\|\mathbf{d}\|^2.
\end{aligned}$$

Define $\nu = \gamma_3 - \frac{(c_{12}k^2+\alpha_2)}{2\gamma_6 k^2}$, $\mu = \frac{\beta^2}{2} + \gamma_5$ and $\delta = k\|Q\| - 2\gamma_6\|P\|^2 - 2\|P\|^2 - \frac{\alpha_1}{2\gamma_6 k^2}$, then

$$\begin{aligned}
&\dot{V}_1(t,\mathbf{z},\mathbf{e}) + \frac{\mu\|\mathbf{y}\|^2}{\rho^2} - \mu\|\mathbf{d}\|^2 \\
&\le\ -\nu\|\mathbf{z}\|^2 - \delta\|\mathbf{e}\|^2 + \left(\gamma_4 + \frac{c_{11}}{2\gamma_6} + \frac{\mu}{\rho^2}\right)\|\mathbf{y}\|^2.
\end{aligned}$$

Step 2: In this step we would like to find a desired signal $\hat{\mathbf{x}}_2^*$ to stabilize the $\hat{\mathbf{x}}_1$ subsystem. First construct a new Lyapunov function $V_2(t,\mathbf{z},\mathbf{e},\hat{\mathbf{x}}_1) = V_1(t,\mathbf{z},\mathbf{e}) + \frac{1}{2}\hat{\mathbf{x}}_1^T\hat{\mathbf{x}}_1$, then

$$\dot{V}_2 = \dot{V}_1 + \hat{\mathbf{x}}_1^T A\hat{\mathbf{x}}_2 + \hat{\mathbf{x}}_1^T kK_1\mathbf{e}_1, \tag{11}$$

a desired $\hat{\mathbf{x}}_2^* = -\frac{A^T}{r(t,\hat{\mathbf{x}})}\hat{\mathbf{x}}_1$ can be designed, where $r(\cdot)$ is a positive scalar function, then (11) becomes $(q > 0)$

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 - \frac{\|A\|^2}{r}\|\hat{\mathbf{x}}_1\|^2 + k\hat{\mathbf{x}}_1^T K_1\mathbf{e}_1 \\
&\le\ \dot{V}_1 - \left(\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q}\right)\|\hat{\mathbf{x}}_1\|^2 + \frac{q}{2}\|\mathbf{e}_1\|^2. \\
\Rightarrow\quad &\dot{V}_2 + \frac{\mu\|\mathbf{y}\|^2}{\rho^2} - \mu\|\mathbf{d}\|^2 \\
&\le\ -\nu\|\mathbf{z}\|^2 - \delta\|\mathbf{e}\|^2 + (\gamma_4 + \frac{c_{11}}{\gamma_6} + \frac{\mu}{\rho^2})\|\mathbf{y}\|^2 \\
&-\left(\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q}\right)\|\hat{\mathbf{x}}_1\|^2 + \frac{q}{2}\|\mathbf{e}_1\|^2 \quad (12)
\end{aligned}$$

If $\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q} \ge\ 0$ is to be ensured, and using $\|\hat{\mathbf{x}}_1\|^2 = \|\mathbf{x}_1 - \mathbf{e}_1\|^2 \ge\ \|\mathbf{x}_1\|^2 - \|\mathbf{e}_1\|^2$ and $\|\mathbf{e}_1\|^2 \le\ \|\mathbf{e}\|^2$, then (12) becomes

$$\begin{aligned}
&\dot{V}_2 + \frac{\mu\|\mathbf{y}\|^2}{\rho^2} - \mu\|\mathbf{d}\|^2 \\
&\le\ -\nu\|\mathbf{z}\|^2 - \left[\delta - \frac{q}{2} - \left(\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q}\right)\right]\|\mathbf{e}\|^2 \\
&-\left[\left(\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q}\right)\right. \\
&\left.-\left(\gamma_4 + \frac{c_{11}}{\gamma_6} + \frac{\mu}{\rho^2}\right)\right]\|\mathbf{y}\|^2,
\end{aligned} \tag{13}$$

where r and q are designed such that

$$\begin{aligned}
\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q} &\ge\ 0, \\
\left[\delta - \frac{q}{2} - \left(\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q}\right)\right] &\ge\ 0, \\
\left[\frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q} - (\gamma_4 + \frac{c_{11}}{\gamma_6} + \frac{\mu}{\rho^2})\right] &\ge\ 0,
\end{aligned}$$

should be satisfied at the same time. The feasibility of the proposed controller is based on whether we can obtain such a solution $r(\cdot)$ to satisfy all the three conditions, i.e,

$$\begin{aligned}
\delta - \frac{q}{2} \ge \frac{\|A\|^2}{r} - \frac{k^2\|K_1\|^2}{2q} \\
\ge\ \gamma_4 + \frac{c_{11}}{\gamma_6} + \frac{\mu}{\rho^2} > 0.
\end{aligned} \tag{14}$$

Hence if we obtain a solution $r(\cdot)$ to satisfy the inequality in (14), then (13) becomes $\dot{V}_2 + \frac{\mu\|\mathbf{y}\|^2}{\rho^2} - \mu\|\mathbf{d}\|^2 \le\ 0$.

Step 3: In this step, we would like to design a robust discontinuous control signal to realize the regulation problem. To begin with, design the switching surface as $\boldsymbol{\sigma} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_2^* = \hat{\mathbf{x}}_2 + \frac{A^T}{r(t,\hat{\mathbf{x}})}\hat{\mathbf{x}}_1 \in \mathcal{R}^m$. If all the states are measurable, the switching surface should be selected as a function of the states $\boldsymbol{\sigma}(\mathbf{x}_1,\mathbf{x}_2)$. However, here the states are not measurable, so we use the estimated one $\hat{\mathbf{x}}$ instead. Define a fourth Lyapunov function $V_3(t,\mathbf{z},\mathbf{e},\hat{\mathbf{x}}_1,\boldsymbol{\sigma}) = V_2(t,\mathbf{z},\mathbf{e},\hat{\mathbf{x}}_1) + \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\sigma}$. Then

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &= D_t\boldsymbol{\sigma} + (D_{\hat{\mathbf{x}}_1}\boldsymbol{\sigma})(A\hat{\mathbf{x}}_2 + kK_1\mathbf{e}_1) \\
&+(D_{\hat{\mathbf{x}}_2}\boldsymbol{\sigma})(B\mathbf{u} + \hat{\mathbf{f}}_2(t,\mathbf{x}_1,\hat{\mathbf{x}}_2) + kK_2\mathbf{e}_1).
\end{aligned}$$

Design $\mathbf{u}$ as in equation (8) in Theorem 1, then $\dot{V}_3 = \dot{V}_2 + \boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}}$. Hence

$$\begin{aligned}
&\dot{V}_3 + \frac{\mu\|\mathbf{y}\|^2}{\rho^2} - \mu\|\mathbf{d}\|^2 \\
&\le\ \boldsymbol{\sigma}^T\left[D_t\boldsymbol{\sigma} + (D_{\hat{\mathbf{x}}_1}\boldsymbol{\sigma})(A\hat{\mathbf{x}}_2 + kK_1\mathbf{e}_1)\right. \\
&\left.+(D_{\hat{\mathbf{x}}_2}\boldsymbol{\sigma})(B\mathbf{u} + \hat{\mathbf{f}}_2(t,\mathbf{x}_1,\hat{\mathbf{x}}_2) + kK_2\mathbf{e}_1)\right] \\
&= -k_s\|\Gamma^T\boldsymbol{\sigma}\| \le\ 0.
\end{aligned} \tag{15}$$

Integrating both sides of (15), and simplifying the definition $V_3(t) = V_3(t,\mathbf{z}(t),\mathbf{e}(t),\hat{\mathbf{x}}_1(t),\boldsymbol{\sigma}(t))$ and $V_3(0) = V_3(0,\mathbf{z}(0),\mathbf{e}(0),\hat{\mathbf{x}}_1(0),\boldsymbol{\sigma}(0))$, we have

$$V_3(t) - V_3(0) \leq \; -\frac{\mu}{\rho^2}\int_0^t \|\mathbf{y}\|^2 d\tau + \mu\int_0^t \|\mathbf{d}\|^2 d\tau, \tag{16}$$

$$\Rightarrow \int_0^t \|\mathbf{y}\|^2 d\tau \leq \; \beta_v(\cdot) + \rho^2 \int_0^t \|\mathbf{d}\|^2 d\tau, \tag{17}$$

where $\beta_v(\cdot) \triangleq \frac{\rho^2}{\mu} V_3(\cdot)$ is a function related to the initial condition only. ■

From (15), we have $\boldsymbol{\sigma}^T\dot{\boldsymbol{\sigma}} \leq \; -k_s\|\Gamma^T\boldsymbol{\sigma}\| < 0$, hence it is straightforward that the sliding manifold will be reached in finite time.

Corollary 1. Under the proposed output feedback controller (8) - (10), we have: (a) if $\mathbf{d} \in L_2[0,\infty)$, all the system states are bounded; (b) if $\mathbf{d} \in L_2[0,\infty) \cap L_\infty[0,\infty)$, $\lim_{t\to\infty}\mathbf{y}(t) = 0$, and $\mathbf{z}$, $\mathbf{x}_2$ are bounded.

Proof: (a) If $\mathbf{d} \in L_2[0,\infty)$, then $\int_0^t \|\mathbf{d}\|^2 d\tau \leq M_d$, where M_d is a finite constant. From (16),

$$\begin{aligned} &V_3(t, \mathbf{z}(t), \mathbf{e}(t), \hat{\mathbf{x}}_1(t), \boldsymbol{\sigma}(t)) \\ &\leq \; V_3(0, \mathbf{z}(0), \mathbf{e}(0), \hat{\mathbf{x}}_1(0), \boldsymbol{\sigma}(0)) + \mu M_d. \end{aligned} \tag{18}$$

Because $V_3(\cdot)$ is radially unbounded in $\mathbf{z}$, $\mathbf{e}$, $\hat{\mathbf{x}}$ and $\boldsymbol{\sigma}$, (18) means that $\mathbf{z}$, $\mathbf{e}$, $\hat{\mathbf{x}}$ and $\boldsymbol{\sigma}$ are bounded. Furthermore $\mathbf{x}_1 = \mathbf{e}_1 + \hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2 = \boldsymbol{\sigma} - \dfrac{A^T}{r}\hat{\mathbf{x}}_1$, $\mathbf{x}_2 = \mathbf{e}_2 + \hat{\mathbf{x}}_2$, hence $\mathbf{x}$ is also bounded.

(b) If $\mathbf{d} \in L_2[0,\infty) \cap L_\infty[0,\infty)$, then we have $\int_0^t \|\mathbf{d}\|^2 d\tau \leq M_d$ and $\|\mathbf{d}\| \leq \epsilon_d$, where ϵ_d is a constant. Inequality (15) becomes $\dot{V}_3 \leq -\frac{\mu}{\rho^2}\|\mathbf{y}\|^2 + \mu\epsilon_d^2$, which shows that $\|\mathbf{y}\| \leq \rho\epsilon_d$ is bounded. Thus from the system dynamic in (2), $\dot{\mathbf{x}}_1$ is bounded and as a result $\mathbf{y} = \mathbf{x}_1$ is uniformly continuous. Note that in (17), $\int_0^t \|\mathbf{y}\|^2 d\tau$ is bounded because $\mathbf{d} \in L_2[0,\infty)$. Using *Barbalat's Lemma* (Narendra and Annaswamy, 1989), it is straightforward to reach the conclusion that $\lim_{t\to\infty}\mathbf{y}(t) = 0$. ■

Note that the discontinuous unit vector control law $\mathbf{u}_s$ in (10) may cause chattering when the system enters the sliding mode in a finite time. In order to eliminate the chattering phenomenon, $\mathbf{u}_s$ can be modified as

$$\mathbf{u}_s = -k_s \frac{\Gamma^T\boldsymbol{\sigma}}{\|\Gamma^T\boldsymbol{\sigma}\| + \varepsilon e^{-\lambda t}}, \tag{19}$$

where ε and λ are positive constants.

Corollary 2. Consider the uncertain nonlinear system in (2), with $\mathbf{d} \in L_2[0,\infty)$, the controller in (8), (9) and (19) guarantees that: (i) a finite L_2 gain performance is achieved; and (ii) all the state variables are bounded. Moreover, if $\mathbf{d} \in L_2[0,\infty) \cap L_\infty[0,\infty)$, then $\mathbf{y}$ converges to zero asymptotically.

Proof: The proof is similar as in Corollary 1. ■

5. ILLUSTRATIVE EXAMPLE

In this section, the nonlinear system as in (2) is considered with the z dynamics of the form

$$\dot{z} = -z^3 + zy_1 + zy_2 + zd_1(t) + zd_2(t), \tag{20}$$

where $\mathbf{y} = \mathbf{x}_1 = [y_1, y_2]^T$, $\mathbf{x}_1 = [x_{11}, x_{12}]^T$, $\mathbf{x}_2 = [x_{21}, x_{22}]^T$, $\mathbf{d} = [d_1(t), d_2(t)]^T = [e^{-0.1t}, e^{-0.5t}]$, $\mathbf{f}_1(t, \mathbf{z}, \mathbf{x}_1) = [h_1(t)x_{11}sin(z), h_2(t)x_{12}cos(z)]^T$, $h_1(t) = 0.1sin(\pi t)$, $h_2(t) = 0.1cos(\pi t)$, $A = B = I_{2\times 2}$, $\mathbf{f}_2(t, \mathbf{z}, \mathbf{x}) = [x_{21}sin(x_{11}), x_{22}sin(x_{12})]^T$, and

$$G_1(t, \mathbf{z}, \mathbf{x}_1) = 0.5\begin{bmatrix} cos(x_{11}) & cos(x_{12}) \\ sin(x_{11}) & sin(x_{12}) \end{bmatrix},$$

$$G_2(t, \mathbf{z}, \mathbf{x}) = 0.5\begin{bmatrix} sin(x_{21}) & sin(x_{22}) \\ cos(x_{21}) & cos(x_{22}) \end{bmatrix}.$$

For z dynamics, $V_0(\mathbf{z}, t) = \frac{1}{2}z^2$ is selected. Hence we have $\gamma_3 = 0.75$, $\gamma_4 = 2$ and $\gamma_5 = 2$. The following parameters can be $\alpha_1 = 1$, $\alpha_2 = 0$, $c_{11} = 1$, $c_{12} = 0$, $q = 4$, $\gamma_6 = 0.5$, $\beta = \sqrt{2}$ and $\mu = 1.5$. In the observer design: $K_1 = K_2 = I_{2\times 2}$, $k = 18$. The minimum disturbance attenuation level in this case is $\rho = \sqrt{3}$. For an identity $Q = I_{2\times 2}$, from $PE_a + E_a^T P = -Q$, we have

$$P = \begin{bmatrix} 1.0 & 0 & -0.5 & 0 \\ 0 & 1.0 & 0 & -0.5 \\ -0.5 & 0 & 1.5 & 0 \\ 0 & -0.5 & 0 & 1.5 \end{bmatrix}.$$

Hence $\boldsymbol{\sigma} = \hat{\mathbf{x}}_2 + \frac{A^T}{r}\hat{\mathbf{x}}_1$, where r is selected as $0.02197 \leq r = 0.0222 \leq 0.02222$. In the controller design, $k_s = 1$ is selected. The initial conditions are $[z(0), \mathbf{x}_1(0)^T, \mathbf{x}_2(0)^T]^T = [1, 5, 4, 3, 2]^T$ and $\hat{\mathbf{x}}(0) = \mathbf{0}$.

Simulation results are shown as the follows. In $Fig.1$, $\mathbf{u} = \mathbf{0}$ is first applied. It is shown that the states diverge which means that the system is not stable without an output feedback stabilization controller. In the following discussion, according to (19), a smooth function is then constructed instead of sign function as $\mathbf{u}_s = -k_s\dfrac{\boldsymbol{\sigma}}{\|\boldsymbol{\sigma}\| + 0.1e^{-0.01t}}$. In $Fig.2(a)(b)$,the estimated states from the observer and the real states are compared. The estimation errors converge to zero asymptotically as shown in $Fig.2(c)(d)$. Since our control target is to minimize the desired L_2 disturbance attenuation level. From $Fig.3$, the output integration term $\int_0^t \|\mathbf{y}(\tau)\|^2 d\tau$ is smaller than the disturbance term $\int_0^t \|\mathbf{d}(\tau)\|^2 d\tau$ which means the real disturbance attenuation level is $\rho_{real} < 1$. Hence it is obvious that the desired attenuation level $\rho = \sqrt{3}$ derived from theory is obtained finally. Also the switching surface profile is as shown in $Fig.4$.

6. CONCLUSIONS

For a class of nonlinear system with unknown systems uncertainties and external disturbances, we have realized an output feedback control law based on a high gain nonlinear observer that achieves desired global Input-to-State Stability with disturbance attenuation. The problem dealt with in this paper has triangular structure which is a general form in dealing with output feedback stabilization problems.

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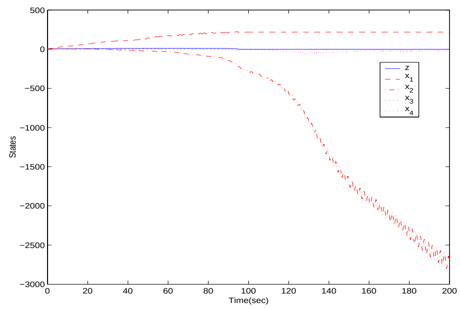

Fig. 1. The system states without controller.

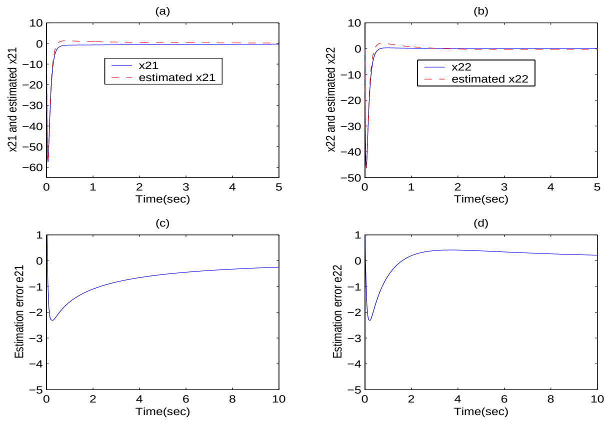

Fig. 2. (a) $x_{21}(t)$ and $\hat{x}_{21}(t)$; (b) $x_{22}(t)$ and $\hat{x}_{22}(t)$; (c) $e_{21}(t) = x_{21}(t) - \hat{x}_{21}(t)$; (c) $e_{22}(t) = x_{22}(t) - \hat{x}_{22}(t)$.

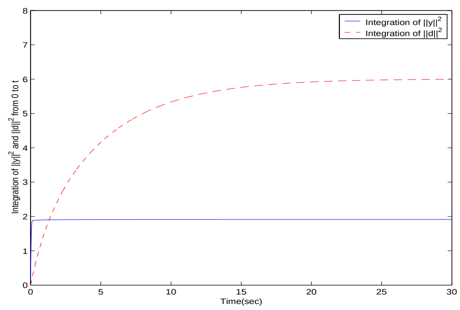

Fig. 3. The integration of $\|\mathbf{y}\|^2$ and $\|\mathbf{d}\|^2$.

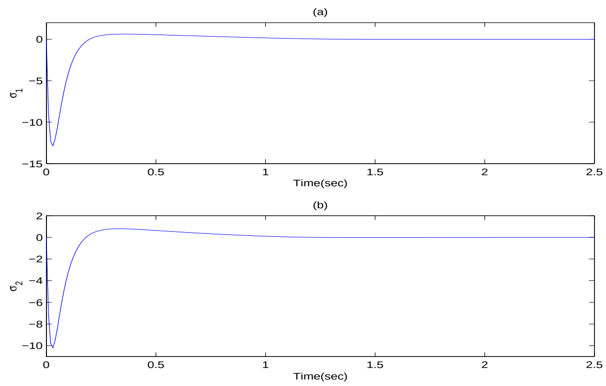

Fig. 4. The evolution of $\boldsymbol{\sigma}(t)$: (a) $\sigma_1(t)$; (b) $\sigma_2(t)$.