WEIGHTED SENSITIVY MINIMIZATION IN THE PRESENCE OF AN UNCERTAIN GAIN

Daniel E. Miller *

* *Department of Electrical and Computer Engineering,*
University of Waterloo, Canada, N2L 3G1.
email: miller@hobbes.uwaterloo.ca
Phone: 519-885-1211 ext. 5215
FAX: 519-746-3077

Abstract: In this paper we consider the use of linear periodic controllers (LPCs) to minimize the weighted sensitivity function in the face of a multiplicative gain uncertainty. We show that, under a technical assumption on the single-input single-output (siso) plant (it is relative degree one), on the weighting function (it is strictly proper), and on the multiplicative gain (it lies in a compact set not including zero), there exists a LPC which can provide a near LTI-optimal weighted sensitivity for every admissible gain.

Keywords: Disturbance rejection, gain margin, robust performance, sampled-data systems, periodic control.

1. INTRODUCTION

In this paper we consider a robust performance problem, namely that of minimizing the weighted sensitivity function in the face of a multiplicative gain uncertainty. It is well known that if a linear time-invariant (LTI) plant is unstable and non-minimum phase, then there is an upper bound on the gain margin achievable using an LTI controller (Khargonekar and Tannenbaum, 1985); however, it is also known that the gain margin can be made arbitrarily large using a LPC (e.g. Francis and Georgiou, 1988; Yang and Kabamba, 1994). On the other hand, it is well known that in carrying out sensitivity minimization when there is no plant uncertainty, there is no advantage to turning to time-varying or nonlinear controllers when using $2-$ norms on signals (Khargonekar and Poolla, 1986); when using the ∞-norm, there is no advantage to using time-varying controllers, at least in the discrete-time case (Shamma and Dahleh, 1991), although there can be an advantage to using nonlinear controllers (Stoorvogel, 1994), at least in the multi-input multi-output case. Furthermore, it is shown in Yan and Anderson (1990) that if one approaches the aforementioned robust performance problem using an LTI controller, then for an unstable nonminimum phase plant the size of sensitivity function (when using the $2-$norm) tends to infinity as the gain margin required tends toward the maximum attainable. This motivated the work of Yan, Anderson, Bitmead (1994) where it is shown that, under suitable assumptions, one can use a LPC to make the gain margin be large as desired while ensuring that the size of the sensitivity function remains bounded. Here our goal is to show that, under suitable assumptions, one can achieve any desired gain margin while ensuring that the size of the weighted sensitivity function is near LTI-optimal; indeed, we allow for a much more general gain, namely that of a compact set not containing zero, which means that it can include ones of both signs. Hence, from a certain point of view there is no cost in terms of performance by allowing an uncertain gain in the plant model. Here we adopt the $\infty-$norm to measure our signal size.

The approach that we adopt is as follows. First, we observe that if the LTI controller $C_0(s)$ stabilizes the nominal plant $P_0(s)$, then the LTI controller $\frac{1}{k}C_0(s)$ stabilizes $kP_0(s)$ and yields the same weighted sensitivity function. Hence, we start with an LTI controller $C_0(s)$ which stabilizes $P_0(s)$ and provides near optimal performance. We then adopt the following periodic control mechanism: during the period $[jT,(j+1)T)$ we have three phases: the Estimation Phase, the Reset Phase, and the Control Phase. During the Estimation Phase we probe the plant and (linearly) estimate the quantity $\frac{1}{k}e(jT)$; during the Reset Phase we approximately erase the effect of the probing on the plant, and in parallel apply the (suitably scaled) estimate of $\frac{1}{k}e(jT)$ to the model of $C_0(s)$; during the Control Phase we first sample the output of $C_0(s)$ and then apply this (suitably scaled) to the plant. We end up with a stable LPC, parameterized by the period T and the approximation parameter ε, which will provide stability for modest values of T and ε, and which will recover the near-optimal performance as $T \to 0$ and $\varepsilon \to 0$. Since the control signal is recomputed during each period, we would expect that the controller will tolerate slow time variations in the uncertain gain, which is a highly desirable feature. The ideas used here in the controller design are related to those used in Miller (2003) in the model reference adaptive control problem.

2. NOTATION

Let $\mathbf{R}$ denote the set of real numbers, $\mathbf{C}$ denote the set of complex numbers, $\mathbf{C}^-$ denote the set of complex numbers with a real part less than zero, and $\mathbf{Z}^+$ denote the set of non-negative integers. We use the Holder ∞-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\|\cdot\|$. We let $\mathcal{P}C_\infty$ denote the set of bounded piecewise continuous signals, [1] , and we measure the size of $f \in \mathcal{P}C_\infty$ by

$$\|f\|_\infty := \sup_{t\geq 0}\|f(t)\|.$$

The norm of a linear operator $G : PC_\infty \to PC_\infty$ is given by

$$\|G\| = \sup_{f\in PC_\infty,\, f\neq 0} \frac{\|Gf\|_\infty}{\|f\|_\infty}.$$

3. PROBLEM FORMULATION

Our nominal siso plant model P_0 is

$$\dot{x}(t) = A\,x(t) + B\,u(t),\;\; x(0) = x_0, \qquad (1)$$
$$y(t) = E\,x(t),$$

[1] We could have used the essentially bounded Lebesgue measureable signals, but it provides a complication: since they are not well-defined pointwise and the norm allows for the occasional extremely large value, we would need to filter the signal before we sample it, which complicates the controller design.

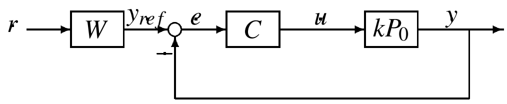

Figure 1: Basic feedback control structure.

with $x(t) \in \mathbf{R}^n$ the state, $u(t) \in \mathbf{R}$ the control input, and $y(t) \in \mathbf{R}$ the plant output. Our standing assumption is

Assumption 1: (A,B) is controllable, (E,A) is observable, and $EB \neq 0$.

We capture uncertainty in the model by supposing that the actual system P is given by

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t),\;\; x(0) = x_0, \\ y(t) &= kE\,x(t), \end{aligned} \qquad (2)$$

with $k \in \mathbf{R}$; we represent this system by the triple (A,B,kE). Our parameter k is assumed to be in a compact set $\mathbf{K}$ not including zero, so our plant model is assumed to lie in

$$\mathcal{P} = \{(A,B,kE) : k \in \mathbf{K}\}.$$

Our feedback configuration is given in Figure 1, with C representing the controller and W representing the weighting function (or filter). The latter is assumed to be finite-dimensional, LTI, low-pass and (naturally) stable: it represents a model of the class of reference signals which are to be tracked - we adopt a state-space representation of

$$\dot{\eta} = K\eta + Lr,\;\; v(0) = v_0, \qquad (3)$$
$$y_{ref} = Mv. \qquad (4)$$

We define closed loop stability in the usual way: with zero initial conditions on the plant, controller and filter, and with w_1 and w_2 fictitious signals introduced at the plant input and output, respectively, closed loop stability means that the map from $\begin{bmatrix} w_1 \\ w_2 \\ r \end{bmatrix} \longmapsto \begin{bmatrix} u \\ y \end{bmatrix}$ has a finite gain. The *sensitivity function* is

$$S_k = (I + kP_0C)^{-1},$$

which represents the map from the reference signal to the tracking error; we define $S := S_1$. The goal is to minimize S_kW in the face of uncertainty in k.

Remark 1. In the gain margin problem we have $\mathbf{K} = [a,b] > 0$. If a controller stabilizes $\mathcal{P}$ then it provides a gain margin of $\frac{b}{a}$ for P_0. It is well-known that there is an upper bound on the gain margin achievable by an LTI controller for a non-minimum phase unstable plant (Khargonekar and Tannenbaum, 1985), although there is no such bound if one uses an LPC, e.g. see (e.g. Francis and Georgiou, 1988; Yang and Kabamba, 1994; Rossi and Miller, 1999). Since we allow gains of both signs here, even when dealing with simple nominal models such as $\frac{1}{s-1}$, one must typically use either a time-varying or nonlinear controller to stabilize $\mathcal{P}$, let alone provide good performance.

Before proceeding, we define

$$\alpha_{lti} := \inf_{C \text{ is LTI and stabilizes } P_0} \|SW\|.$$

For $k \neq 0$, it is clear that C stabilizes P_0 iff $\frac{1}{k}C$ stabilizes kP_0, so

$$\inf_{C \text{ is LTI and stabilizes } kP_0} \|S_k W\| = \alpha_{lti},$$

i.e. the optimal cost is the same for all k. We will be able to prove the following:

Theorem 1. For every $\gamma > 0$ there exists a linear periodic controller C_{LPC} which stabilizes $\mathcal{P}$ and provides the following performance bound:

$$\sup_{k\in\mathbf{K}} \|(I + kP_0 C_{LPC})^{-1} W\| \leq \alpha_{lti} + \gamma.$$

Remark 2. This means that we can tolerate an uncertain gain and still achieve near-optimal LTI performance.

The idea behind the controller goes as follows. First, we start with a stabilizing LTI controller (possibly near optimal) C_{lti} for P_0; the goal is to apply a good approximation of $\frac{1}{k}C_{lti}$. Since k is uncertain, some form of estimation is required. We use a periodic controller of period T which will achieve the objective if T is small enough.

The controller that we adopt here has two components - a continuous-time part and a sampled-data part. We let C_{lti} denote any finite-dimensional LTI control law which stabilizes P_0:

$$\dot{z} = Fz + Ge,\ z(0) = z_0 \in \mathbf{R}^l$$
$$u = Hz + Je.$$

Since we are not implementing this controller directly we rename its input and output:

$$\begin{aligned} \dot{z} &= Fz + Ge^0 \\ u^0 &= Hz + Je^0. \end{aligned} \tag{5}$$

Before defining the sampled-data component we group the plant, C_{lti}, and the filter W together:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & K \end{bmatrix}}_{=:\bar{A}} \underbrace{\begin{bmatrix} x \\ z \\ \eta \end{bmatrix}}_{=:\bar{x}} + \underbrace{\begin{bmatrix} B & 0 \\ 0 & G \\ 0 & 0 \end{bmatrix}}_{=:\bar{B}_2} \underbrace{\begin{bmatrix} u \\ e^0 \end{bmatrix}}_{=:\bar{u}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix}}_{\bar{B}_1} r$$

$$\underbrace{\begin{bmatrix} u^0 \\ e \end{bmatrix}}_{=:\bar{e}} = \underbrace{\begin{bmatrix} 0 & H & 0 \\ -kE & 0 & M \end{bmatrix}}_{=:\bar{E}_k} \begin{bmatrix} x \\ z \\ \eta \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & J \\ 0 & 0 \end{bmatrix}}_{=:\bar{D}} \begin{bmatrix} u \\ e^0 \end{bmatrix}.$$

Notice that if we wish to apply the nominal controller C_{lti} we simply set

$$\bar{u} = \bar{e};$$

however, to apply $\frac{1}{k}C_{lti}$, we would set

$$\bar{u} = \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix} \bar{e}. \tag{6}$$

Of course, since k is unknown we need to do some form of estimation. To do this we use the following sampled-data component:

$$\begin{aligned} \bar{z}[j+1] &= \bar{F}[j]\bar{z}[j] + \bar{G}[j]\bar{e}(jh), z[0] = z_0 \in \mathbf{R}^{\bar{l}} \\ \psi[j] &= \bar{H}[j]\bar{z}[j] + \bar{J}[j]\bar{e}(jh) \\ \bar{u}(t) &= \psi[j],\ t \in [jh, (j+1)h). \end{aligned} \tag{7}$$

Here the controller gains $(\bar{F}, \bar{G}, \bar{H}, \bar{J})$ are periodic of period $p \in \mathbf{N}$; hence, the controller is periodic of period $T := ph$ and we associate it with the 6−tuple $(\bar{F}, \bar{G}, \bar{H}, \bar{J}, h, p)$. Note that (7) can be implemented with a sampler, a zero-order-hold, and an $\bar{l}^{th}$ order periodic discrete-time system of period p .

The idea behind the sampled-data part of the controller (7) goes as follows. First, notice that the control law

$$\bar{u}(t) = \begin{bmatrix} u(t) \\ e^0(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix} \bar{e}(jT) = \begin{bmatrix} u^0(jT) \\ \frac{1}{k}e(jT) \end{bmatrix},$$
$$t \in [jT, (j+1)T) \tag{8}$$

should achieve our objective if T is small enough, since it is clearly a good approximation to (6). Second, with $T_1 \in (0, T)$ and $T_2 \in (T_1, T)$ notice that the control law

$$\bar{u}(t) = \begin{bmatrix} u(t) \\ e^0(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & t \in [jT, jT + T_1) \\ \begin{bmatrix} 0 \\ \frac{T}{k(T_2 - T_1)} e(jT) \end{bmatrix}, & t \in [jT + T_1, jT + T_2) \\ \begin{bmatrix} \frac{T}{T - T_2}[u^0(jT) + \frac{1}{k}Je(jT)] \\ 0 \end{bmatrix}, & t \in [jT + T_2, (j+1)T) \end{cases}$$

should achieve our objective if T is small enough. (Notice that $u^0(jT) = Hz(jT)$.) It turns out that we can design (7) in such a way as to approximate the above. We split each interval $[jT, (j+1)T)$ into three phases, described below in open loop:

- **Estimation Phase**: On the interval $[jT, jT + T_1)$ we probe the plant in order to estimate $\frac{1}{k}e(jT)$.

- **Reset Phase**: On the interval $[jT+T_1, jT+T_2)$ we probe the plant in such a way as to (almost) cancel the effect of the Estimation Phase and to apply the above (suitably scaled) estimate to the LTI compensator C_{lti}.
- **Control Phase**: On the interval $[jT+T_2,(j+1)T)$ we apply a (suitably scaled) output of the LTI compensator C_{lti} to the plant.

At this point we need to further elaborate on the three phases, especially the problematic first phase. Once this is done we will write down our proposed sampled-data controller parameters and then prove Theorem 1.

4. THE CONTROLLER DESIGN

4.1 Phase 1 - Estimation

In this phase we would like to estimate the quantity $\frac{1}{k}e(jT)$. It will turn out that we can estimate $k^i e(jT)$ quite accurately, so the first step is to approximate $\frac{1}{k}$ by a polynomial on $\mathbf{K}$. From the Stone-Weierstrass Approximation Theorem we know that we can approximate it arbitrarily well over $\mathbf{K}$. Indeed, for every $\varepsilon>0$ we can choose a polynomial $\phi_\varepsilon(k)=\sum_{i=0}^q c_i k^i$ so that

$$|1-k\phi_\varepsilon(k)|<\varepsilon,\ \ k\in\mathbf{K}. \quad (9)$$

Proposition 1. There exist constants $\bar{\varepsilon}>0$ and $c>0$ so that, for every $\varepsilon\in(0,\bar{\varepsilon})$ and $k\in\mathbf{K}$, the controller $\phi_\varepsilon(k)C_{lti}$ stabilizes kP_0 and satisfies

$$\|[1+kP_0\phi_\varepsilon(k)C_{lti}]^{-1}W-[1+P_0C_{lti}]^{-1}W\|\le c\varepsilon.$$

Proof: This follows easily from perturbation analysis.

QED

At this point we freeze $\varepsilon\in(0,\bar{\varepsilon})$, $q\in\mathbf{N}$, and $c_i\in\mathbf{R}$ so that

$$|1-k\underbrace{\sum_{i=0}^q c_i k^i}_{=:\phi_\varepsilon(k)}|<\varepsilon,\ \ k\in\mathbf{K}.$$

Hence, the goal becomes that of estimating $\sum_{i=0}^q c_i k^i e(jT)$. The following lemma proves useful.

Lemma 1. (Probing Lemma) With $\delta\in(0,1)$ and $\rho>0$, there exist constants $c>0$ and $\bar{h}>0$ so that for every $\bar{u}\in\mathbf{R}$ and $h\in(0,\bar{h})$, if the control signal

$$u(t)=\rho h^{-\delta}\bar{u},\ \ t\in[t_0,t_0+h]$$

is applied to the plant/filter combination (2)-(4), we have

$$|e(t_0+h)-e(t_0)+\rho kEBh^{1-\delta}\bar{u}|\le$$
$$ch\|\eta(t_0)\|+ch\|x(t_0)\|+ch\|r\|_\infty+ch^{2-\delta}\|\bar{u}\|.$$

Proof: This follows easily from direct analysis of the system equations.

QED

This result provides a mechanism to carry out estimation. With $\rho>0$ a scaling parameter and $\delta\in(0,1)$, on the interval $[jT, jT+h)$ we set

$$u(t)=\rho h^{-\delta}e(jT).$$

We see from above that we should define

$$\text{Est}[ke(jT)]:=\frac{-1}{\rho EB}h^{\delta-1}[e(jT+h)-e(jT)];$$

recalling the definition of e, it follows that the error in our estimate is

$$O(h^\delta)[\|\eta(jT)\|+\|x(jT)\|+\|r\|_\infty]$$

Indeed, if we recursively set

$$u(t)=\rho h^{-\delta}\text{Est}[k^i e(jT)],\ \ t\in[jT+ih, jT+(i+1)h),$$

for $i=1,...,q-1$, then it follows that we should define

$$\text{Est}[k^{i+1}e(jT)]:=$$
$$\frac{-1}{\rho EB}h^{\delta-1}[e(jT+(i+1)h)-e(jT+ih)].$$

At $t=jT+qh$, we have good estimates of

$$k^i e(jT),\ \ i=0,...,q,$$

from which we can form a good estimate of $\frac{1}{k}e(jT)$ for all $k\in\mathbf{K}$. At this point we freeze $p\in\mathbf{N}$ satisfying

$$p>q+1.$$

4.2 The Reset Phase

At the end of the Estimation Phase we have a good estimate of $\frac{1}{k}e(jT)$, but we have disturbed the plant state by approximately

$$\int_{jT}^{jT+qh} e^{A(qh-\tau)} Bu(jT+\tau)\,d\tau \approx \int_{jT}^{jT+qh} Bu(jT+\tau)\,d\tau$$

$$\approx B\rho h^{1-\delta} \sum_{i=0}^{q} \mathrm{Est}[k^i e(jT)].$$

This can be largely undone by setting

$$u(t) = -\rho h^{-\delta} \sum_{i=0}^{q} \mathrm{Est}[k^i e(jT)],$$

for $t \in [jT+qh, jT+(q+1)h)$.

Now we turn to the controller C_{lti}. We define

$$\mathrm{Est}[\frac{1}{k}e(jT)] := \sum_{i=0}^{q} c_i \mathrm{Est}[k^i e(jT)].$$

We would like to apply an input to C_{lti} over the interval $[jT+qh, jT+(q+1)h)$ (and zero for the remainder of the interval $[jT,(j+1)T)$) so that

$$z((j+1)T) \approx e^{FT} z(jT) + \int_0^T e^{F(T-\tau)} G e^0(jT+\tau)\,d\tau$$

which we can achieve by setting

$$e^0(t) = \frac{T}{h}\mathrm{Est}[\frac{1}{k}e(jT)] = p\mathrm{Est}[\frac{1}{k}e(jT)]$$

for $t \in [jT+qh, jT+(q+1)h)$.

4.3 The Control Phase

On the rest of the period, namely $[jT+(q+1)h,(j+1)T)$, we simply apply a control signal to the plant. Given that it is active for only a fraction of the period, it must be scaled accordingly. We would like to set

$$u(t) = \frac{p}{p-q-1}\{Hz(t) + J\frac{1}{k}e(t)\}$$

for $t \in [jT+(q+1)h,(j+1)T)$; instead we use a good approximation, namely

$$u(t) = \frac{p}{p-q-1}\{u^0(jT) + J\mathrm{Est}[\frac{1}{k}e(jT)]\},$$

for $t \in [jT+(q+1)h,(j+1)T)$. (Notice from Section 4.2 that $u^0(jT) = Hz(jT)$.)

4.4 The Proposed Sampled-Data Compensator

At this point we can use the previous three subsections to construct a controller. We first write down the complete control signal and procedure in open loop. To this end, with $\mathrm{Est}[e(jT)] := e(jT)$, we set

$$u(t) = \begin{cases} \rho h^{-\delta}\mathrm{Est}[k^i e(jT)], \\ \qquad t \in [jT+ih, jT+(i+1)h), \\ \qquad i = 0,1,...,q-1 \\ -\rho h^{-\delta}\sum_{i=0}^{q}\mathrm{Est}[k^i e(jT)], \\ \qquad t \in [jT+qh, jT+(q+1)h) \\ \frac{p}{p-q-1}\{u^0(jT)+ \\ \qquad J\sum_{i=0}^{q} c_i\mathrm{Est}[k^i e(jT)]\}, \\ \qquad t \in [jT+(q+1)h, jT+ph), \end{cases} \tag{10}$$

$$e^0(t) = \begin{cases} 0, \\ \qquad t \in [jT, jT+qh) \\ p\sum_{i=0}^{q} c_i\mathrm{Est}[k^i e(jT)], \\ \qquad t \in [jT+qh, jT+(q+1)h) \\ 0, \\ \qquad t \in [jT+(q+1)h, jT+ph), \end{cases} \tag{11}$$

with $\mathrm{Est}[k^i e(jT)]$, $i = 1,...,q$, given by

$$\mathrm{Est}[k^i e(jT)] := \frac{-1}{\rho EB} h^{\delta-1}[e(jT+ih) - e(jT+(i-1)h)]. \tag{12}$$

Now we can turn to a state-space model. We need one of dimension $q+2$:

- $\bar{z}_1$ of dimension $q+1$ is used to hold the samples of $e(jT+ih)$, $i = 0,1,...,q$, for the period.
- $\bar{z}_2$ of dimension one is used to hold the sample of $u^0(jT)$ for the period.

It is routine, though tedious, to construct the parameters $(\bar{F},\bar{G},\bar{H},\bar{J})$ so that the corresponding controller (7) implements the estimation, reset and control phases described above and encapsulated in (10)-(12); for space reasons the details are not included here. With $\rho > 0$ we label this controller $C_{lpc}(h)$. The next step is to prove that the proposed controller achieves our objective. To do so we need the following proposition.

Proposition 2. There exist constants $\bar{h} > 0$ and $c > 0$ so that the controller $C_{lpc}(h)$ stabilizes kP_0 for every $h \in (0,\bar{h})$ and $k \in \mathbf{K}$ and satisfies

$$\|[1+kP_0\phi_\varepsilon(k)C_{lti}]^{-1}W - [1+kP_0C_{lpc}(h)]^{-1}W\|$$

$$\le c(h^\delta + h^{1-\delta}),\ \ h \in (0,\bar{h}),\ \ k \in \mathbf{K}.$$

Proof:

We first analyse the closed system behaviour at integer multiples of $T = ph$, and prove that it is very close to that provided by $\phi_\varepsilon(k)C_{lti}$. At this point we can prove the same about the inter-sample behaviour.

QED

Proof of Theorem 1:

First, let $\gamma > 0$. Choose C_{lti} to be a finite-dimensional LTI controller which stabilizes P_0 and ensures that

$$\|(I+P_0C_{lti})^{-1}W\| \leq \alpha_{lti} + \gamma/3.$$

From Proposition 1 we know that there exist constants $\bar{\varepsilon} > 0$ and $c > 0$ so that, for every $\varepsilon \in (0,\bar{\varepsilon})$, the controller $\phi_\varepsilon(k)C_{lti} = (\sum_{i=0}^{q} c_i k^i)C_{lti}$ stabilizes kP_0 for every $k \in \mathbf{K}$ and satisfies

$$\|[1+kP_0\phi_\varepsilon(k)C_{lti}]^{-1}W - [1+P_0C_{lti}]^{-1}W\| \leq c\varepsilon$$

for $k \in \mathbf{K}$. Choose $\varepsilon \in (0,\bar{\varepsilon})$ so that $c\varepsilon < \gamma/3$. Fix $\rho > 0$ and $p > q+1$. From Proposition 2 there exists an $\bar{h} > 0$ and $\bar{c} > 0$ so that the controller $C_{lpc}(h)$ stabilizes kP_0 for every $h \in (0,\bar{h})$ and $k \in \mathbf{K}$ and satisfies

$$\|[1+kP_0\phi_\varepsilon(k)C_{lti}]^{-1}W - [1+kP_0C_{lpc}(h)]^{-1}W\| \leq \bar{c}(h^\delta + h^{1-\delta})$$

for $h \in (0,\bar{h})$ and $k \in \mathbf{K}$. Choose $h \in (0,\bar{h})$ so that

$$\bar{c}(h^\delta + h^{1-\delta}) < \gamma/3.$$

Using the triangle inequality, it follows that $C_{lpc}(h)$ stabilizes kP_0 for every $k \in \mathbf{K}$ and ensures that

$$\|[1+kP_0C_{lpc}(h)]^{-1}W\| \leq \alpha_{lti} + \gamma,\ k \in \mathbf{K},$$

as desired, so we set $C_{LPC} = C_{lpc}(h)$.

QED

5. SUMMARY AND CONCLUSIONS

In this paper we have considered the problem of minimizing the weighted sensitivity in the face of an uncertain multiplicative gain uncertainty. We show that, under a technical assumption on the plant (it is relative degree one), on the weighting function (it is strictly proper), and on the multiplicative gain (it lies in a compact set not including zero), there exists a LPC which can provide a near LTI-optimal weighted sensitivity for every admissible gain. The controller consists of two parts: an LTI controller which is near optimal for the nominal plant, together with a sampled-data linear periodic controller which carries out some probing. While we have used the ∞-norm on our signals here, we expect that the approach should translate to the $2-$norm case with a proper choice of the controller parameters. We are presently working on removing the plant relative degree restriction and on proving that the controller tolerates slow time-variations in the gain.

6. ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada via a research grant.

REFERENCES

Francis, B.A. and T.T. Georgiou (1988). Stability Theory for Linear Time-Invariant Plants with Periodic Digital Controllers. *IEEE Transactions on Automatic Control*, **AC-33**, 820–832.

Khargonekar, P.P. and K.R. Poolla (1986). Uniformly optimal control of linear time-varying plants: nonlinear time-varying controllers. *Systems and Control Letters*, **5**, 303 – 308.

Khargonekar, P.P. and A. Tannenbaum (1985). Non-euclidean metrics and the robust stabilization of systems with parameter uncertainty. *IEEE Transactions on Automatic Control*, **AC-30**, 1005–1013.

Miller, D.E. (2003). A New Approach to Model Reference Adaptive Control. *IEEE Transactions on Automatic Control*, **AC-48**, 743-757.

Rossi, M. and D.E. Miller (1999). Gain/Phase Margin Improvement Using Static Sampled-Data Hold Functions. *Systems & Control Letters*, **37**, 163 – 172.

Shamma, J.S. and M.A. Dahleh (1991). Time-varying versus time-invariant compensation for rejection of persistent bounded disturbances and robust stabilization. *IEEE Transactions on Automatic Control*, **AC-36**, 838 – 847.

Stoorvogel, A. (1994). Nonlinear L_1 Optimal Controllers for Linear Systems. *IEEE Transactions on Automatic Control*, **AC-40**, 694 – 696.

Yan, W. and B.D.O. Anderson (1990). The Simultaneous Optimization Problem for Sensitivity and Gain Margin. *IEEE Transactions on Automatic Control*, **AC-35**, 558 – 563.

Yan, W., B.D.O. Anderson, and R.R. Bitmead (1994). On the Gain Margin Improvement Using Dynamic Compensation Based on Generalized Sampled-Data Hold Functions. *IEEE Transactions on Automatic Control*, **AC-39**, 2347 – 2354.

Yang, C. and P.T. Kabamba (1994). Multi-channel Output Gain Margin Improvement Using Generalized Sampled-Data Hold Functions. *IEEE Transactions on Automatic Control*, **AC-39**, 657 – 661.