ON SOLVING DISCRETE-TIME PERIODIC RICCATI EQUATIONS

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Abstract: Two numerically reliable algorithms to compute the periodic nonnegative definite stabilizing solution of discrete-time periodic Riccati equations are proposed. The first method represents an extension of the periodic QZ algorithm to non-square periodic pairs, while the second method represents an extension of a quotient-product swapping and collapsing "fast" algorithm. Both approaches are completely general being applicable to periodic Riccati equations with time varying dimensions as well as with singular control weighting. For the "fast" method, reliable software implementation is available in a recently developed PERIODIC SYSTEMS Toolbox. *Copyright©2005 IFAC*

Keywords: Periodic systems, Riccati equation, deadbeat control, numerical methods.

1. INTRODUCTION

This paper deals with the efficient computation of the unique symmetric stabilizing N-periodic solution X_k, $k = 1, \dots, N$ of the *periodic reverse discrete-time algebraic Riccati equation* (PRDARE)

$$\begin{aligned} X_k = Q_k + A_k^T X_{k+1} A_k - (A_k^T X_{k+1} B_k + S_k) \\ \times (R_k + B_k^T X_{k+1} B_k)^{-1} (A_k^T X_{k+1} B_k + S_k)^T \end{aligned} \quad (1)$$

where $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m_k}$, $Q_k \in \mathbb{R}^{n_k\times n_k}$, $R_k \in \mathbb{R}^{m_k\times m_k}$ and $S_k \in \mathbb{R}^{n_k\times m_k}$ are N-periodic matrices ($N \geq 1$). All Q_k and R_k are assumed symmetric matrices.

Equation (1) arises, for example, when solving the LQ optimal control problem for the linear periodic system

$$x_{k+1} = A_k x_k + B_k u_k \quad (2)$$

by minimizing the quadratic cost functional

$$J = \frac{1}{2}\sum_{k=0}^{\infty} [\, x_k^T \; u_k^T \,] \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \quad (3)$$

where usually it assumed that $\begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \geq 0$. Provided a nonnegative stabilizing solution X_k, $k = 1, \dots, N$ of PRDARE (1) is known, the periodic state-feedback matrix F_k in the optimal control law $u_k^* = F_k x_k$, which minimizes the performance index (3), results as

$$F_k = -(B_k^T X_{k+1} B_k + R_k)^{-1} (A_k^T X_{k+1} B_k + S_k)^T \quad (4)$$

A solution X_k of (1) is called a *stabilizing solution* if the corresponding feedback (4) ensures that $\Phi_{A+BF}(N+1, 1) \subset \mathbb{C}^-$, where $\Phi_A(j,i) := A_{j-1} \cdots A_{i+1} A_i$, with $\Phi_A(i,i) := I_{n_i}$, denotes the transition matrix between time moments i and j.

[1] Partially supported via the Swedish Strategic Research Foundation Grant "Matrix Pencil Computations in Computer-Aided Control System Design: Theory, Algorithms and Software Tools".

Any method to solve the PRDARE (1) can be also employed to solve the dual *periodic forward discrete-time algebraic Riccati equation* (PFDARE)

$$X_{k+1} = Q_k + A_k X_k A_k^T - (A_k X_k C_k^T + S_k) \times (R_k + C_k X_k C_k^T)^{-1} (A_k X_k C_k^T + S_k)^T \quad (5)$$

by solving the PRDARE (1) with the replacements

$$A_k \leftarrow A_{N-k}^T,\ B_k \leftarrow C_{N-k}^T,$$
$$Q_k \leftarrow Q_{N-k},\ S_k \leftarrow S_{N-k}^T,\ R_k \leftarrow R_{N-k}$$

From the solution X_k of the PRDARE, the solution of PFDARE is recovered with the replacements

$$X_k \leftarrow X_{N-k+2},\ k = 2, ..., N$$

The solution of PRDARE for constant dimensions has been considered by several authors (Bojanczyk *et al.*, 1992; Hench and Laub, 1994; Benner *et al.*, 2002), but PRDAREs with time-varying dimensions have been considered only recently in (Chu *et al.*, 2004). A standard assumption in all proposed algorithms is the invertibility of R_k. The methods proposed in (Bojanczyk *et al.*, 1992; Hench and Laub, 1994) rely on the periodic QZ decomposition to compute orthogonal bases for suitable periodic deflating subspaces from which the solution results. The main computational ingredient for these methods is the computation of an ordered periodic QZ decomposition of a pair of periodic square matrices with constant dimensions. Algorithms to perform this decomposition have been proposed in (Bojanczyk *et al.*, 1992; Hench and Laub, 1994), but presently there is no robust numerical software implementing these methods for pairs of real periodic matrices (Kressner, 2004).

The methods proposed in (Benner *et al.*, 2002; Chu *et al.*, 2004) belong to the family of "fast" methods, by reducing the problem to a single DARE satisfied say by X_1. This DARE can be then solved using standard deflating subspace methods. The rest of solution X_k for $k = N, N-1, \ldots, 2$ can be obtained by a convergent direct iteration on (1). The quotient-product swapping and collapsing method of (Benner *et al.*, 2002; Benner and Byers, 2001) has been reinterpreted to perform locally orthogonal compressions on an extended regular pencil to obtain a subpencil whose stable deflating subspace generates the stabilizing solution at a certain time moment (Van Dooren, 1999). This method performs substantially less operations than the periodic QZ decomposition based approach, and this justifies the term "fast" used to label it (Van Dooren, 1999). The method of (Chu *et al.*, 2004) performs essentially the same reduction of the extended pencil by using structure preserving non-orthogonal block compressions. In the resulting final subpencil the "matrices" of a standard problem can be identified which define the solution at a certain time moment. To compute the solution, a structure-preserving iterative doubling algorithm with quadratic convergence is employed in (Chu *et al.*, 2004), but any alternative approach for standard systems can be employed as well.

In this paper, extensions of the above methods are proposed which can address the most general case of time-varying dimensions and singular input weighting matrices R_k. Problems with possibly time-varying dimensions are to be expected when solving spectral and inner-outer factorization problems for periodic systems using deflation based approaches similar to those proposed in (Oară and Varga, 1999) for standard systems. Problems with singular R_k arise when solving, for example, dead-beat control problems via an LQ-optimization based approach (Emami-Naeini and Franklin, 1982). Both of proposed methods are able to address problems with null characteristic multipliers. The first method extends the periodic QZ algorithm to handle non-square periodic pairs with time-varying dimensions. To apply the standard periodic QZ algorithm algorithm, a certain periodic pair is preprocessed by deflating first its infinite characteristic multipliers and then isolating the core finite characteristic multipliers. The second method performs structure exploiting orthogonal transformations to reduce an extended regular pencil to a regular subpencil, whose deflating subspaces produce the solution of PDARE at a fixed time moment. This method extends the quotient-product swapping and collapsing approach of (Benner *et al.*, 2002) along the lines of the reduction technique employed in (Varga and Van Dooren, 2003). The main advantage of this approach is that its implementation is straightforward using available robust numerical software. For the "fast" method, reliable software implementation is already available in a recently developed PERIODIC SYSTEMS Toolbox (Varga, 2005).

2. EXTENDED PERIODIC QZ ALGORITHM

Consider the periodic matrix pairs

$$M_k = \begin{bmatrix} A_k & O & B_k \\ -Q_k & I_{n_k} & -S_k \\ S_k^T & O & R_k \end{bmatrix},\ L_k = \begin{bmatrix} I_{n_{k+1}} & O & O \\ O & A_k^T & O \\ O & -B_k^T & O \end{bmatrix} \quad (6)$$

where $M_k \in \mathbb{R}^{(n_k+n_{k+1}+m_k)\times(2n_k+m_k)}$ and $L_k \in \mathbb{R}^{(n_k+n_{k+1}+m_k)\times(2n_{k+1}+m_{k+1})}$. With this pairs, the PRDARE (1) and (4) can be written as

$$M_k \begin{bmatrix} I_{n_k} \\ X_k \\ F_k \end{bmatrix} = L_k \begin{bmatrix} I_{n_{k+1}} \\ X_{k+1} \\ F_{k+1} \end{bmatrix} \Psi_k$$

for $\Psi_k := A_k + B_k F_k \in \mathbb{R}^{n_{k+1}\times n_k}$. This alternative form of the PRDARE leads to a straightforward extension of the methods of (Bojanczyk *et al.*, 1992; Hench and Laub, 1994) to compute the stabilizing periodic solution X_k:

1. Compute the N-periodic matrices V_k and Z_k such that for $k = 1, \ldots, N$

$$V_k M_k Z_k = \begin{bmatrix} H_{k,11} & H_{k,12} \\ O & H_{k,22} \end{bmatrix},$$

$$V_k L_k Z_{k+1} = \begin{bmatrix} T_{k,11} & T_{k,12} \\ O & T_{k,22} \end{bmatrix},$$

where $H_{k,11} \in \mathbb{R}^{n_{k+1}\times n_k}$, $T_{k,11} \in \mathbb{R}^{n_{k+1}\times n_{k+1}}$ is invertible, and $\Phi_{T_{11}^{-1}H_{11}}(N+1,1) \in \mathbb{C}^-$.

2. Partition Z_k to conform with the partitioning of matrices M_k and $V_k M_k Z_k$

$$Z_k = \begin{bmatrix} Z_{k,11} & Z_{k,12} \\ Z_{k,21} & Z_{k,22} \\ Z_{k,31} & Z_{k,32} \end{bmatrix}$$

If all $Z_{k,11}$ are invertible, compute $X_k = Z_{k,21}Z_{k,11}^{-1}$ and $F_k = Z_{k,31}Z_{k,11}^{-1}$.

Note that in the case of time-varying dimensions, the direct application of the periodic QZ algorithm at Step 1 is not possible. It is shown in what follows how to overcome this difficulty with recently developed algorithms to reduce periodic matrix pairs.

The computation of V_k and Z_k at Step 1 can be performed in three main steps. In the first step a finite-infinite spectral separation is performed by using the recently developed algorithm to compute Kronecker-like forms of periodic pairs (Varga, 2004*b*). By applying this algorithm to the dual periodic pair (M_{N-k}^T, L_{N-k}^T), orthogonal matrices V_k^1 and Z_k^1 are determined such that

$$V_k^1 M_k Z_k^1 = \begin{bmatrix} M_k^f & M_k^{f,\infty} \\ O & M_k^\infty \end{bmatrix},$$

$$V_k^1 L_k Z_{k+1}^1 = \begin{bmatrix} L_k^f & L_k^{f,\infty} \\ O & L_k^\infty \end{bmatrix},$$

where L_k^f and M_k^∞ are nonsingular matrices. The pair (M_k^f, L_k^f) contains the finite characteristic multipliers, while the pair (M_k^∞, L_k^∞) contains the infinite characteristic multipliers. Note that the finite characteristic multipliers at time k are the eigenvalues of the product $(L_{k+N}^f)^{-1}M_{k+N}^f \cdots (L_k^f)^{-1}M_k^f$ and the nilpotent matrix product $(M_k^\infty)^{-1}L_k^\infty \cdots (M_{k+N}^\infty)^{-1}L_{k+N}^\infty$ characterizes the infinite characteristic multipliers. The problem is not solvable if the above separation can not be performed (i.e., the resulting Kronecker-like form contains parts which correspond to a left or right Kronecker structure).

In the second step, orthogonal matrices V_k^2 and Z_k^2 are determined such that

$$V_k^2 M_k^f Z_k^2 = \begin{bmatrix} M_{k,11}^f & M_{k,12}^f \\ O & M_{k,22}^f \end{bmatrix},$$

$$V_k^2 L_k^f Z_{k+1}^2 = \begin{bmatrix} L_{k,11}^f & L_{k,11}^f \\ O & L_{k,22}^f \end{bmatrix},$$

where the pair $(M_{k,22}^f, L_{k,22}^f)$ has constant dimensions and is in a periodic generalized Hessenberg form (Hench and Laub, 1994), $L_{k,11}^f \in \mathbb{R}^{(n_{k+1}-\underline{n})\times(n_{k+1}-\underline{n})}$ is upper triangular and $M_{k,11}^f \in \mathbb{R}^{(n_{k+1}-\underline{n})\times(n_k-\underline{n})}$ is upper trapezoidal, with $\underline{n} = \min n_k$. The pair $(M_{k,11}^f, L_{k,11}^f)$ has only null characteristic multipliers. This reduction can be performed by extending the generalized periodic Hessenberg reduction procedures of (Bojanczyk *et al.*, 1992; Hench and Laub, 1994) to the case of non-constant dimensions similarly as done in (Varga, 1999) for the periodic Hessenberg form.

Finally, orthogonal matrices V_k^3 and Z_k^3 are computed such that

$$V_k^3 M_{k,22}^f Z_k^3 = \begin{bmatrix} M_k^s & M_k^{s,u} \\ O & M_k^u \end{bmatrix},$$

$$V_k^3 L_{k,22}^f Z_{k+1}^3 = \begin{bmatrix} L_k^s & L_k^{s,u} \\ O & L_k^u \end{bmatrix},$$

where the pair (M_k^s, L_k^s) has only stable characteristic multipliers, and the pair (M_k^u, L_k^u) has only unstable characteristic multipliers. Because of the symplectic nature of the eigenvalue problem, the nonzero eigenvalues must appear in reciprocal pairs. For this step, the algorithms of (Bojanczyk *et al.*, 1992; Hench and Laub, 1994) to compute and reorder the periodic QZ decomposition can be used.

The final transformation matrix Z_k is obtained as

$$Z_k = Z_k^1 \mathrm{diag}\,(I, Z_k^2)\mathrm{diag}\,(I, Z_k^3)$$

To compute X_k and F_k, the accumulation of the left transformations V_k is not necessary.

Each computational step of the above algorithm is numerically stable and has a computational complexity of $O(N(2n+m)^3)$, where n and m are the maximum problem dimensions for n_k and m_k, respectively.

In the case of constant dimensions, a similar approach to that proposed for standard systems (Van Dooren, 1981), can be used to deflate trivial infinite characteristic multipliers first. For this purpose, orthogonal U_k are chosen such that

$$U_k \begin{bmatrix} B_k \\ -S_k \\ R_k \end{bmatrix} = \begin{bmatrix} O \\ O \\ \overline{R}_k \end{bmatrix}$$

where all $\overline{R}_k \in \mathbb{R}^{m\times m}$ are nonsingular. Note that this is a necessary condition for the existence of a solution of the PRDARE. Let define the reduced periodic pair $(\overline{M}_k, \overline{L}_k)$ from

$$U_k M_k = \begin{bmatrix} \overline{M}_k & O \\ * & \overline{R}_k \end{bmatrix}, \quad U_k L_k = \begin{bmatrix} \overline{L}_k & O \\ * & O \end{bmatrix},$$

where $\overline{M}_k \in \mathbb{R}^{2n\times 2n}$, $\overline{L}_k \in \mathbb{R}^{2n\times 2n}$. Then the N-periodic orthogonal matrices $\overline{V}_k$ and $\overline{Z}_k$ are computed such that for $k = 1,\ldots,N$

$$\overline{V}_k \overline{M}_k \overline{Z}_k = \begin{bmatrix} \overline{H}_{k,11} & \overline{H}_{k,12} \\ O & \overline{H}_{k,22} \end{bmatrix},$$

$$\overline{V}_k\overline{L}_k\overline{Z}_{k+1} = \begin{bmatrix} \overline{T}_{k,11} & \overline{T}_{k,12} \\ O & \overline{T}_{k,22} \end{bmatrix},$$

where $\overline{H}_{k,11} \in \mathbb{R}^{n\times n}$, $\overline{T}_{k,11} \in \mathbb{R}^{n\times n}$ is invertible, and $\Phi_{\overline{T}_{11}^{-1}\overline{H}_{11}}(N+1,1) \in \mathbb{C}^-$. Finally, let partition $\overline{Z}_k$ to conform with the partitioning of matrix $\overline{V}_k\overline{M}_k\overline{Z}_k$

$$\overline{Z}_k = \begin{bmatrix} \overline{Z}_{k,11} & \overline{Z}_{k,12} \\ \overline{Z}_{k,21} & \overline{Z}_{k,22} \end{bmatrix}$$

and, provided all $\overline{Z}_{k,11}$ are invertible, compute

$$X_k = \overline{Z}_{k,21}\overline{Z}_{k,11}^{-1}$$

The state-feedback matrix F_k results from (4).

3. FAST ALGORITHM

The second algorithm can be explained by defining the extended pencil

$$H - zT := \begin{bmatrix} M_1 & -L_1 & O & \cdots & O \\ O & M_2 & -L_2 & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & & & M_{N-1} & -L_{N-1} \\ -zL_N & O & \cdots & O & M_N \end{bmatrix} \quad (7)$$

of order $\mu = \sum_{i=1}^{N}(2n_i + m_i)$. In what follows it is assumed that this pencil is regular and has no eigenvalues on the unit circle.

The proposed algorithm implicitly constructs a stable deflating subspace of the pencil (7) by employing structure exploiting reductions similar to that employed in (Varga and Van Dooren, 2003) for computing periodic systems zeros. The method can be seen as an extension of the *swapping and collapsing* approach proposed in (Benner and Byers, 2001; Benner *et al.*, 2002) of quotient-products. The basic reduction is performed as follows.

Consider the $(n_1 + 2n_2 + n_3 + m_1 + m_2)$-th order orthogonal transformation matrix U_1 compressing the rows of the matrix $\begin{bmatrix} -L_1 \\ M_2 \end{bmatrix}$ to $\begin{bmatrix} R_1 \\ O \end{bmatrix}$, where R_1 is a nonsingular matrix of order $2n_2 + m_2$. Applying U_1 to the first two blocks rows of $H - zT$ we obtain for the nonzero elements

$$U_1 \begin{bmatrix} M_1 & -L_1 & O \\ O & M_2 & -L_2 \end{bmatrix} = \begin{bmatrix} \widetilde{M}_1 & R_1 & -\widetilde{L}_1 \\ \widehat{M}_2 & O & -\widehat{L}_2 \end{bmatrix}$$

which defines the new matrices $\widehat{M}_2$ and $\widehat{L}_2$ with $n_1 + n_3 + m_1$ rows.

Then construct the $(n_1+2n_{i+1}+n_{i+2}+m_1+m_{i+1})$-th order orthogonal transformations U_i for $i = 2, \ldots N{-}1$ such that

$$U_i \begin{bmatrix} \widehat{M}_i & -\widehat{L}_i & O \\ O & M_{i+1} & -L_{i+1} \end{bmatrix} = \begin{bmatrix} \widetilde{M}_i & R_i & -\widetilde{L}_i \\ \widehat{M}_{i+1} & O & -\widehat{L}_{i+1} \end{bmatrix}$$

where R_i are invertible matrices of order $2n_{i+1} + m_{i+1}$. This recursively defines the new matrices $\widehat{M}_{i+1}$ and $\widehat{L}_{i+1}$ with $n_1 + n_{i+2} + m_1$ rows.

Applying the transformations U_i successively to the i-th and $(i+1)$-th block rows of the transformed pencil $H - zT$, the reduced pencil

$$\overline{H} - z\overline{T} = \left[\begin{array}{c|cccc} \widetilde{M}_1 & R_1 & -\widetilde{L}_1 & O & O \\ \widetilde{M}_2 & O & R_2 & \ddots & O \\ \vdots & \vdots & \vdots & \ddots & -\widetilde{L}_{N-2} \\ \widetilde{M}_{N-1} - z\widetilde{L}_{N-1} & O & O & \cdots & R_{N-1} \\ \hline \widehat{M}_N - z\widehat{L}_N & O & O & \cdots & O \end{array}\right] \quad (8)$$

is obtained, which is orthogonally similar to $H - zT$. Since the matrices R_i are nonsingular, the regular subpencil $\widehat{M}_N - z\widehat{L}_N$ of order $2n_1 + m_1$ will contain all finite eigenvalues of the original pencil. To check the regularity of the extended pencil, the reciprocal condition numbers of the upper triangular matrices R_i can be cheaply estimated to detect possible rank losses.

To compute the solution X_1, orthogonal V_1 and Z_1 are determined such that

$$V_1(\widehat{M}_N - z\widehat{L}_N)Z_1 = \begin{bmatrix} H_{11} - zT_{11} & H_{12} - zT_{12} \\ O & H_{22} - zT_{22} \end{bmatrix}$$

where $H_{11} - zT_{11}$ has only finite and stable eigenvalues. Partition Z_1 conformably in two block columns and three block rows

$$Z_1 = \begin{bmatrix} Z_{1,11} & Z_{1,12} \\ Z_{1,21} & Z_{1,22} \\ Z_{1,31} & Z_{1,32} \end{bmatrix}$$

such that $Z_{1,11} \in \mathbb{R}^{n_1\times n_1}$. Provided $Z_{1,11}$ is invertible, the solution X_1 and feedback F_1 can be computed as $X_1 = Z_{1,21}Z_{1,11}^{-1}$ and $F_1 = Z_{1,31}Z_{1,11}^{-1}$. The rest of the solution X_k for $k = N, \ldots, 2$ is computed iteratively with (1). Since the process to iterate directly on the equation (1) is convergent, this iteration produces virtually no errors.

To estimate the computational effort of the fast approach, let assume constant dimensions n and m for the matrices $A_k \in \mathbb{R}^{n\times n}$ and $B_k \in \mathbb{R}^{n\times m}$. The reduction of the pencil $H-zT$ can be done by computing successively $N-1$ QR decompositions of $(4n + 2m) \times (2n + m)$ matrices and applying the transformation to two sub-blocks of the same dimensions. The reduction step has thus a computational complexity of $O((N-1)(2n+m)^3)$. Since the last step, the computation of stable deflating subspace of the reduced pencil $\widehat{M}_N - z\widehat{L}_N$, has a complexity of $O((2n + m)^3)$, it follows that the overall computational complexity of the proposed approach corresponds to what is expected for a satisfactory algorithm for periodic systems. This approach is substantially more efficient than the periodic QZ algorithm based approach and this is why, the proposed algorithm belongs to the family of so-called *fast* algorithms (Van Dooren, 1999).

Since the main reduction consists of successive QR-decompositions, it can be shown (Golub and Van Loan, 1989) that the matrices of the computed reduced pencil $\overline{H} - \lambda \overline{T}$ satisfy

$$\|UX - \overline{X}\|_2 \leq \epsilon_M f(2n+m)\|X\|_2, \quad X = H,\ T$$

where U is the matrix of accumulated left orthogonal transformations, ϵ_M is the relative machine precision, and $f(2n+m)$ is a quantity of order of $2n+m$. The subsequent computational step is performed using the algorithm of (Misra *et al.*, 1994) and is also based exclusively on orthogonal transformations. This second step is numerically stable as well. Overall, it is thus guaranteed that the computed solution is exact for a slightly perturbed extended pencil. It follows that the proposed algorithm to compute zeros is *numerically backward stable*.

Since the structure of the perturbed extended pencil is not preserved in the reduction, we can not say however that the computed zeros are exact for slightly perturbed original data (i.e., the algorithm is not *strongly* stable). In spite of this weaker type of stability, the proposed algorithm is the first numerically reliable procedure able to solve PRDAREs of the most general form.

4. DEADBEAT CONTROL

The LQ optimization based approach proposed in (Emami-Naeini and Franklin, 1982) for deadbeat control can be easily extended to periodic systems. To this end, a periodic LQ optimization problem with $R_k = 0$, $S_k = 0$ and $Q_k = C_k^T C_k$ is solved, where $C_k \in \mathbb{R}^{m_k \times n_k}$ are chosen such that the periodic system (A_k, B_k, C_k) has no finite zeros. For standard systems the choice of C_k is straightforward and involves the computation of the Kalman controllability form (Emami-Naeini and Franklin, 1982). The same approach can be used for periodic systems by computing the periodic Kalman reachability form $(\widetilde{A}_k, \widetilde{B}_k)$ of the periodic pair (A_k, B_k) using the algorithm proposed in (Varga, 2004*a*). For a completely reachable pair (A_k, B_k), this algorithm computes orthogonal N-periodic transformation matrices Z_k such that

$$\widetilde{A}_k = Z_{k+1}^T A_k Z_k, \quad \widetilde{B}_k = Z_{k+1}^T B_k$$

where each matrix $[\,\widetilde{B}_k\ \widetilde{A}_k\,]$ is in a staircase form

$$[\,\widetilde{B}_k \mid \widetilde{A}_k\,] = \left[\begin{array}{c|ccccc} A_{k;1,0} & A_{k;1,1} & A_{k;1,2} & \cdots & & A_{k;1,\ell} \\ O & A_{k;2,1} & A_{k;2,2} & \cdots & & A_{k;2,\ell} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ O & O & O & & A_{k;\ell,\ell-1} & A_{k;\ell,\ell} \end{array}\right]$$

where $A_{k;i,i} \in \mathbb{R}^{\nu_{k+1}^{(i)} \times \nu_k^{(i)}}$, $i = 1, \ldots, \ell$; $A_{k;i,i-1} \in \mathbb{R}^{\nu_{k+1}^{(i)} \times \nu_k^{(i-1)}}$ and rank $A_{k;i,i-1} = \nu_{k+1}^{(i)}$, $i = 1, \ldots, \ell$. C_k can be chosen in the form

$$C_k = \text{diag}\,(C_{k,1}, \cdots, C_{k,\ell-1}, I_{\nu_k^{(\ell)}})Z_k^T$$

where $C_{k,i-1} \in \mathbb{R}^{(\nu_k^{(i-1)} - \nu_{k+1}^{(i)}) \times \nu_k^{(i-1)}}$ contains in each line a single nonzero element (e.g., set to 1) in the position corresponding to a linearly dependent column of $A_{k;i,i-1}$. For example, in the simple case when all sub-diagonal blocks are invertible, C_k can be simply chosen as

$$C_k = [\,O\ \cdots\ O\ I_{\nu_k^{(\ell)}}\,]Z_k^T$$

5. NUMERICAL EXPERIMENTS

Two examples are considered to illustrate the capabilities of the proposed fast algorithm. The computations have been performed using MATLAB based implementations relying on the tools available in the recently developed PERIODIC SYSTEMS Toolbox (Varga, 2005). To assess the accuracy of the results, the residuals (assuming $S_k = 0$)

$$r_k = \|X_k - Q_k - A_k^T X_{k+1}(A_k + B_k F_k)\|_F$$

can be computed from which a *total residual* can be defined as

$$\text{Residual} = \left(\sum_{j=1}^{N} r_j^2\right)^{1/2} \tag{9}$$

Example 1. This is an example with time-varying dimensions, defined by the 3-periodic matrices

$$A_1 = \begin{bmatrix} -3 & 2 & 9 \\ 0 & 0 & -4 \end{bmatrix},\ A_2 = \begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix},\ A_3 = \begin{bmatrix} 2 & -3 \\ 4 & -15 \\ -2 & 9 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For this problem the deadbeat control problem has been solved by choosing $R_1 = R_2 = R_3 = 0$ and

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix},\ Q_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},\ Q_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The resulting total residual of the computed solution is $2.1 \cdot 10^{-12}$. For reference purposes, the exact periodic deadbeat state-feedback is given below

$$F_1 = \begin{bmatrix} 6 & -4 & -22 \end{bmatrix}, F_2 = \begin{bmatrix} -\frac{80}{33} & \frac{40}{33} \end{bmatrix}, F_3 = \begin{bmatrix} \frac{8}{5} & -\frac{32}{5} \end{bmatrix}$$

Example 2. This example has been used in (Varga and Pieters, 1998) to design periodic output feedback controllers for periodic systems with relatively large periods. The discrete-time periodic system originates from a continuous-time periodic model of a spacecraft pointing and attitude system described in (Pittelkau, 1993). This system has state and input dimensions $n = 4$, $m = 1$, respectively, and a period of $T = 6073.8$ seconds. The discretized system for different sampling periods T/N has been used in (Varga and Pieters, 1998) to design periodic output feedback controllers for this system. The matrices of the discrete-time periodic system can be computed explicitly for arbitrary values of N.

The deadbeat control problem has been solved for different sampling times by choosing $R_k = 0$, $S_k = 0$ and $Q_k = C_k^T C_k$ with $C_k = [0\ 0\ 0\ 1]Z_k^T$, where Z_k is the N-periodic orthogonal transformation matrix which brings the periodic pair (A_k, B_k) into the periodic Kalman reachability form $(\widetilde{A}_k, \widetilde{B}_k)$. Note that in this form each $\widetilde{B}_k$ has only the (1,1) entry nonzero and each $\widetilde{A}_k$ is an upper Hessenberg matrix.

The table below presents the following results obtained for different values of N: the total residual (9), ITER – the number of iterations performed on PRDARE (1) to achieve the limiting accuracy, and STDEG – the achieved closed-loop stability degree (i.e., the maximum modulus of closed-loop characteristic multipliers). Note that $N = 600$ corresponds to a typical sampling period of about 10 seconds used to control satellites on low Earth orbits.

N	40	120	360	600
Residual	$1.0 \cdot 10^{-11}$	$7.7 \cdot 10^{-14}$	$8.4 \cdot 10^{-12}$	$2.4 \cdot 10^{-11}$
ITER	2	2	2	6
STDEG	$2 \cdot 10^{-155}$	$5 \cdot 10^{-324}$	$1 \cdot 10^{-322}$	$2 \cdot 10^{-322}$

6. CONCLUDING REMARKS

The proposed algorithms to solve PRDAREs are completely general, numerically reliable and computationally efficient. For the fast algorithm, a robust numerical implementation is available in the PERIODIC SYSTEMS Toolbox (Varga, 2005). This implementation underlies user-friendly software to solve periodic LQ control and filtering related problems. This software used in conjunction with software tools to compute periodic Kalman forms allows to solve periodic deadbeat control problems in the most general setting (i.e., for controllable systems with time-varying dimensions). The ultimate *structure preserving and numerically stable* approach to solve PRDAREs is still needed to be developed, and very likely, it will be based on structure preserving orthogonal symplectic reductions.

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