REGULATION OF DISCRETE-TIME LINEAR SYSTEMS WITH POSITIVE STATE AND CONTROL CONSTRAINTS AND BOUNDED DISTURBANCES [⋆]

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Abstract: A variety of control problems require the control action and/or state to be positive. Typical applications include situations where the operating point maximizes (steady state) efficiency so that the steady state control and/or the steady state itself lie on the boundaries of their respective constraint sets. Any deviation of the control and/or state from its steady state value must therefore be directed to the interior of its constraint set. To address these problems, we characterize a novel family of the robust control invariant sets for linear systems under positivity constraints. The existence of a constraint admissible member of this family can be checked by solving a *single linear or quadratic programming problem.* The solution of this optimization problem yields the corresponding controller. These results are then used to devise a robust time–optimal control scheme for regulation of uncertain linear systems under positivity constraints. Robust finite–time attractivity of an appropriately chosen member of this family is also established.

Keywords: Positive constraints, Constrained control, Set invariance, Linear systems.

1. INTRODUCTION

Most studies in the theory of constrained control include the assumption that the origin is in the interior of constraint sets; see for example (Mayne *et al.*, 2000; Bemporad and Morari, 1999) and references therein. This assumption is not always satisfied in practice. In some practical problems, the controlled system is required to operate as close as possible to or at, the boundary of the constraint set. This issue has been discussed in (Rao and Rawlings, 1999; Pannocchia *et al.*, 2003). Here we consider a more general problem – the regulation problem for discrete-time linear systems with positive state and control constraints subject to additive and bounded disturbances. Control under positivity constraints raises interesting problems that are amplified if the system is subject to additive and bounded disturbances.

In this paper we exploit recent results on robust control invariance (Raković, 2005) and standard ideas in robust time-optimal control (Bertsekas and Rhodes, 1971; Blanchini, 1992; Mayne and Schroeder, 1997; Kerrigan and Mayne, 2002) to devise an efficient control algorithm.

⋆ Research supported by the Engineering and Physical Sciences Research Council, UK.

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Instead of controlling the system to the desired reference couple $(\hat{x}, \hat{u})$ that lie on the boundary of the constraint set, we control the system to a robust control invariant set centered at an equilibrium point $(\bar{x}, \bar{u})$ while minimizing an appropriate distance from the reference couple $(\hat{x}, \hat{u})$. The first subproblem is the construction of a suitable target set, that is an appropriately computed robust control invariant set centered at a suitable equilibrium $(\bar{x}, \bar{u})$. An adequate terminal set can, in principle, be computed as the maximal robust control (positively) invariant set by employing standard recursive set computations of viability (or set invariance) theory (Aubin, 1991; Kolmanovsky and Gilbert, 1998; Blanchini, 1999). However, in general case finite time computation of such a set is not guaranteed. In contrast to the standard results in set invariance, we provide a novel characterization of a family of the polytopic robust control invariant sets. The existence of a constraint admissible member of this family as well as the computation of the corresponding feedback controller can be efficiently realized by solving a *single linear or quadratic programming problem* (LP or QP). This set is then used to implement a standard robust time-optimal control scheme. We also remark that the recent methodology of *parametric programming* (Bemporad *et al.*, 2002; De Doná and Goodwin, 2000; Mayne and Raković, 2003) can be used to obtain low complexity controllers (Grieder *et al.*, 2003) that ensure robust constraint satisfaction as well as robust time–optimal convergence to the target set.

This paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 addresses the robust control invariance issue for linear systems with positive constraints. Section 4 gives an algorithm for the construction of a non-decreasing sequence of the robust control invariant sets – level sets of the value function for the robust time–optimal control problem. Section 5 presents an interesting numerical example. Finally, Section 6 contains concluding remarks.

NOTATION: Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$ and $\mathbb{N}_q \triangleq \{0, 1, \ldots, q\}$. Let $\mathbf{0}$ denote vector (or matrix) of zeros and I identity matrix of appropriate dimensions. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is a closed and bounded polyhedron. Given a polyhedral set $\mathcal{P} \triangleq \{z \mid C_p z \leq c_p\}$, the set $\mathcal{P}_{\varepsilon_p} \triangleq \{z \mid C_p z \leq c_p - \varepsilon_p\}$. Let $\mathbb{B}_p^q(z, r) \triangleq \{x \in \mathbb{R}^q \mid |x - z|_p \leq r\}$ be a p-norm ball in $\mathbb{R}^q$ centered at z, where $r \geq 0$ and $|\cdot|_p$ denotes the vector p-norm. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski set addition is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v \mid u \in \mathcal{U},\ v \in \mathcal{V}\}$ and the Pontryagin set difference by: $\mathcal{U} \ominus \mathcal{V} \triangleq \{x \mid x \oplus \mathcal{V} \subseteq \mathcal{U}\}$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^b$, we define $\bigoplus_{i=a}^b \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \cdots \oplus \mathcal{U}_b$. We use $2^{\mathcal{U}}$ to denote the power set (set of all subsets) of $\mathcal{U}$ and $d(u, \mathcal{U}) \triangleq \inf_{v \in \mathcal{U}} |u - v|$.

2. PRELIMINARIES

We consider the following discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + Bu + w, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown disturbance and $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. The disturbance w is persistent, but contained in a convex and compact set $W \subset \mathbb{R}^n$ that contains the origin. We make the standing assumption that the couple (A, B) is controllable. The system (2.1) is subject to the following set of hard state and control constraints:

$$(x, u) \in X \times U \tag{2.2}$$

where $X \subseteq \mathbb{R}_+^n$ and $U \subseteq \mathbb{R}_+^m$ are polyhedral and polytopic sets respectively. We denote by $\phi(k; x, \pi, \mathbf{w}(\cdot))$ the solution to (2.1) at time instant k if the initial state is x at time 0, the control policy $\pi \triangleq \{\mu_i(\cdot),\ i \in \mathbb{N}_{N-1}\}$ (where for each $i \in \mathbb{N}_{N-1}$, $\mu_i(\cdot) :\ X \to U$) and $\mathbf{w}(\cdot)$ is an admissible infinite disturbance sequence.

Let the set $\mathcal{F}$ denote the set of equilibrium points for the nominal part of difference equation (2.1) (i.e. $x^+ = Ax + Bu$) :

$$\mathcal{F} \triangleq \{(\bar{x}, \bar{u}) \mid (A - I)\bar{x} + B\bar{u} = \mathbf{0}\} \tag{2.3}$$

If $A - I$ is invertible than $\bar{x}(\bar{u}) \triangleq -(A - I)^{-1} B\bar{u}$ is a singleton for any $\bar{u} \in \mathbb{R}^m$. We need the following definitions in the sequel:

Definition 1. The set $\Omega \subset \mathbb{R}^n$ is a *robust control invariant* (RCI) set for the system (2.1) and constraint set (X, U, W) if $\Omega \subseteq X$ and for all $x \in \Omega$ there exists a $u \in U$ such that $Ax + Bu + w \in \Omega$ for all $w \in W$.

Definition 2. The set Ω is robust asymptotically (finite-time) attractive with domain of attraction Ψ iff, for all $x(0) \in \Psi$, $d(x(i), \Omega) \to 0$ as $i \to \infty$ (there exists a time I such that $x(i) \in \Omega$ for all $i \geq I$) for all admissible disturbance sequences.

3. ROBUST CONTROL INVARIANCE ISSUE

First, we characterize a family of the polytopic RCI sets for the system (2.1) for *unconstrained* case, for constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$, by extending a relevant result recently established in (Raković, 2005).

Let $M_i \in \mathbb{R}^{m\times n},\ i \in \mathbb{N}$ and for each $k \in \mathbb{N}$ let $\mathbf{M}_k \triangleq (M_0, M_1, \ldots, M_{k-2}, M_{k-1})$. An appropriate characterization of a family of RCI sets for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$ is given by the following sets for $k \geq n$:

$$R_k(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} D_i(\mathbf{M}_k)W \tag{3.1}$$

where the matrices $D_i(\mathbf{M}_k),\ i \in \mathbb{N}_k,\ k \geq n$ are defined by:

$$D_0(\mathbf{M}_k) = I,\ D_i(\mathbf{M}_k) \triangleq A^i + \sum_{j=0}^{i-1} A^{i-1-j}BM_j, i \geq 1 \tag{3.2}$$

providing that $\mathbf{M}_k$ satisfies:

$$D_k(\mathbf{M}_k) = \mathbf{0}. \tag{3.3}$$

Since the couple (A, B) is assumed to be controllable such a choice exists for all $k \geq n$. Let $\mathbb{M}_k$ denote the set of all matrices $\mathbf{M}_k$ satisfying condition (3.3):

$$\mathbb{M}_k \triangleq \{\mathbf{M}_k \mid D_k(\mathbf{M}_k) = \mathbf{0}\} \tag{3.4}$$

Theorem 1. (Raković, 2005) Given any $\mathbf{M}_k \in \mathbb{M}_k$ and the corresponding set $R_k(\mathbf{M}_k)$ there exists a control law $\nu : R_k(\mathbf{M}_k) \to \mathbb{R}^m$ such that $Ax + B\nu(x) \oplus W \subseteq R_k(\mathbf{M}_k),\ \forall x \in R_k(\mathbf{M}_k)$, i.e. the set $R_k(\mathbf{M}_k)$ is RCI for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$.

The family of RCI sets (3.1) for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$ is merely a subset of a richer family of RCI sets for system (2.1) defined by the following sets for $k \geq n$:

$$S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \triangleq \bar{x} \oplus R_k(\mathbf{M}_k) \tag{3.5}$$

and for all triples $(\bar{x}, \bar{u}, \mathbf{M}_k) \in \mathcal{F} \times \mathbb{M}_k$.

Theorem 2. (Raković, 2005) Given any triple $(\bar{x}, \bar{u}, \mathbf{M}_k) \in \mathcal{F} \times \mathbb{M}_k$ and the corresponding set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ there exists a control law $\mu : S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \to \mathbb{R}^m$ such that $Ax + B\mu(x) \oplus W \subseteq S_k(\bar{x}, \bar{u}, \mathbf{M}_k),\ \forall x \in S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$, i.e. the set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ is RCI for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$.

PROOF. Let $x \in S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ so that $x = \bar{x} + y$ for some $(\bar{x}, \bar{u}, y) \in \mathcal{F} \times R_k(\mathbf{M}_k)$. Let $\mu(x) = \bar{u} + \nu(y)$, where $\nu(\cdot)$ is the control law of Theorem 1. $x^+ \in A(\bar{x}+y) + B(\bar{u}+\nu(y)) \oplus W$. Since $(\bar{x}, \bar{u}) \in \mathcal{F}$, $\bar{x} = A\bar{x} + B\bar{u}$. Also $Ay + B\nu(y) \oplus W \subseteq R_k(\mathbf{M}_k)$ by Theorem 1. Hence, $Ax + B\mu(x) \oplus W = A\bar{x} + B\bar{u} + Ay + B\nu(y) \oplus W = \bar{x} + Ay + B\nu(y) \oplus W \subseteq \bar{x} \oplus R_k(\mathbf{M}_k) = S_k(\bar{x}, \bar{u}, \mathbf{M}_k),\ \forall x \in S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$.

The feedback control law $\mu :\ S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \to \mathbb{R}^m$ in Theorem 2 is a selection from the set valued map:

$$\mathcal{U}(x) \triangleq \bar{u} + \mathbf{M}_k \mathbf{W}(x) \tag{3.6}$$

where $\mathbf{M}_k \in \mathbb{M}_k$ and the set of *disturbance sequences* $\mathbf{W}(x)$ is defined for each $x \in S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ by:

$$\mathbf{W}(x) \triangleq \{\mathbf{w} \mid \mathbf{w} \in \mathbf{W}^k,\ \bar{x} + D\mathbf{w} = x\}, \tag{3.7}$$

where $\mathbf{W}^k \triangleq W \times W \times \ldots \times W$ and $D = [D_{k-1}(\mathbf{M}_k)\ \ldots\ D_0(\mathbf{M}_k)]$. A $\mu(\cdot)$ satisfying Theorem 2 can be defined, for instance, as follows:

$$\mu(x) \triangleq \bar{u} + \mathbf{M}_k \mathbf{w}^0(x) \tag{3.8a}$$

$$\mathbf{w}^0(x) \triangleq \arg\min_{\mathbf{w}} \{|\mathbf{w}|^2 \mid \mathbf{w} \in \mathbf{W}(x)\} \tag{3.8b}$$

The function $\mathbf{w}^0(\cdot)$ is piecewise affine, being the solution of a *parametric* quadratic programme; it follows that the feedback control law $\mu : S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \to \mathbb{R}^m$ is piecewise affine (being an affine map of a piecewise affine function). Implementation of $\mu(\cdot)$ can be simplified by noticing that $\mathbf{w}^0(x)$ in (3.8) can be replaced by any disturbance sequence $\mathbf{w} \triangleq \{w_0, w_1, \ldots, w_{k-1}\} \in \mathbf{W}(x)$.

Theorem 2 states that for any $k \geq n$ the RCI set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$, *finitely determined by* k, is easily computed if W is a polytope. The set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ is parametrized by the couple $(\bar{x}, \bar{u})$ and the matrix $\mathbf{M}_k$; this allows us to formulate an LP or QP that yields the set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ while minimizing an appropriate norm of the set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ or the standard Euclidean distance of the couple $(\bar{x}, \bar{u})$ from the desired *reference couple* $(\hat{x}, \hat{u})$ in case of the hard *positive* state and control constraints.

3.1 Optimized Robust Control Invariance Under Positivity Constraints

We consider the following case frequently encountered in practice:

$$W \triangleq \{Ed + f \mid |d|_\infty \leq \eta\} \tag{3.9}$$

where $d \in \mathbb{R}^t$, $E \in \mathbb{R}^{n\times t}$ and $f \in \mathbb{R}^n$ and:

$$\mathring{\mathcal{F}} \neq \emptyset \tag{3.10}$$

where $\mathring{\mathcal{F}} \triangleq \mathcal{F} \cap (\text{interior}(X) \times \text{interior}(U))$. We illustrate that in this case, one can formulate an LP or QP, whose feasibility establishes existence of a RCI set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ for the system (2.1) and constraint set (X, U, W). The control law $\mu(x)$ satisfies $\mu(x) \in U(\bar{x}, \bar{u}, \mathbf{M}_k)$ for all $x \in S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ where:

$$U(\bar{x}, \bar{u}, \mathbf{M}_k) \triangleq \bar{u} \oplus \bigoplus_{i=0}^{k-1} M_i W \tag{3.11}$$

The constraints (2.2) are satisfied if:

$$S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \subseteq X_{\varepsilon_x},\ U(\bar{x}, \bar{u}, \mathbf{M}_k) \subseteq U_{\varepsilon_u} \tag{3.12}$$

where $(\varepsilon_x, \varepsilon_u) \geq \mathbf{0}$.

Let $\gamma \triangleq (\bar{x}, \bar{u}, \mathbf{M}_k, \varepsilon_x, \varepsilon_u, \alpha, \beta)$ and

$$\Gamma \triangleq \{\gamma \mid (\bar{x}, \bar{u}, \mathbf{M}_k) \in \mathcal{F} \times \mathbb{M}_k, \\ S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \subseteq X_{\varepsilon_x} \cap \mathbb{B}_p^n(\hat{x}, \alpha), \\ U(\bar{x}, \bar{u}, \mathbf{M}_k) \subseteq U_{\varepsilon_u} \cap \mathbb{B}_p^m(\hat{u}, \beta), \\ (\varepsilon_x, \varepsilon_u, \alpha, \beta) \geq \mathbf{0}\} \tag{3.13}$$

where $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ and $U(\bar{x}, \bar{u}, \mathbf{M}_k)$ are given by (3.5) and (3.11), respectively, and $(\hat{x}, \hat{u})$ is the desired reference couple.

Let

$$d_1(\gamma) \triangleq q_\alpha \alpha + q_\beta \beta \\ d_2(\gamma) \triangleq |\bar{x} - \hat{x}|_Q^2 + |\bar{u} - \hat{u}|_R^2 \tag{3.14}$$

where the couple (q_α, q_β) and the positive definite weighting matrices Q and R are design variables. Consider the following minimization problems:

$$\mathbb{P}_k^i : \gamma^0 = \arg\min_{\gamma} \{d_i(\gamma) \mid \gamma \in \Gamma\}, \ \ i = 1, 2 \tag{3.15}$$

where $\gamma^0 \triangleq (\bar{x}, \bar{u}, \mathbf{M}_k, \varepsilon_x, \varepsilon_u, \alpha, \beta)^0$. It is easy to establish (Raković, 2005) that the the problem $\mathbb{P}_k^1$ is an LP and the problem $\mathbb{P}_k^2$ is a QP providing that $p = 1$ or $p = \infty$ in (3.13).

If the set $\Gamma \neq \emptyset$, there exists a RCI set $S_k = S_k(\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$ for the system (2.1) and constraint set (X, U, W), and corresponding control law $\mu(\cdot)$ defined by (3.7) and (3.8) with $(\bar{x}, \bar{u}, \mathbf{M}_k) = (\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$. Furthermore, there might exist more than one set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ that yields the optimal cost. The cost function might be modified. For instance, an appropriate choice for the cost function is a positively weighted quadratic norm of the decision variable γ that yields a unique solution, since in this case problem becomes a quadratic programming problem of the form $\min_\gamma \{|\gamma|_P^2 \mid \gamma \in \Gamma\}$, where P is positive definite and it represents a suitable weight. A relevant observation is:

Proposition 1. Suppose that the problem $\mathbb{P}_k^1$ ($\mathbb{P}_k^2$) is feasible for some $k \in \mathbb{N}$ and the optimal value of d_{1k} (d_{2k}) is d_{1k}^0 (d_{2k}^0), then, for every integer $s \geq k$, the problem $\mathbb{P}_s^1$ ($\mathbb{P}_s^2$) is also feasible and the corresponding optimal value of d_{1s} (d_{2s}) satisfies $d_{1s}^0 \leq d_{1k}^0$ ($d_{2s}^0 \leq d_{2k}^0$).

Thus we conclude that the first subproblem – checking the existence and the construction of a suitable target set that is robust control invariant set for the system (2.1) and constraint set (X, U, W) and the computation of the corresponding feedback controller can be efficiently realized by solving a *single* LP or QP (if necessary for sufficiently large $k \in \mathbb{N}$). The crucial advantage of our method lie in the fact that the hard *positive* state and control constraints are incorporated directly into the optimization problem allowing for the construction of an appropriate RCI set (target set) with a local piecewise affine feedback control law $\mu : S_k(\bar{x}, \bar{u}, \mathbf{M}_k) \to U$ that renders the set $S_k(\bar{x}, \bar{u}, \mathbf{M}_k)$ RCI. These results can be used in the synthesis of the robust time–optimal controller as we illustrate next.

4. ROBUST TIME–OPTIMAL CONTROL

In this section we assume that there exists a set $T \triangleq S_k(\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$ obtained by solving the problem $\mathbb{P}_k^1$ or ($\mathbb{P}_k^2$) for some $k \in \mathbb{N}$; T is compact, RCI for the system (2.1) and constraint set (X, U, W) and contains the point $\bar{x}^0$ in its interior; this set is a suitable target set.

The robust time–optimal control problem $\mathbb{P}(x)$ is defined, as usual, by:

$$N^0(x) \triangleq \inf_{\pi, N} \{N \mid (\pi, N) \in \Pi_N(x) \times \mathbb{N}_{N_{max}}\}, \tag{4.1}$$

where $N_{\max} \in \mathbb{N}$ is an upper bound on the horizon and $\Pi_N(x)$ is defined as follows:

$$\Pi_N(x) \triangleq \{\pi \mid (x_i, u_i) \in X \times U, \forall i \in \mathbb{N}_{N-1}, \\ \forall \mathbf{w}(\cdot), \ \ x_N \in T\} \tag{4.2}$$

where for each $i \in \mathbb{N}$, $x_i \triangleq \phi(i; x, \pi, \mathbf{w}(\cdot))$ and $u_i \triangleq \mu_i(\phi(i; x, \pi, \mathbf{w}(\cdot)))$. It should be observed that the solution is sought in the class of the state feedback control laws because of the additive disturbances, i.e. π is a control policy ($\pi = \{\mu_i(\cdot), \ i \in \mathbb{N}_{N-1}\}$, where for each $i \in \mathbb{N}_{N-1}$, $\mu_i(\cdot) : \ X \to U$). The solution to $\mathbb{P}(x)$ is

$$\big(\pi^0(x), N^0(x)\big) \triangleq \\ \arg\inf_{\pi, N} \{N \mid (\pi, N) \in \Pi_N(x) \times \mathbb{N}_{N_{max}}\}. \tag{4.3}$$

Note that, the value function of the problem $\mathbb{P}(x)$ satisfies $N^0(x) \in \mathbb{N}_{N_{max}}$ and for any integer i, the robust controllable set $X_i \triangleq \{x \mid N^0(x) \leq i\}$ is the set of initial states that can be *robustly* steered (steered for all $\mathbf{w}(\cdot)$) to the target set T, in i steps or less while satisfying all state and control constraints for all admissible disturbance sequences. Hence $N^0(x) = i$ for all $x \in X_i \setminus X_{i-1}$.

The robust controllable sets $\{X_i\}$ and the associated robust time-optimal control laws $\kappa_i : X_i \to 2^U$ can be computed by the following standard recursion (Mayne and Schroeder, 1997):

$$X_i \triangleq \{x \in X \mid \exists u \in U \text{ s.t.} \\ Ax + Bu \oplus W \subseteq X_{i-1}\} \tag{4.4}$$

$$\kappa_i(x) \triangleq \{u \in U \mid Ax + Bu \oplus W \subseteq X_{i-1}\}, \\ \forall x \in X_i \tag{4.5}$$

for $i \in \mathbb{N}_{N_{max}}$ with the boundary condition $X_0 = T = S_k(\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$.

The time-invariant control law $\kappa^0 : X_{N_{\max}} \to 2^U$ defined, for all $i \in \mathbb{N}_{N_{max}}$, by

$$\kappa^0(x) \triangleq \begin{cases} \kappa_i(x), & \forall x \in X_i \setminus X_{i-1},\ i \geq 1 \\ \mu(x), & \forall x \in X_0 \end{cases} \tag{4.6}$$

where the feedback control law $\mu(\cdot)$ is defined by (3.7)– (3.8) with $(\bar{x}, \bar{u}, \mathbf{M}_k) = (\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$, robustly steers any $x \in X_i$ to X_0 in i steps or less to X_0, while satisfying state and control constraints, and thereafter maintains the state in X_0. We now recall a standard result (Mayne and Schroeder, 1997) in robust time–optimal control:

Proposition 2. Suppose $X_0 = S_k(\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0) \neq \emptyset$, then the set sequence $\{X_i\}$ computed using the recursion (4.4) is a non-decreasing sequence of compact RCI sets (for the system (2.1) and constraint set (X, U, W), i.e. $X_i \subseteq X_{i+1} \subseteq X$ for all $i \in \mathbb{N}_{N_{max}}$; moreover for each $i \in \mathbb{N}_{N_{max}}$, X_i contains the point $\bar{x}^0$ in its interior.

The following property of the set-valued control law $\kappa^0(\cdot)$ defined in (4.6) follows directly from the construction of $\kappa^0(\cdot)$:

Theorem 3. The target set X_0 is robustly finite-time attractive for the closed-loop system $x^+ \in Ax + B\kappa^0(x) \oplus W$ with a region of attraction $X_{N_{\max}}$.

We observe that for any $i \in \mathbb{N}_{N_{max}}$ an appropriate selection of the control law $\kappa_i(x)$ for all $x \in X_i \setminus X_{i-1}$ can be obtained by employing the *parametric mathematical programming* as we briefly demonstrate next. For each $i \geq 1,\ i \in \mathbb{N}_{N_{max}}$ let:

$$Z_i \triangleq \{(x, u) \in X \times U \mid Ax + Bu \in X_{i-1} \ominus W\} \tag{4.7}$$

and let $V_i(x, u)$ be any linear or quadratic (strictly convex) function in (x, u), for instance:

$$V_i(x, u) \triangleq |Ax + Bu|_Q^2 \tag{4.8}$$

Since Z_i is a polyhedral set and since $V_i(x, u)$ is a linear or a quadratic (strictly convex) function it follows that for each $i \geq 1,\ i \in \mathbb{N}_{N_{max}}$ the optimization problem $\mathbb{P}_i(x)$:

$$\theta_i^0(x) \triangleq \arg\inf_u \{V_i(x, u) \mid (x, u) \in Z_i\} \tag{4.9}$$

is a *parametric* linear/quadratic problem. As is well known (Bemporad *et al.*, 2002; De Doná and Goodwin, 2000; Mayne and Raković, 2003; Bemporad *et al.*, 2003), the solution takes the form of a piecewise affine function of state $x \in X_i$:

$$\theta_i^0(x) = S_{i,j}x + s_{i,j},\ x \in R_{i,j}, j \in \mathbb{N}_{l_i} \tag{4.10}$$

where l_i is a finite integer and the union of polyhedral sets $R_{i,j}$ partition the set X_i, i.e. $X_i = \bigcup_{j \in \mathbb{N}_{l_i}} R_{i,j}$.

If we let:

$$i^0(x) \triangleq \arg\min_i \{i \in \mathbb{N}_{N_{max}} \mid x \in X_i\} \tag{4.11}$$

it follows that $\theta^0_{i^0(x)}(x) \in \kappa_i(x)$ for all $i \geq 1,\ i \in \mathbb{N}_{N_{max}}$.

Our final remark is that the presented results are also applicable, with a minor set of appropriate modifications, when the hard control and state constraints are arbitrary polytopes not necessarily satisfying $X \times U \subseteq \mathbb{R}^n_+ \times \mathbb{R}^m_+$.

5. NUMERICAL EXAMPLE

Our numerical example is the second order unstable system that is a linearized model of a flight vehicle sampled every 0.2 s.:

$$x^+ = \begin{bmatrix} 0.9625 & -0.1837 \\ 0.3633 & 0.8289 \end{bmatrix} x + \begin{bmatrix} 0.0618 \\ -0.5990 \end{bmatrix} u + w \tag{5.1}$$

where $w \in W \triangleq \{w \in \mathbb{R}^2 \mid |w|_\infty \leq 0.05\}$. The following set of hard semi–positive state and positive control constraints is required to be satisfied:

$$\begin{aligned} X =& \{x \mid 0 \leq x^1 \leq 10,\ -0.5 \leq x^2 \leq 10\}, \\ U =& \{u \mid 0 \leq u \leq 1\} \end{aligned} \tag{5.2}$$

where x^i is the i^{th} coordinate of a vector x. The control objective is to bring the system as close as possible to the origin, i.e. $(\hat{x}, \hat{u}) = (\mathbf{0}, 0)$ that is a point on the boundary of the constraint sets. The appropriate target set is constructed from the solution of the modified version of the problem $\mathbb{P}_k^1$, in which $p = \infty$ and $(\varepsilon_x, \varepsilon_u)$ were set to $\mathbf{0}$ and with the following design parameters:

$$(k, q_\alpha, q_\beta) = (9, 1, 1), \tag{5.3}$$

The optimal values of a triple $(\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$ are as follows: $\bar{x}^0 = (0.2421,\ -0.0000)'$, $\bar{u}^0 = 0.1468$ and:

$$\mathbf{M}_k{}^0 = \begin{bmatrix} -0.0627 & 1.4081 \\ -0.0000 & 0.0000 \\ -0.0000 & 0.0000 \\ -0.0000 & 0.0000 \\ 0.0000 & -0.0000 \\ 0.0000 & -0.0000 \\ 1.1753 & -0.0212 \\ 0.1611 & -0.1083 \\ 0.0000 & 0.0000 \end{bmatrix} \tag{5.4}$$

The RCI set $X_0 = S_k(\bar{x}^0, \bar{u}^0, \mathbf{M}_k^0)$ is shown together with the RCI set sequence $\{X_i\},\ i \in \mathbb{N}_{13}$ computed by (4.4) in Figure 1.

6. CONCLUSIONS

The main contribution of this note is a novel characterization of a family of RCI sets for which

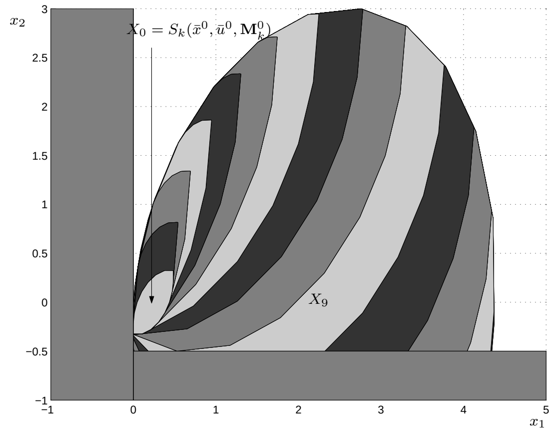

Fig. 1. RCI Set Sequence $\{X_i\},\ i \in \mathbb{N}_{13}$

the corresponding control law is non-linear (piecewise affine) enabling better results to be obtained compared with existing methods where the control law is linear. Construction of a member of this family that is constraint admissible can be obtained from the solution of an appropriately specified LP or QP. The *optimized robust control invariance* algorithms were employed to devise robust–time optimal controller that is illustrated by an example. The results can be extended to the case when disturbance belongs to an arbitrary polytope.

An appropriate and relatively simple extension of the presented results allows for efficient robust model predictive control of linear discrete time systems subject to positive state and control constraints and additive, but bounded disturbances. This relevant extension will be presented elsewhere.

ACKNOWLEDGMENT

The authors gratefully acknowledge helpful discussions with Dr. Alessandro Astolfi. All set computations were performed with the Geometric Bounding Toolbox (Veres, 2003).

REFERENCES

Aubin, J. P. (1991). *Viability theory*. Systems & Control: Foundations & Applications. Birkhauser. Boston, Basel, Berlin.

Bemporad, A. and M. Morari (1999). *Robustness in Identification and Control*. Chap. Robust model predictive control: A survey, pp. 207–226. Number 245 In: *Lecture Notes in Control and Information Sciences*. Springer-Verlag. Survey paper.

Bemporad, A., F. Borrelli and M. Morari (2003). Min-max control of constrained uncertain discrete-time linear systems. *IEEE Trans. Automatic Control* **48**(9), 1600–1606.

Bemporad, A., M. Morari, V. Dua and E.N. Pistikopoulos (2002). The explicit linear quadratic regulator for constrained systems. *Automatica* **38**, 3–20.

Bertsekas, D. P. and I. B. Rhodes (1971). On the minimax reachability of target sets and target tubes. *Automatica* **7**, 233–247.

Blanchini, F. (1992). Minimum-time control for uncertain discrete-time linear systems. In: *Proc. 31st IEEE Conference on Decision and Control*. Vol. 3. Tuczon AZ, USA. pp. 2629–34.

Blanchini, F. (1999). Set invariance in control. *Automatica* **35**, 1747–1767. survey paper.

De Doná, José A. and Graham G. Goodwin (2000). Elucidation of the state-space regions wherein model predictive and anti-windup strategies achieve identical control policies. In: *Proceedings of the American Control Conference*. Chicago, Illinois. pp. 1924–1928.

Grieder, P., P. Parillo and M. Morari (2003). Robust receding horizon control - analysis & synthesis. In: *CDC03*. Maui, Hawaii, USA.

Kerrigan, Eric. C. and David Q. Mayne (2002). Optimal control of constrained piecewise affine systems with bounded disturbances. In: *Proceedings of the IEEE 2002 Conference on Decision and Control*. Las Vegas, USA.

Kolmanovsky, I. and E. G. Gilbert (1998). Theory and computation of disturbance invariance sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications* **4**, 317–367.

Mayne, D. Q. and S. Raković (2003). Optimal control of constrained piecewise affine discrete-time systems. *Journal of Computational Optimization and Applications* **25**, 167–191.

Mayne, D. Q. and W. R. Schroeder (1997). Robust time-optimal control of constrained linear systems. *Automatica* **33**, 2103–2118.

Mayne, D. Q., J. B. Rawlings, C. V. Rao and P. O. M. Scokaert (2000). Constrained model predictive control: Stability and optimality. *Automatica* **36**, 789–814. Survey paper.

Pannocchia, G., S. J. Wright and J. B. Rawlings (2003). Existence and computation of infinite horizon model predictive control with active steady-state input constraints. *IEEE Trans. Automatic Control* **48**(6), 1002–1006.

Raković, Saša V. (2005). Robust Control of Constrained Discrete Time Systems: Characterization and Implementation. PhD thesis. Imperial College London, London, United Kingdom.

Rao, C. V. and J. B. Rawlings (1999). Steady states and constraints in model predictive control. *AIChe* **45**(6), 1266–1278.

Veres, S. M. (2003). Geometric Bounding Toolbox (GBT) for MATLAB. Official website: http://www.sysbrain.com.