

# MODELLING OF DISTURBANCE PROPAGATION IN ANISOTROPIC MEDIA USING DIFFERENTIAL GAME METHODS

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Abstract: The paper is devoted to the modelling of the propagation of acoustic waves at high frequency in anisotropic media. In this case, the WKB approximation results in eikonal equations whose Hamiltonians are neither convex nor concave in the impulse variable so that the Fermat principle fails. In this paper, methods of differential games are adopted for the analysis of wave propagation. If the Hamiltonian of a differential game approximates the Hamiltonian of the eikonal equation, then the solution to the game approximates the phase function satisfying the eikonal equation. Classification of singularities in differential games associated with wave propagation problems is done. Numerical results are presented for wave velocity surfaces typical for anisotropic quartz crystals. *Copyright©2005 IFAC*

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## 1. INTRODUCTION

The study of questions related to the propagation of acoustic waves in anisotropic media is very important for many applications such as biological sensors whose operating principle is based on the excitation and detection of acoustic waves in crystals. Propagation of acoustic waves of high frequency in isotropic media is well investigated and mathematically described by means of geometric optics and the Fermat principle. The latter expresses the fact that the rays, energy propagation trajectories, are minimizers of a functional which is the curve integral of some Lagrangian. In the

case of short waves in anisotropic media, one can also utilize geometric optics using WKB approximations. This leads to Hamilton-Jacobi (eikonal) equations describing dynamics of the wave phase (see e.g. (Barles, 1985)).

Many publications are devoted to the computation of ray fields and wave fronts in the case of convex in the impulse variable Hamiltonians (see, for example (Cerveny, 1985; Benamou, 1996; Sethian, 2002)). However, typical Hamiltonians in the anisotropic case are neither convex, nor concave so that one can not find a proper Lagrangian that would realize the Fermat principle.

This paper considers the possibility to approximate the Hamiltonians through conflict-controlled

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systems where the first player chooses the current velocity from an admissible set of velocities with the goal to minimize the signal propagation time, whereas the second, opposite, player strives to maximize this time. Although such an approach might look artificial, the point is that solutions to problems with close Hamiltonians must be also close each to other. Thus, the value function of the associated time-optimal differential game satisfies the eikonal equation of the wave propagation problem. Therefore, level sets of the value function represent wave fronts (first arrival) propagating in the anisotropic medium.

Similar to the Fermat principle, optimal trajectories in the associated differential game can apparently deliver information about the rays representing the energy flow. Note, that the structure of optimal trajectories in differential games is rather complicated because of the so-called singular surfaces that can attract, repulse and break the trajectories. In the paper, a preliminary analysis of possible types of singular surfaces is performed for Hamiltonians that correspond to typical phase velocity contours related to anisotropic crystals.

## 2. LAGRANGIAN AND HAMILTONIAN FORMALISM IN RAY TRACING

The rays along which disturbances propagate in a medium can be computed as the extremals of the functional representing the propagation time (Fermat' principle):

$$T = \int_{t_0}^{t_1} L(x, \dot{x}) dt, \quad L(x, \dot{x}) = \frac{|\dot{x}|}{V(x, \dot{x}/|\dot{x}|)} \quad (1)$$

where  $V$  is the energy wave velocity which specifies the *wave surface*  $\mathcal{V}$  at each point of the medium (Auld, 1972). Equivalently, the surface  $\mathcal{V}$  can be given as  $L(x, v) = 1$ .

The wave front propagation can be found by solving the Hamilton-Jacobi (eikonal) equation

$$H(x, \frac{\partial T}{\partial x}) = 1, \quad H(x, p) = |p|W(x, p/|p|), \quad (2)$$

where the  $W$  is the phase wave velocity that specifies the *velocity surface*  $\mathcal{W}$  at each point of the medium. The specifier  $N(x, e)$  of the *slowness surface*  $\mathcal{N}$  is defined through the relation:  $N(x, e)W(x, e) = 1$  for all unit vectors  $e$ . The surface  $\mathcal{N}$  can be also given as  $H(x, n) = 1$ .

The Lagrangian  $L(x, \dot{x})$  and the Hamiltonian  $H(x, p)$  are positively homogeneous of degree one in the second variable and admit the representation (Rund, 1966):

$$L^2(x, \dot{x}) = \langle G(x, \dot{x})\dot{x}, \dot{x} \rangle, \quad H^2(x, p) = \langle Q(x, p)p, p \rangle,$$

where  $G$  and  $Q$  are positively homogeneous of degree zero symmetric matrices.

Lagrangian and Hamiltonian can be transformed to each other through the formulas

$$Q = G^{-1}(x, \omega(x, p)), \quad G = Q^{-1}(x, \phi(x, \dot{x})),$$

if the following condition is fulfilled:

$$\det G(x, \dot{x}) = \det \left[ \frac{1}{2}(L^2)_{\dot{x}\dot{x}} \right] = \det Q^{-1}(x, p) \neq 0,$$

which means that the Hamiltonian and Lagrangian are strictly convex or concave. The function  $\dot{x} = \omega(x, p)$  is defined implicitly by

$$p = \phi(x, \dot{x}) = \frac{1}{2}(L^2)_{\dot{x}} = LL_{\dot{x}} = G(x, \dot{x})\dot{x}.$$

For anisotropic media the Hamiltonians may be convex-concave like as in differential games. In this case, one can not transform the Hamiltonian to a single-valued Lagrangian. Constructions based on Hamilton-Jacobi equations are nevertheless possible, though solutions may contain singularities known in differential games.

## 3. HAMILTON-JACOBI EQUATION IN GEOMETRICAL OPTICS

Eikonal equation (2) with  $W$  being the phase wave velocity arises, for example, when applying the WKB approximation to the elasticity equations

$$\rho u_{itt} - \frac{\partial}{\partial x_j} \left( C_{ijkl}(x) \frac{\partial u_l}{\partial x_k} \right) = 0, \quad i, j, k, l = 1, 2, 3.$$

describing anisotropic inhomogeneous media. Here  $u_i$  are the components of the displacement vector,  $\rho$  is the density, and  $C_{ijkl}$  is the elastic stiffness tensor. The summation over repeated indices is assumed.

One looks for solutions of the form

$$u_j^\varepsilon(t, x) = \varepsilon e^{iS(t, x)/\varepsilon} v_j^\varepsilon(t, x)$$

with

$$v_j^\varepsilon(t, x) = v_j^0(t, x) + \varepsilon v_j^1(t, x) + \dots,$$

and the initial conditions

$$u_j^\varepsilon(0, x) = \varepsilon e^{iS(0, x)/\varepsilon} \phi_j(x), \quad x \in R^3,$$

$$u_{jt}^\varepsilon(0, x) = \psi_j(x), \quad x \in R^3.$$

Here  $\varepsilon$  is a small parameter,  $\varepsilon = (\text{length scale} \times \text{wave number})^{-1}$ . Assuming  $\varepsilon \ll 1$ , restrict the analysis to the first term of the expansion for  $v_j^\varepsilon$ .

Inserting this ansatz into the elasticity equations and collecting terms of the order  $1/\varepsilon$  and 1, respectively, yields the following non-stationary eikonal and transport equations

$$\det \left[ \frac{1}{\rho} C_{ijkl}(x) \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_k} - \left( \frac{\partial S}{\partial t} \right)^2 I \right] = 0,$$

$$\rho S_{tt} v_i^0 + 2\rho S_{tv} v_{it}^0 - C_{ijkl}(x) \frac{\partial^2 S}{\partial x_j \partial x_k} v_l^0 - 2C_{ijkl}(x) \frac{\partial S}{\partial x_j} \frac{\partial v_l^0}{\partial x_k} - \left[ \frac{\partial}{\partial x_j} C_{ijkl}(x) \right] \frac{\partial S}{\partial x_k} v_l^0 = 0,$$

for  $i = 1, 2, 3$ . Let functions  $W_\alpha(x, n)$ ,  $\alpha = 1, 2, 3$ , be solutions to the eigenvalue problem

$$\det \left| \frac{1}{\rho} C_{ijkl}(x) n_j n_k - W^2 I \right| = 0$$

where  $|n| = 1$ . The functions  $W_\alpha(x, n)$  are known to be the phase velocities for three types of waves propagating in anisotropic media in the direction  $n$ . The non-stationary eikonal equation is obviously equivalent to the following three equations

$$S_{\alpha t} - |\nabla S_\alpha| W_\alpha \left( x, \frac{\nabla S_\alpha}{|\nabla S_\alpha|} \right) = 0, \quad \alpha = 1, 2, 3.$$

The substitution  $S_\alpha(t, x) = t + T_\alpha(x)$  leads to the eikonal equation

$$|\nabla T_\alpha(x)| W_\alpha \left( x, \frac{\nabla T_\alpha(x)}{|\nabla T_\alpha(x)|} \right) = 1, \quad \alpha = 1, 2, 3,$$

which determines the propagation times  $T_\alpha(x)$  for different wave modes (see Section 4).

#### 4. DESCRIPTION OF WAVES PROPAGATION USING DIFFERENTIAL GAME APPROACH

Generally, if the *slowness surface*  $\mathcal{N}$  is non-convex, it is impossible to transform the *velocity surface*  $\mathcal{W}$  to the *wave surface*  $\mathcal{V}$  to obtain the variational problem (1) with a single valued Lagrangian. Therefore, it is necessary to work with the original Hamiltonian (2) constructed on the base of the *velocity surface*  $\mathcal{W}$ . This Hamiltonian is convex-concave whenever the *slowness surface*  $\mathcal{N}$  is non-convex. Thus, equation (2) can be associated with a differential game (Isaacs, 1965) but not with an optimal control problem. When solving differential games, the main goal is to find the set of optimal trajectories in its complexity, i.e. the optimal phase portrait. Note that these trajectories are not arranged in a regular way as they are in the calculus of variations or even in the theory of optimal control. There are several specific trajectories, called singular lines (Isaacs, 1965; Melikyan, 1998) which match regions filled by regular trajectories. The knowledge of types and locations of singularities can deliver important information about the behavior of rays in the associated wave propagation problem.

The idea of this section is to formulate a differential game whose Hamiltonian coincides with or is close to the Hamiltonian of the corresponding wave propagation problem. It occurs that a game

with the so-called “simple dynamics” is appropriate for our purposes:

$$\dot{x} = u - v, \quad 0 \leq t \leq T, \quad u \in P, \quad v \in Q.$$

Here the vectors  $u$  and  $v$  are controls of two players  $P_1$  and  $P_2$ , respectively. The controls assume values from convex sets  $P$  and  $Q$  which can depend on the state vector  $x$ , i.e.  $P = P(x)$ ,  $Q = Q(x)$ . Such dependence reflects inhomogeneity of the elastic media. The goal of the first player is to bring the vector  $x(t)$  to a given terminal surface  $M$  as soon as possible, i.e. to minimize the time  $t$  of the first event  $x(t) \in M$ . The second player strives to maximize the time of reaching  $M$  using his controls  $v$ . In other words, the payoff of the game is the time of attaining  $M$ . The value function  $T(x)$  is the optimal guaranteed time of attaining  $M$  when starting from point  $x$ .

The function  $T(x)$  satisfies the following first-order PDE, (Isaacs, 1965):

$$\max_{u \in P} \min_{v \in Q} \langle T_x, -u + v \rangle = 1$$

for all points  $x$  where  $T$  is differentiable. At points of non-differentiability it satisfies the above equations in a viscosity sense (Subbotin, 1995; Fleming and Soner, 1993). Therefore, if the sets  $P$  and  $Q$  are constructed such that the left hand side of the last equation coincides with the Hamiltonian of the wave propagation problem, then  $T(x)$  is the time of the first arrival of the excitation at  $x$ . Therefore, the wave fronts are level sets of the value function  $T$ .

Assuming that  $P$  and  $Q$  are symmetric about the origin yields:

$$H(x, p) = \max_u \min_v \langle p, -u + v \rangle = \max_u \langle p, u \rangle - \max_v \langle p, v \rangle, \quad u \in P(x), v \in Q(x).$$

Obviously, the resulting Hamiltonian  $H(x, p)$ , generally, is not purely convex or concave with respect to  $p$ .

The theory of viscosity solutions provides the existence of piece-wise smooth solutions to the Hamilton-Jacobi equation

$$H \left( x, \frac{\partial S}{\partial x} \right) = 1.$$

Generally, the singular surfaces (lines in two dimensions) are of the following types: dispersal, equivocal, and focal, see Fig. 1. The dispersal surface does not contain trajectories, while the equivocal and focal surfaces consist of singular trajectories. The arrows in Fig. 1 show the motion of the phase point of the game in direct time. For the ray propagation problem the directions of all arrows must be reversed. Recall that the terminal surface  $M$  “absorbs” trajectories in a differential game, whereas it irradiates rays in the related wave propagation problem.

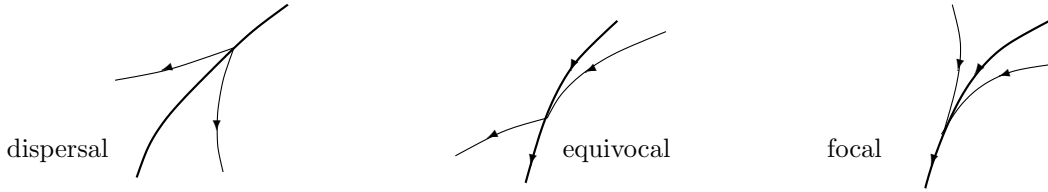


Fig. 1. Singular lines in direct time for game (backward time for rays).

The gradient  $\partial S/\partial x$  has a jump on singular lines. Denote the gradient on different sides of a singular line by  $p$  and  $q$ ,  $p \neq q$ . In (Melikyan, 1998), using the definition of viscosity solutions, the following necessary conditions for possible values of  $p$  and  $q$  are derived in the form of inequalities depending on the character of the gradient jump:

$$H(x, \lambda q + (1 - \lambda)p) \geq 1, \quad S = \max\{S^+, S^-\},$$

$$H(x, \lambda q + (1 - \lambda)p) \leq 1, \quad S = \min\{S^+, S^-\}.$$

Here  $\lambda$  runs in the interval  $[0, 1]$ ;  $S^+$  and  $S^-$  are smooth branches of the solution  $S$ .

Perform a geometrical analysis (see Fig. 2) of the above inequalities using slowness and velocity surfaces. Bear in mind a differential game with the Hamiltonian  $H(x, p) = |p|W(x, p/|p|)$ . Fix some vectors  $p, q \in \mathcal{N}$  and consider the segment  $AB$  given by  $\lambda q + (1 - \lambda)p$ ,  $0 \leq \lambda \leq 1$ . Consider a point  $D$  of this segment and the ray (half-line) passing through  $O$  and  $D$ . Let  $R$  and  $R'$  be points of the intersection of the ray with the surfaces  $\mathcal{N}$  and  $\mathcal{W}$ . Let  $d$ ,  $r$  and  $r'$  be the lengths of the vectors  $OD$ ,  $OR$  and  $OR'$ , respectively. Since  $H(x, p) = |p|W(x, p/|p|)$ , it holds:  $H = dr'$ . Note that  $rr' = 1$  by definition of the surfaces  $\mathcal{V}$  and  $\mathcal{W}$ . Since  $d > r$  for the point considered, the relation  $dr' > 1$  holds, and, consequently,  $H > 1$ . This consideration shows that the part  $A'B'$  of the segment  $AB$  satisfies the inequality  $H(x, p) \geq 1$ , the parts  $AA'$  and  $B'B$  satisfy the inequality  $H(x, p) \leq 1$ .

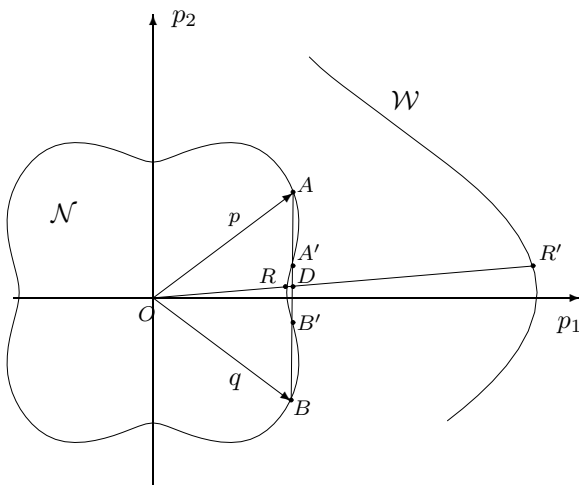


Fig. 2. Geometric construction showing that singular lines are possible.

This property allows us to specify candidates vectors  $p, q$  for a potential singular line. Figure 3 demonstrates possible configurations of  $p, q$  for dispersal and equivocal type; for the focal type one has tangency at the both points  $p$  and  $q$ . One can see that unique values of  $p, q$  for focal surface can be found through convexification of the surface  $\mathcal{N}$ . The tangency of trajectories in Fig. 1 leads to the tangency of the segments at a single point for the equivocal case, and at both points for the focal case provided that the Hamiltonian is smooth in neighborhoods of  $(x, p)$  and  $(x, q)$ , see (Melikyan, 1998). Note, that these singular lines are only potentially allowed by the necessary conditions. Which surfaces will appear in the solution depends on boundary conditions.

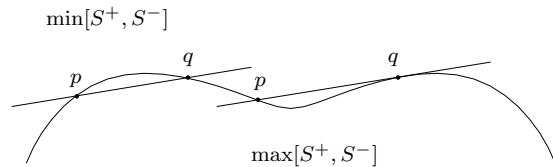


Fig. 3. Dispersal and equivocal situations.

## 5. EQUIVALENT DIFFERENTIAL GAMES, NUMERICAL RESULTS

This section demonstrates application of differential games to the problem of propagation of surface acoustic waves in a quartz wafer covered by a thin film made of the isotropic silicon dioxide. Such a structure is typical for acoustic sensors whose operation principle is based on the piezoelectric excitation of surface acoustic waves and the detection of the phase shift in the waves that arises because of deposition of an additional mass on the sensor surface. The wave propagation velocity in such multi-layered structures is obtained using numerical treatment of dispersion relations derived via substituting plane waves into material equations and matching the interface conditions between the layers, (Botkin *et al.*, 2004).

In Figure 4, the phase velocity contour for shear surface acoustic waves propagating in an ST-quartz wafer covered by a  $5 \mu\text{m}$   $\text{SiO}_2$  film is shown. The contour is symmetric with respect to the origin. In fact, surface shear waves exist

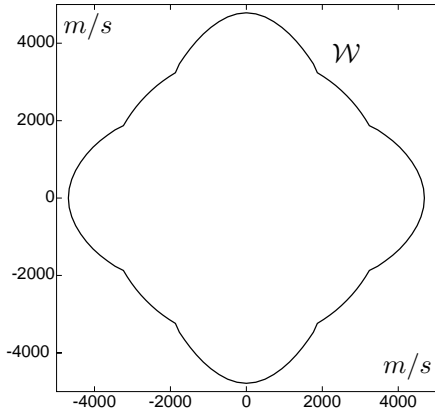


Fig. 4. Velocity contour for shear acoustic waves in ST-quartz covered by a  $5 \mu\text{m}$   $\text{SiO}_2$  film.

for directions from the set  $\Omega = \{(\cos \phi, \sin \phi) : \phi \in [-\pi/6 + k\pi/2, \pi/6 + k\pi/2], k = \overline{0, 3}\}$ . Such directions will be called feasible. For all non-feasible directions, the velocity value is the same and equals to  $3739.79 \text{ m/s}$ , which corresponds to the velocity of shear bulk waves in the structure considered.

Our next goal is to find constraint sets  $P$  and  $Q$  for the differential game

$$\dot{x} = u - v, \quad u \in P, \quad v \in Q,$$

so that, for every  $\ell \in \mathbb{R}^2$ ,  $|\ell| = 1$ , the value of the Hamiltonian

$$H(\ell) := \max_{u \in P} \min_{v \in Q} \langle \ell, -u + v \rangle$$

coincides with or is close to the value  $W(\ell)$  taken from the velocity surface  $\mathcal{W}$  shown in Fig. 4.

Assuming that  $P$  and  $Q$  are symmetric with respect to the origin yields

$$H(\ell) = \max_{u \in P} \langle \ell, u \rangle - \max_{v \in Q} \langle \ell, v \rangle.$$

Evidently, there exist many sets  $P$  and  $Q$  satisfying the relation

$$\max_{u \in P} \langle \ell, u \rangle - \max_{v \in Q} \langle \ell, v \rangle = W(\ell), \quad \ell \in \mathbb{R}^2.$$

To approximate the set  $\mathcal{W}$  along feasible directions, some parametric families  $P^{\alpha, \beta}$ ,  $\alpha \geq 1$ ,  $\beta \geq 1$ , and  $Q^\gamma$ ,  $\gamma > 0$ , having a simple analytical description are used. Optimization in  $\alpha, \beta, \gamma$  yields the optimal values  $\alpha_*, \beta_*, \gamma_*$ . Construct the set  $\tilde{P}^{\alpha_*, \beta_*}$  that deviates from  $P^{\alpha_*, \beta_*}$  along four directions close to  $(\cos(\pi/4 + k\pi/2), \sin(\pi/4 + k\pi/2))$ ,  $k = \overline{0, 3}$ . The result is represented in Fig. 5.

Consider now a game where the first player having the control parameter  $u \in P = \tilde{P}^{\alpha_*, \beta_*}$  at his disposal minimizes the time of attaining a given terminal set  $M$ , whereas the second player whose control parameter is  $v \in Q = Q^{\gamma_*}$  maximizes this time. The isochrones or level sets of the value function of this problem are wave fronts in the wave propagation problem with the velocity contour

$\mathcal{W}$  provided that  $M$  generates shear waves with waves vectors locally transversal to its boundary. Using a further development of algorithms (see (Patsko and Turova, 1995), (Patsko and Turova, 2001)) for computation of level sets of value functions for time-optimal game problems, one can obtain a portrait of the propagation of wave fronts for sources with very complicated geometries. The propagation time can also be found very precisely which might be helpful when estimating the sensitivity of acoustic sensors.

In Fig. 6, numerically computed wave fronts propagating from a curved source  $M$  in  $x_1$ -direction are shown. Corner points of the fronts, being connected, form singular lines. These lines are classified using the above mentioned algorithm. Black lines in Fig. 6 are equivocal, white lines are focal, and gray lines are dispersal. In Fig. 7 the graph of  $H(x, \lambda q + (1-\lambda)p)$  versus  $\lambda$  is depicted for the point  $x = O$ , see Fig. 6, where two equivocal lines start, compare also with Fig. 3.

One can see that the dispersal lines correspond to the intersection of rays (they can be generally continued beyond the first arrival); equivocal lines absorb rays from one side, guide them and radiate to the other side; focal lines guide rays and radiate them to the both sides.

Regular characteristics are known to represent physical rays. According to the authors' conjecture the same is true for singular characteristics so that the optimal phase portrait of such a game represents the complete set of rays of first arrival in the corresponding physical problem. The global structure of rays depends, of course, upon the initial conditions in addition to the Hamiltonian. This knowledge can be useful for the study of focusing and beam-steering properties of specially designed transducers.

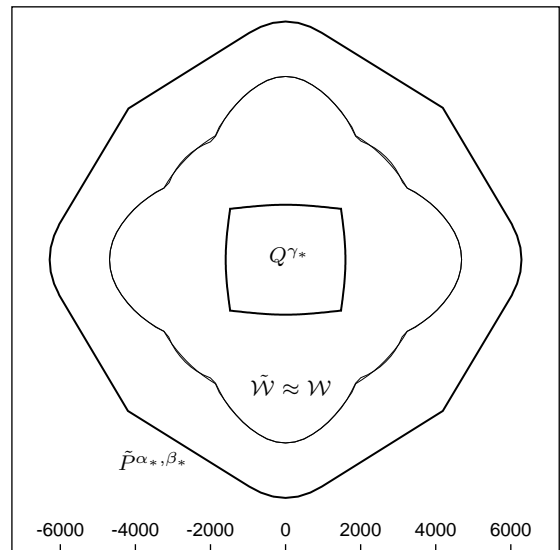


Fig. 5. Sets  $\tilde{P}^{\alpha_*, \beta_*}$ ,  $Q^{\gamma_*}$ ,  $\mathcal{W}$ , and  $\tilde{\mathcal{W}}$ .

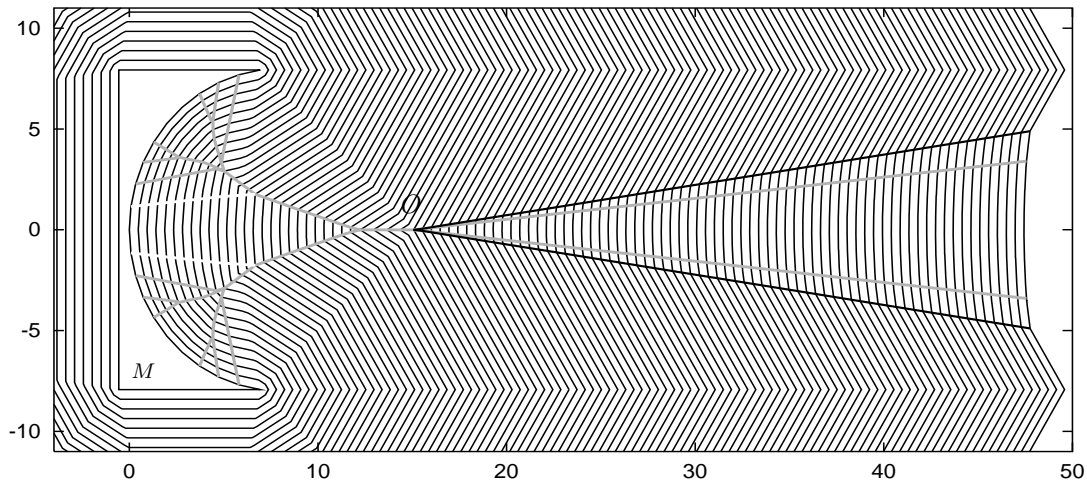


Fig. 6. Wave fronts and singular lines for acoustic sensor-like structure.

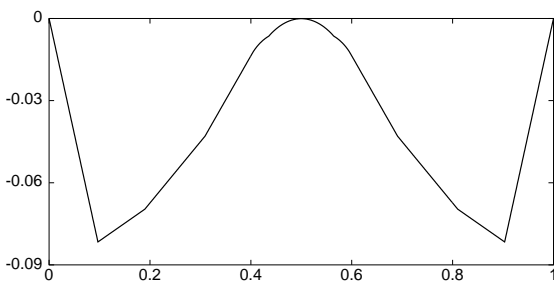


Fig. 7. Hamiltonian's behavior at the starting point for two equivocal lines.

## 6. CONCLUSION

Convex-concave Hamiltonians inherent to differential games can be used to approximate the Hamilton-Jacobi equations describing the propagation of waves in anisotropic media. With this approach, the wave fronts are given by level sets of the value function and the rays can be obtained through construction of singular lines.

An interesting question for further research is related to physical interpretation of the objectives of the players (minimization and maximization of the propagation time). Physical experiments confirming the structure of ray fields computed seem to be very useful.

## REFERENCES

- Auld, B. A. (1972). *Acoustic Fields and Waves in Solids. I*. Krieger Publishing Company. Malabar.
- Barles, G. (1985). Remarks on a flame propagation model. Technical Report 464. INRIA, Sophia Antipolis.
- Benamou, J.-D. (1996). Big ray tracking: Multivalued travel time field computation using viscosity solutions of eikonal equation. *J. of Comp. Physics* **128**, 463–474.
- Botkin, N.D., K.-H. Hoffmann, O.A. Pykhteev and V.L. Turova (2004). Numerical computation of dispersion relations for multi-layered anisotropic structures. In: *Proceedings of the Nanotechnology conference and trade show*. Vol. 2. Boston. pp. 411–414.
- Cerveny, V. (1985). The application of ray tracing to the numerical modeling of seismic wave fields in complex structures. In: *Seismic Shear Waves: Handbook of Geophysical Exploration* (G. Dohr, Ed.). Vol. 15A. pp. 1–124. Geophysical Press. London-Amsterdam.
- Fleming, W. H. and H. M. Soner (1993). *Controlled Markov Processes and Viscosity Solutions*. Springer Verlag. New York.
- Isaacs, R. (1965). *Differential Games*. John Wiley. New York.
- Melikyan, A. A. (1998). *Generalized Characteristics of First Order PDEs: Applications in Optimal Control and Differential Games*. Birkhäuser. Boston.
- Patsko, V. S. and V. L. Turova (1995). Numerical solution of two-dimensional differential games. Technical report. Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences. Ekaterinburg.
- Patsko, V. S. and V. L. Turova (2001). Level sets of the value function in differential games with the homicidal chauffeur dynamics. *International Game Theory Review* **3**(1), 67–112.
- Rund, H. (1966). *The Hamilton-Jacobi Theory in the Calculus of Variations*. D. Van Nostrand Company LTD. London, New York.
- Sethian, J.-A. (2002). Fast algorithms for optimal control, anisotropic front propagation and multiple arrival. In: *Proceedings of the ICM*. Vol. 3. Beijing: Higher Education Press. pp. 735–746.
- Subbotin, A. I. (1995). *Generalized Solutions of First Order PDEs: the Dynamical Optimization Perspective*. Birkhäuser. Boston.