AN ALGORITHM FOR COMPUTING HETEROCLINIC ORBITS AND ITS APPLICATION TO CHAOS SYNTHESIS IN THE GENERALIZED LORENZ SYSTEM

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Abstract: In this paper, an algorithm for computing heteroclinic orbits of nonlinear systems, which can have several hyperbolic equilibria, is suggested and analyzed both analytically and numerically. The method is based on a representation of the invariant manifold of a hyperbolic equilibrium via a certain exponential series expansion. The algorithm for computing the series coefficients is derived and the uniform convergence of the series is theoretically proved. The algorithm is then applied to computing heteroclinic orbits numerically in the generalized Lorenz system, thereby theoretically justifying the previously demonstrated existence of chaotic oscillations in this important class of dynamical systems.

Keywords: Generalized Lorenz canonical form, Heteroclinic orbit, Ši'lnikov criterion

1. INTRODUCTION

To theoretically define and prove the existence of chaotic behaviors in a particular system is perhaps the most difficult task in the study of chaotic dynamics. It is therefore common to study chaotic systems numerically, based on time series data, especially in the fields of engineering and applied physics. To analyze chaotic systems theoretically, an important role is played by the so-called homoclinic and heteroclinic orbits, thanks to the now-classical result of Ši'lnikov (Shilnikov, 1965; Shi'lnikov, 1970; Silva, 1993; Wiggins, 1988), who characterized the existence of chaos in terms of some simple properties of hyperbolic equilibria and the existence of heteroclinic or homoclinic orbits. Some studies on the heteroclinic or homoclinic orbits can be found in (Vakakis et al, 1998; Robinson1, 1989; Robinson2, 1992; Robinson3, 2000; Huang, 2003; Hassard and Zhang, 1994; Lassoued and Mathlouthi, 1992; Champneys et al, 1996; Bai and Champneys, 1996) and some applications of this criterion can be found, for example, in (Zhou and Chen, 2004; Zhou et al, 2004; Zhou et al, 2003; Zhou et al, 2005).

In this paper, an algorithm is developed for computing heteroclinic orbits. Loosely speaking, a heteroclinic orbit of a pair of hyperbolic equilibria is a solution of the system that has a one-sided limit as $t \to \infty$, with another one-sided limit as $t \to -\infty$. Geometrically, each heteroclinic orbit is the intersection of a stable manifold of one equilibrium with an unstable invariant manifold of another equilibrium. Therefore,

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the key ingredient of this new approach to finding a heteroclinic orbit in the proposed algorithm is to compute these invariant manifolds and their dependence on the system parameters. To this end, one is then able to determine for which parameters a heteroclinic connection exists.

Rather than presenting detailed and exhausting mathematical proofs for the existence, based on various conditions, this paper describes clearly an easily-implementable algorithm that can determine whether or not a heteroclinic orbit exists and, if so, providing a good approximation of it. In such a way, the algorithm contributes to computer-aided analysis of various chaotic systems.

2. PRELIMINARIES

To specify the systems which the proposed algorithm is applicable to, the following definition is needed.

Definition 1. *Consider a smooth nonlinear dynamical system,*

$$\dot{x} = f(x), \quad x \in R^n. \tag{1}$$

This system is said to be almost everywhere (a.e.) real-analytic eliminable, if there exists a smooth function, $h : R^n \mapsto R$, an almost everywhere real-analytic mapping[2] $\psi : R^n \mapsto R^n$, *such that*

$$x = \psi(y, \dot{y}, \ddot{y}, \cdots, y^{(n-1)}), \tag{2}$$

where

$$y^{(0)} := y := h(x), \;\; y^{(1)} := \dot{y} := \frac{\partial h}{\partial x} f(x),$$

$$y^{(j)} := \frac{\partial y^{(j-1)}}{\partial x} f(x), \; j = 1, 2, \ldots . \tag{3}$$

Note that $x(t)$ is a solution of (1) if and only if $y(t)$ is a solution of

$$y^{(n)} + \varphi(y, \dot{y}, ..., y^{(n-1)}) = 0,$$

$$\varphi := -\left(\frac{\partial y^{(n-1)}}{\partial x} f\right)\left(\psi^{-1}(y, \dot{y} \ldots, y^{(n-1)})\right). \tag{4}$$

If all the above mappings are globally real-analytic on R^n, then the system is said to be real-analytic eliminable. Below, (4) is referred to as the (a.e.) real-analytic eliminated form, while (4) is called the (a.e.) real-analytic elimination transformation.

[2] Functions or mappings are said to be almost everywhere analytic if for every bounded subset $\mathcal{S}$ of R^n there exists a finite collection of open and mutually nonintersecting sets, the union of their closures being $\mathcal{S}$, such that on each of them the corresponding function or mapping has a converging Taylor series.

Typical (a.e.) real-analytical eliminable systems include quadratic systems, like the Lorenz and Rössler systems (Rössler, 1996; Sparrow, 1982; Tucker, 1999). It includes the generalized Lorenz system (Čelikovský and Chen, 2002; Čelikovský and Vaněček, 1994; Vaněček, and Čelikovský, 1996; Zhou et al, 2005), as shown below.

Example 1. Generalized Lorenz system in the heteroclinic canonical form (Zhou et al, 2005)

$$\begin{cases} \dfrac{dx_1}{dt} = \xi x_1 + x_2 \\ \dfrac{dx_2}{dt} = (\xi(\lambda_1 + \lambda_2 - \xi) - \lambda_1\lambda_2)\, x_1 \\ \qquad + (\lambda_1 + \lambda_2 - \xi)x_2 - x_1 x_3 \\ \dfrac{dx_3}{dt} = \lambda_3 x_3 + x_1 x_2, \end{cases} \tag{5}$$

where $\xi \in (-\infty, 0)$. This system is (a.e.) real-analytic eliminable, since for $y = h(x) := x_1$

$$x_1 = y, \quad x_2 = \dot{x}_1 - \xi x_1, \tag{6}$$

$$x_3 = -\frac{\ddot{x}_1 - (\lambda_1 + \lambda_2)\dot{x}_1}{x_1} - \lambda_1\lambda_2, \tag{7}$$

$$\frac{d}{dt}\left(\frac{\ddot{x}_1 - (\lambda_1 + \lambda_2)\dot{x}_1}{x_1}\right) - \lambda_3\left(\frac{\ddot{x}_1 - (\lambda_1 + \lambda_2)\dot{x}_1}{x_1}\right)$$
$$+ x_1\dot{x_1} - \xi x_1^2 - \lambda_1\lambda_2\lambda_3 = 0. \tag{8}$$

One can proceed in a straightforward way to obtain the corresponding eliminated form:

$$y^{(3)} - (\lambda_1 + \lambda_2 + \lambda_3)\ddot{y} + \lambda_3(\lambda_1 + \lambda_2)\dot{y} - \lambda_1\lambda_2\lambda_3 y$$
$$-\xi y^3 + y^2\dot{y} - \frac{\ddot{y}\dot{y} + (\lambda_1 + \lambda_2)\dot{y}^2}{y} = 0. \tag{9}$$

The last expression can be decomposed on any bounded set into two converging Taylor series. For any $A > 0$, one has

$$\frac{1}{y} = \begin{cases} \dfrac{1}{A}\displaystyle\sum_{k=0}^{\infty}\left(\frac{A - y}{A}\right)^k & \text{for } \; y \in (0, 2A) \\ \dfrac{1}{-A}\displaystyle\sum_{k=0}^{\infty}\left(\frac{A + y}{A}\right)^k & \text{for } \; y \in (-2A, 0) \end{cases}$$

where both series converge in the given range, so for large enough value of A the above statement holds.

Example 2. Lur'e system *The special case of the Lur'e system, considered in (Alvarez et al, 2002) for synthesis and synchronization of chaos, has the following form:*

$$\dot{x}_1 = x_2, \; \ldots \; , \dot{x}_{n-1} = x_n, \; \dot{x}_n = \alpha(x_1, \ldots, x_n),$$

which is (a.e.) real-analytic eliminable if ϕ is an (a.e.) real-analytical function. The corresponding elimination transformation is

$$y = x_1, \;\; \dot{y} = x_2, \;\; \ddot{y} = x_3, \;\; \ldots, \;\; y^{(n-1)} = x_n,$$

resulting in the following elimination form:

$$y^{(n)} - \alpha(y, \dot{y}, \dots, y^{(n-1)}) = 0.$$

The following Lemma is straightforward.

Lemma 1. *Consider an equilibrium x_E of system (1) and suppose that its eliminated form as well as elimination transformation are well defined at this point (i.e., the denominators of the corresponding rational expressions are nonzero). Then, every eigenvalue of the approximate linearization of system (1) at x_E is a solution of the characteristic equation of the linear part of system (4) w.r.t. the point*

$$y = h(x_E), \quad \dot{y} = 0,$$
$$\ddot{y} = 0, \quad \cdots, \quad y^{(n)} = 0, \quad \varphi(h(x_E), 0, \cdots, 0) = 0.$$

3. THE ALGORITHM FOR COMPUTING THE HETEROCLINIC ORBIT

3.1 Computation of stable and unstable manifolds

Suppose that system (1) has a hyperbolic equilibrium x_E and let $y_E = h(x_E)$ be the corresponding equilibrium of its (a.e.) real-analytic eliminated form (4) such that both the eliminated form and the elimination transformation are well defined at x_E, y_E. To demonstrate the key ideas in the computational algorithm, for brevity, assume that there are n real distinct eigenvalues, while the cases of multiple and complex eigenvalues may be treated analogously. Assume that there are n_s negative and n_u positive eigenvalues, due to the hyperbolicity, with $n_s + n_u = n$. Denote also by $\lambda_1, ..., \lambda_{n_s}$ the corresponding negative eigenvalues, while by $\lambda_{n_s+1}, ..., \lambda_n$, the positive ones.

It is now to discuss how to compute a stable invariant manifold. The same algorithm will then be applied to computing an unstable invariant manifold by replacing $t \mapsto -t$ and negative eigenvalues $\mapsto$ positive eigenvalues throughout the algorithm.

First, search for a particular trajectory $\phi(t)$, belonging to the stable manifold of the equilibrium x_E, as follows:

$$\phi(t) = y_E + \sum_{k_1,\dots,k_{n_s}=1}^{\infty} a_{k_1,\dots k_{n_s}} \exp\left(t \sum_{i=1}^{n_s} \lambda_i k_i\right), \qquad (10)$$

where $a_{k_1,\dots k_{n_s}}$, $k_1, ..., k_{n_s} = 1, 2, ...$ are undetermined coefficients.

Then, it is straightforward but tedious to show that one has to use the above basis of the expansion if a series expansion with negative exponents is to be used. As a matter of fact, first-order terms in the expansion constitute the solution of the linearized system of (4) at y_E, and higher-order terms represent the nonlinearity.

It is important to note that the undetermined coefficients can be determined recursively, thanks to the (a.e.) real-analyticity of the corresponding eliminated form, by comparing coefficients in the same exponentials. In other words, if one lets

$$a_1 := a_{1,0,\dots,0}, \dots, a_{n_S} := a_{0,\dots 0,1}, \qquad (11)$$

then each coefficient $a_{k_1,\dots,k_{n_S}}$, $k_1, ..., k_{n_s} = 2, 3, ...$, is completely determined by the system parameters, $a_1, ..., a_{n_S}$, in the following way:

$$a_{k_1,\dots k_{n_s}} = \phi_{k_1,\dots k_{n_s}} \prod_{i=1}^{n_s} a_1^{k_1} ... a_{n_S}^{k_{n_s}}, \qquad (12)$$

where $\phi_{k_1,\dots,k_m}$ are some known constants depending on the systems parameters. Both a's and ϕ's may be determined by the following scheme:

Algorithm for solving the undetermined exponential series:

First, let $P_{char}(\cdot)$ be the characteristic polynomial of the linear part of the corresponding eliminated form (4) at y_E, and denote by ν the operator given by higher-order terms of this eliminated form in Taylor expansion at y_E. Then, the undetermined coefficients are recursively computed as follows:

(1) **Initial step.** *Choose arbitrary*

$$a_1 := a_{1,0,\dots,0}, \dots, a_{n_S} := a_{0,\dots 0,1}.$$

(2) **Recursive (inductive) step.** *Suppose (12) holds for all $a_{k_1,\dots,k_{n_s}}$ such that $\sum_{i=1}^{n_S} k_i \le j$, and consider any $a_{k'_1,\dots,k'_{n_s}}$ with $\sum_{i=1}^{n_S} k'_i = j+1$. Substituting (10) into the corresponding eliminated form of (1) and re-grouping the terms, one has*

$$\begin{aligned}
&P_{char}\left(\sum_{i=1}^{n_s} \lambda_i k'_i\right) a_{k'_1,\dots,k'_{n_s}} \exp\left(t \sum_{i=1}^{n_s} \lambda_i k'_i\right) \\
&\quad + \sum_{k_1,\dots,k_{n_s}=1}^{\sum_{i=1}^{n_S} k_i = j} P_{char}\left(\sum_{i=1}^{n_s} \lambda_i k_i\right) a_{k_1,\dots,k_{n_s}} \\
&\qquad \times \exp\left(t \sum_{i=1}^{n_s} \lambda_i k_i\right) \\
&= \nu\Big\{ \sum_{k_1,\dots,k_{n_s}=1}^{\sum_{i=1}^{n_S} k_i \le j} a_{k_1,\dots,k_{n_s}} \exp\big(t \sum_{i=1}^{n_s} \lambda_i k_i\big) \\
&\quad + \sum_{k_1,\dots,k_{n_s}=1}^{\sum_{i=1}^{n_S} k_i > j} a_{k_1,\dots,k_{n_s}} \exp\big(t \sum_{i=1}^{n_s} \lambda_i k_i\big) \Big\} \\
&= \nu\left(\sum_{k_1,\dots,k_{n_s}=1}^{\sum_{i=1}^{n_S} k_i \le j} a_{k_1,\dots,k_{n_s}} \exp\left(t \sum_{i=1}^{n_s} \lambda_i k_i\right)\right) + \beta,
\end{aligned}$$

where in β there are only coefficients $a_{k_1'',\ldots,k_{n_s}''}$ with $\sum_{i=1}^{n_S} k_i'' > j+1$. Therefore, $a_{k_1',\ldots,k_{n_s}'}$ with $\sum_{i=1}^{n_S} k_i' = j+1$ may be computed from $a_{k_1,\ldots,k_{n_s}}$ such that $\sum_{i=1}^{n_S} k_i \le j$, and so the recursive step is well defined.

In addition to its usefulness for computing the appropriate coefficients, this algorithm may also serve as a theoretical proof by induction of the solvability for an eliminated form in terms of the above exponential series. In doing so, it can be rigorously proved that the expansion (10) is uniformly convergent for all $t \ge 0$, which is omitted here.

Notice also that in the expansion (10), there are n_s free parameters. This is quite natural, since the stable manifold has dimension n_s.

3.2 Computing the heteroclinic orbit

A heteroclinic orbit is by definition the intersection of a stable invariant manifold of one equilibrium and an unstable invariant manifold of another equilibrium.

Suppose that we are given two equilibria, $x_{E1},\ x_{E2}$, with the corresponding $y_{E1},\ y_{E2}$, both hyperbolic with only distinct real eigenvalues. First, compute the stable-manifold and unstable-manifold expressions as discussed in the previous subsection, yielding

$$\phi^S(t) = y_{E1} + \sum_{k_1,\ldots,k_{n_s}=1}^{\infty} a^S_{k_1,\ldots,k_{n_s}} \exp\left(t \sum_{i=1}^{n_s} \lambda_i^S k_i\right), \quad (13)$$

$$\phi^U(t) = y_{E2} + \sum_{k_1,\ldots,k_{n_U}=1}^{\infty} a^U_{k_1,\ldots,k_{n_U}} \exp\left(-t \sum_{i=1}^{n_U} \lambda_i^U k_i\right). \quad (14)$$

Now, the remaining step is to compute the conditions under which both stable and unstable manifolds mutually intersect. For simplicity, consider the case when the obtained eliminated form is a global one. [3] Then, both (13) and (14) should define **the same time function**, *i.e.*, for all $t \in (-\infty, \infty)$ one should have $\phi^S(t) = \phi^U(t)$. As a matter of fact, this gives n equations to be satisfied by differentiating (after $n-1$ differentiations; further relations are consequence of the previous ones, due to the nth-order eliminated form). Thanks to the "nice" structure of the parameters, [4] one of them, say a_1, can be "sacrificed," leading to

$$a_1 = \exp(t\lambda_1), \;\; t = \frac{\log(a_1)}{\lambda_1}.$$

[3] For the almost-everywhere situation, it is analogous but more technical, as one has to glue both solutions on a singular boundary between two Taylor representations by limiting arguments.
[4] The power of some parameter a_i is the same as the k_i in the corresponding exponent.

This eliminates the time variable t from all equations. Thus, one has $n_S + n_U - 1$ free parameters and n equations to be satisfied.

There are only two cases to consider: transversal and non-transversal.

In the transversal case, the sum of dimensions of the stable manifold of one equilibrium and the unstable manifold of another equilibrium is strictly greater than n. In this case, $n_S + n_U - 1 \ge n$, and one may guarantee the existence and computability of the heteroclinic connection for all system parameters on an open set in the parameter space. Such a heteroclinic connection is also said to be structurally stable, as small system parameter perturbations may slightly change the heteroclinic orbit, but not break it completely.

In the non-transversal case, $n_S + n_U = n$. In this case, one has to use some additional system parameters to fulfill the required equality, so that a heteroclinic orbit exists only for some special values of the system parameters. This case is called a structurally unstable heteroclinic connection, since it is completely broken by some arbitrary small variations of system parameters.

In addition, the above computational method for heteroclinic orbits can be similarly applied to compute homoclinic orbits.

The proposed **Algorithm** is summarized as follows:

Step 1. Calculate the system equilibria, assuming that they are hyperbolic saddle foci.

Step 2. Use the undetermined coefficient method to calculate the corresponding coefficients of the time-expansion series of the solutions belonging to stable and unstable manifolds with $n_S + n_U$ free parameters.

Step 3. Eliminate the time variable and find parameters that solve numerically the finite-order truncations of the corresponding equations up to the selected order.

Step 4. Use these parameters to determine the heteroclinic orbit approximation.

4. AN APPLICATION TO GENERALIZED LORENZ SYSTEM IN CANONICAL FORM

To demonstrate the above-developed algorithm, consider the generalized Lorenz canonical form (GLCF) (5) and its rational form (9). It can be easily seen that (5) has two hyperbolic saddle foci, denoted by $E_1(y_0,\, -\xi y_0,\, -\lambda_1\lambda_2)$ and
$E_2(-y_0,\, \xi y_0,\, -\lambda_1\lambda_2)$, where $y_0 = \sqrt{\frac{-\lambda_1\lambda_2\lambda_3}{\xi}}$. According to the above discussion, it is assumed that

$$y(t) \equiv \varphi(t) = -y_0 + \sum_{k=1}^{\infty} a_k e^{k\alpha t}, \quad (15)$$

where $\alpha < 0$ is an undetermined constant, so are all a_k $(k \geq 1)$.

Substituting (15) into (9) and then comparing the coefficients of $e^{k\alpha t}$ $(k \geq 1)$ of the same power terms, one finds that the exponent α satisfies

$$\alpha^3 - (\lambda_1 + \lambda_2 + \lambda_3)\alpha^2$$
$$+\lambda_3 \left(\lambda_1 + \lambda_2 - \frac{\lambda_1 \lambda_2}{\xi}\right) \alpha + 2\lambda_1 \lambda_2 \lambda_3 = 0, \quad (16)$$

while coefficients a_k can be recursively determined (see (Zhou and Chen, 2004) for more details), as

$$a_n = \varphi_n \cdot a_1^n, \quad n > 1, \qquad (17)$$

where φ_n $(n > 1)$ are some known functions depending on α, λ_1, λ_2, λ_3 and ξ, and a_1 is a free parameter.

To this end, part of the stable manifold of the heteroclinic orbit is determined. Similarly, part of its unstable manifold can be determined with a free parameter, denoted by b_1. Thus, one component of the corresponding heteroclinic orbit has the form

$$\varphi(t) = \begin{cases} -\gamma + \sum_{k=1}^{\infty} a_k e^{k\alpha t} & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ \gamma + \sum_{k=1}^{\infty} b_k e^{-k\alpha t} & \text{for } t < 0. \end{cases} \qquad (18)$$

The remaining question is how to glue these two parts together, so that they tranversally intersect. For this purpose, one needs to impose the following conditions:

$$\varphi(0_-) = \varphi(0_+),$$
$$\varphi'(0_-) = \varphi'(0_+), \quad \varphi''(0_-) = \varphi''(0_+), \qquad (19)$$

which can be validated by suitably utilizing the two free parameters a_1 and b_1.

Finally, a numerical simulation result is shown in Figure 1.

5. CONCLUSIONS AND RESEARCH OUTLOOKS

An algorithm for computing heteroclinic orbits has been developed. Existence of heteroclinic orbits is known to be an important indicator of a possible chaotic attractor in a system. Therefore, the new algorithm facilitates chaos synthesis, i.e., an active design of systems parameters via some control action to create chaotic behaviors purposefully. The algorithm has been tested on the generalized Lorenz system (Čelikovský and Chen, 2002), but still needs further investigation on its numerical properties for more general classes of systems. Nevertheless, the generalized Lorenz system is already general enough with rich

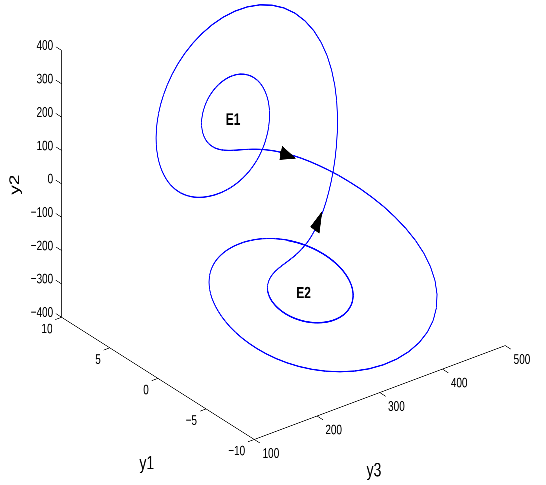

Fig. 1. The heteroclinic orbit in the GLCF (5) with $\lambda_1 = \frac{-11+\sqrt{1201}}{2}$, $\lambda_2 = -\frac{11+\sqrt{1201}}{2}$, $\lambda_3 = \frac{-8}{3}$ and $\xi = -12.5$.

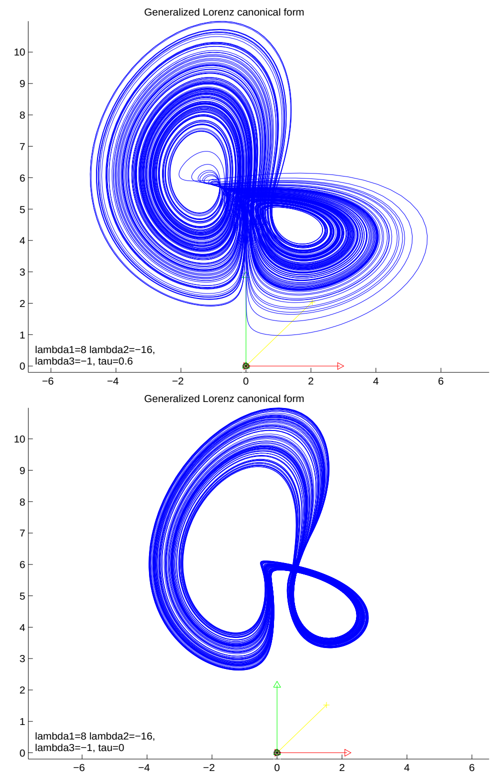

Fig. 2. Two typical chaotic attractors of the generalized Lorenz system.

chaotic behaviors, as demonstrated in (Čelikovský and Chen, 2002), as also shown in Figure 2. The present work is the first step towards establishing more theoretical fundamentals for chaos existence in the generalized Lorenz system and perhaps also in some other complex systems.

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