CONTROLLABILITY AND QUADRATIC STABILIZATION OF A CLASS OF DISCRETE LINEAR REPETITIVE PROCESSES

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Abstract: Repetitive processes are a distinct class of two-dimensional systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or two-dimensional (2D) systems theory. In this paper we define a new model for these processes necessary to represent dynamics which arise in some applications areas and which are not included in the currently used model. Then we proceed to define quadratic stability for this case and develop the first results on a control theory in the form of pass controllability and the design of physically based control laws.

Keywords: repetitive processes, 2D systems, spatio-temporal dynamics, controllability, stability and stabilization, LMI

1. INTRODUCTION

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < \infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p)$, $0 \le t \le \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynam-

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ics of the next pass profile $y_{k+1}(p)$, $0 \leq p \leq \alpha$, $k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (Edwards, 1974). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control schemes (Amann *et al.*, 1998) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle (Roberts, 2000). Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass.

In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between, in particular, the class of so-called discrete linear repetitive processes and 2D linear systems described by the extensively studied Roesser (Roesser, 1975) or Fornasini-Marchesini (Fornasini and Marchesini, 1978) state space models. Discrete linear repetitive processes are distinct from such 2D linear systems in the sense that information propagation in one of the two separate directions (along the pass) only occurs over a finite duration. As a result, recourse to solving control problems by this route has only proved of limited use and leaves many key systems theoretic and control design issues to be addressed by the development of suitable theory and algorithms.

In this paper, we first propose a new model for discrete linear repetitive processes which captures features of the dynamics of these processes which are excluded from the currently used model(s) but which arise in a number of potential applications areas. Then we proceed to develop results on the controllability properties of this model and also on so-called quadratic stabilization. Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix. We use $(*)$ to denote the transpose of matrix blocks in some of the LMIs employed (which are required to be symmetric).

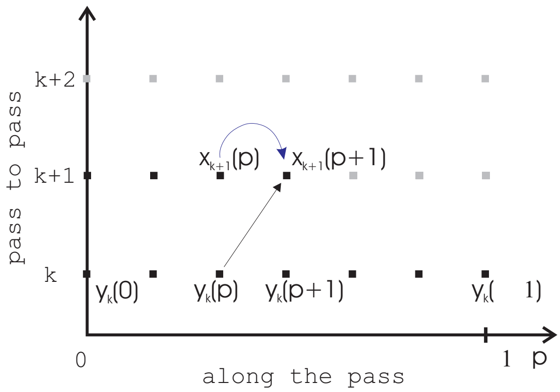

Fig. 1. Illustrating the updating structure of (1).

2. BACKGROUND AND PRELIMINARY ANALYSIS

The most basic discrete linear repetitive process state space model (Rogers and Owens, 1992) has the following form over $0 \leq p \leq \alpha$, $k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p). \end{aligned} \tag{1}$$

Here on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector,$y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and$u_k(p) \in \mathbb{R}^r$ is the vector of control inputs.To complete the process description, it is necessary to specify the boundary conditions, i.e. the state initial vector on each pass and the initial pass profile. Here no loss of generality arises from assuming

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \qquad k \geq 0 \\ y_0(p) &= f(p), \qquad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{2}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known.

The dynamics of (1) and, in particular, the updating structure can be visualized as in Fig. 1. This model however cannot be used in some practically related situations. For example, in (Rogers *et al.*, 2002) it was shown that the structure of the pass state initial vector sequence alone can cause instability in the sense of (Rogers and Owens, 1992) (this reference gives the stability theory for linear repetitive processes which is based on ensuring that the oscillations which increase in amplitude cannot occur in either pass to pass or along the pass). The route was to add terms to $x_{k+1}(0)$ in (2) which are an explicit function of the previous pass profile. In this paper we consider the following model over $k \geq 0$ and $0 \leq p \leq \alpha - 1$ is

$$\begin{aligned} x_{k+1}(p) &= A_0 x_k(p) + A_1 x_k(p+1) \\ &\quad + B_0 u_k(p) + B_1 u_k(p+1) \end{aligned} \tag{3}$$

with boundary conditions

$$\begin{aligned} x_0(p) &= g(p), \qquad 0 \leq p \leq \alpha \\ x_k(\alpha) &= g_k, \qquad k \geq 0 \end{aligned} \tag{4}$$

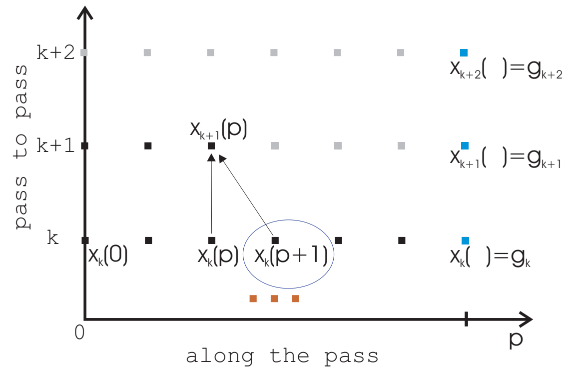

Fig. 2. Illustrating the updating structure of (3).

where now $g(p)$ is an $n \times 1$ vector whose entries are known functions of p over $[0, \alpha]$, and g_k is an $n \times 1$ vector with known constant entries.

The dynamics of (3) and (4) can be visualized as in Fig. 2 and this diagram together with Fig 1 can be used to explain the core differences between these two models (i.e. the one given by (1) and (2) and the second by (3) and (4)). In 2D systems terms, we see that the first model is quarter plane causal and the second is not. Also this new model does not have any joint updating of variables in both directions. Moreover, the boundary conditions are radically different, since in the first model the pass initial conditions are independently specified (i.e. the sequence $d_{k+1}, k \geq 0$) and in the second these are the end pass value $x_k(\alpha) = g_k$ from the previous pass. The causal structure in the first model can be the source of difficulty in modelling many practically relevant cases in, for example, robotics (Arimoto *et al.*, 1984) and (Yamakita and Furuta, 1991), multidimensional signal processing (Jain, 1981), or the discretization of partial differential equations (Levy *et al.*, 1990).

This model can be also considered as a discrete counterpart of the partial differential equation model

$$\frac{\partial x(t,\tau)}{\partial \tau} = \widehat{A}_0 x(t,\tau) + \widehat{A}_1 \frac{\partial x(t,\tau)}{\partial t}$$

which finds numerous engineering applications in modelling and control of transport-reaction processes, see e.g. (Panagiotis and Christofides, 2001). Assume now that $x_k(p)$ is a discrete approximation of $x(t,\tau)$ with $x_k(p) = x(kT_2, pT_1)$, where T_1 and T_2 are discretization periods along the directions τ and t respectively. Now introduce

$$\frac{\partial x(t,\tau)}{\partial \tau} := \frac{x_{k+1}(p) - x_k(p)}{T_1}$$

and

$$\frac{\partial x(t,\tau)}{\partial t} := \frac{x_k(p+1) - x_k(p)}{T_2}$$

which yield the following model of the form considered in this paper

$$x_k(p+1) = \left[I + T_1\widehat{A}_0 - \frac{T_1}{T_2}\widehat{A}_1\right] x_k(p) + \frac{T_1}{T_2}\widehat{A}_1 x_k(p+1).$$

The analysis in this paper will make extensive use of the well known Schur's complement formula for matrices and the following well known result.

Lemma 1. For any appropriately dimensioned matrices Σ_1, Σ_2 and F such that $F^T F \leq I$ together with a scalar $\epsilon > 0$:

$$\Sigma_1 F \Sigma_2 + \Sigma_2^T F \Sigma_1^T \leq \epsilon^{-1}\Sigma_1\Sigma_1^T + \epsilon\Sigma_2^T\Sigma_2. \quad (5)$$

3. 1D EQUIVALENT MODEL AND CONTROLLABILITY

Consider a process described (1) and (2). Then (see (Rogers and Owens, 1992)) it has turned out that a powerful approach to some (but by no means all) control related problems is to exploit their inherent 2D linear systems structure and, in effect, adapt tools/results first developed for 2D linear systems described by the extensively studied Roesser (Roesser, 1975) and Fornasini Marchesini (Fornasini and Marchesini, 1978) state space models. In cases where this approach is not applicable, e.g., pass controllability (Gałkowski *et al.*, 1998) and the presence of so-called dynamic boundary conditions (Owens and Rogers, 1999) which have no Roesser or Fornasini Marchesini model equivalents, the 1D equivalent model has provided the analysis basis on which to solve the problems being considered. Here we show that the 1D equivalent model does extend directly to the class of discrete linear repetitive processes considered here and, as a first use, we show that it immediately allows us to characterize the very important property of pass controllability, i.e. the ability to use causal control action to force the process to produce a specified pass profile on a specified pass number. (An alternative version does not pre-specify the pass number but only that a specified pass profile is produced on some pass during the processes evolution.) In actual fact, the construction of the 1D equivalent model mirrors very closely that reported previously (see, for example (Gałkowski *et al.*, 1998)) and hence only the main steps are detailed here.

First define the so-called global state, input and pass profile vectors of dimensions $n\alpha \times 1$, $m\alpha \times 1$, and $l\alpha \times 1$ respectively for (3)

$$X(k) = \begin{bmatrix} x_k(0) \\ x_k(1) \\ x_k(2) \\ \vdots \\ x_k(\alpha-1) \end{bmatrix}, \; U(k) = \begin{bmatrix} u_k(0) \\ u_k(1) \\ u_k(2) \\ \vdots \\ u_k(\alpha) \end{bmatrix}. \quad (6)$$

Then the 1D equivalent model for the dynamics of discrete linear repetitive processes described by (3)and (4) is given by

$$X(k+1) = \Gamma X(k) + \Sigma U(k) + \Psi_0 x_k(\alpha) \quad (7)$$

where

$$\Psi_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ A_1 \end{bmatrix}, \Sigma = \begin{bmatrix} B_0 & B_1 & & & 0 \\ & B_0 & B_1 & & \\ & & B_0 & \ddots & \\ & & & \ddots & \ddots \\ 0 & & & & B_0 \; B_1 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} A_0 & A_1 & & & 0 \\ & A_0 & A_1 & & \\ & & A_0 & \ddots & \\ & & & \ddots & A_1 \\ 0 & & & & A_0 \end{bmatrix}. \quad (8)$$

The formal definition of pass profile controllability is as follows.

Definition 2. Let k^* be an arbitrarily chosen pass number which satisfies $k^* \leq M$, $M = n\alpha$ for a discrete linear repetitive process described by (3) and (4). Then such processes are said to be pass profile controllable if there exists control input vectors defined over $0 \leq p \leq \alpha$, $0 \leq k \leq k^*$ which will drive the process to a pre-define pass profile on pass k^*.

It has been shown in (Gałkowski *et al.*, 1998) and (Rogers *et al.*, 2002) that pass profile controllability for processes described by (1) can be completely characterized by use of the 1D equivalent model. It is a routine exercise to conclude that this is also true for processes described by also holds for processes described by (3) and (4). This is stated formally as follows.

Theorem 3. Discrete linear repetitive processes described by (3) and (4) are pass profile controllable in the sense of Definition 2 if, and only if, the matrix pair (Γ, Σ) is controllable in the 1D discrete linear systems sense, or equivalently if, and only if, matrix pair $(A_0, \begin{bmatrix} B_0 & B_1 \end{bmatrix})$is controllable in the 1D discrete linear systems sense.

4. QUADRATIC STABILITY AND STABILIZATION

The stability theory (Rogers and Owens, 1992) for linear repetitive processes which includes an example described by (1) and (2) as a special case consists of two distinct concepts but in the vast majority of cases, it is the stronger of these which is required. This is termed stability along the pass and several equivalent sets of necessary and sufficient conditions for processes described by, for example, (1) and (2) to have this property are known. In effect, stability along the pass demands that bounded inputs produce bounded sequences of pass profiles independent of the pass length (here boundedness is defined in terms of the norm on the underlying function space). Mathematically, the way to treat this is to allow $\alpha \to \infty$.

In this paper we use another definition of stability for discrete linear repetitive processes, including those modelled by (3) and (4). This is termed quadratic stabilization and here we will develop it to the stage of producing computable stability tests and control law design algorithms. The basic idea (recall again the unique control problem for these processes) is to define a quadratic Lyapunov energy function for each pass with $\alpha \to \infty$, sum over all passes to give the so-called total Lyapunov function, and then quadratic stabilization demands that the process dissipates this energy from pass-to-pass.

The total Lyapunov function is given by

$$V(k) = \sum_{p=0}^{\infty} x_k^T(p)\widetilde{P}x_k(p) \quad (9)$$

where $\widetilde{P} > 0$, and

$$V(0) < \infty \quad (10)$$

and the following is the formal definition of quadratic stability.

Definition 4. A discrete linear repetitive process described by (3) and (4) is said to be quadratically stable if, and only if, there exists a matrix $\widetilde{P} > 0$ such that

$$V(k+1) < V(k), \; \forall\, k \geq 0 \quad (11)$$

and (10) holds.

The following result gives an LMI based interpretation of this property which forms the basis of the analysis in the rest of this paper. Note that this condition is sufficient but not necessary and hence there is a degree of conservativeness associated with its use. (In the case of, for example, discrete linear repetitive processes described by (1) and (2) it is known that, of the methods currently available, it is only LMI based sufficient conditions which allow control law design for stability and/or performance as against just supplying conditions for stability under control action.)

Theorem 5. A discrete linear repetitive process described by (3) and (4) is quadratically stable if $\sum_{p=0}^{\infty} \|x_0(p)\| < \infty$ and $\exists$ matrices $P > 0$ and $Q > 0$ such that the following LMI is feasible

$$\begin{bmatrix} A_0^T(P+Q)A_0 - P & A_0^T(P+Q)A_1 \\ A_1^T(P+Q)A_0 & A_1^T(P+Q)A_1 - Q \end{bmatrix} < 0. \quad (12)$$

Proof From (9) and (3) we have that

$$V(k+1)=\sum_{p=0}^{\infty} x_{k+1}^T(p)\widetilde{P}x_{k+1}(p)$$

$$=\sum_{p=0}^{\infty}(A_0x_k(p)+A_1x_k(p+1))^T\widetilde{P}(A_0x_k(p)+A_1x_k(p+1))$$

$$=\sum_{p=0}^{\infty}\Big[x_k^T(p)A_0^T\widetilde{P}A_0x_k(p)+x_k^T(p)A_0^T\widetilde{P}A_1x_k(p+1)$$
$$+x_k^T(p+1)A_1^T\widetilde{P}A_0x_k(p)+x_k^T(p+1)A_1^T\widetilde{P}A_1x_k(p+1)\Big]$$

where $\widetilde{P}=P+Q$. Also (9) can be rewritten as

$$V(k)=x_k^T(0)\widetilde{P}x_k(0)+\sum_{p=1}^{\infty}x_k^T(p)\widetilde{P}x_k(p)$$
$$=x_k^T(0)\widetilde{P}x_k(0)+\sum_{p=0}^{\infty}x_x^T(p+1)\widetilde{P}x_k(p+1)$$

and we can now write

$$V(k)=\sum_{p=0}^{\infty}\Big[x_k(p)^TPx_k(p)+x_k^T(p+1)Qx_k(p+1)\Big]$$
$$+x_k(0)^TQx_k(0)\widehat{=}x_k(0)^TQx_k(0)+S(k).$$

Now, it is straightforward to see that the requirement of (12) is equivalent to $V_{k+1}<S_k$ and hence (noting that $x_k(0)^TQx_k(0)>0\forall x_k(0)\in\mathbb{R}^n$) the proof is completed by noting that the infinite sum condition ensures (10) holds. □

4.1 Stabilization by state feedback

Here we produce the first results on the control of processes described by (3) and (4) under the action of the following control law over $k\geq 0$ and $0\leq p\leq\alpha$

$$u_k(p)=Kx_k(p). \tag{13}$$

In effect, this control law is state feedback applied at the current point (also termed memoryless in the repetitive process literature). Forming the closed loop process and applying the result of Theorem 5 to the resulting state space model gives closed loop quadratic stability if there exists a matrix K and matrices $P>0$ and $Q>0$ such that the following LMI is feasible

$$\begin{bmatrix} (A_0+B_0K)^T(P+Q)(A_0+B_0K)-P & (A_0+B_0K)^T(P+Q)(A_1+B_1K) \\ (A_1+B_1K)^T(P+Q)(A_0+B_0K) & (A_1+B_1K)^T(P+Q)(A_1+B_1K)-Q \end{bmatrix}<0. \tag{14}$$

The following result now gives a sufficient condition for the existence of a quadratically stabilizing control law of the form (13).

Theorem 6. A discrete linear repetitive process described by (3) and (4) is quadratically stable by a control law of the form (13) if $\exists$ matrices N, $Y>0$, $Z>0$, and $U>0$, such that the following LMI is feasible

$$\begin{bmatrix} -Z & 0 & (A_0Y+B_0N)^T \\ 0 & -U & (A_1Y+B_1N)^T \\ A_0Y+B_0N & A_1Y+B_1N & -Y \end{bmatrix}<0 \tag{15}$$

and

$$Z+U=Y. \tag{16}$$

If (15) holds, then a stabilizing K in the control law (13) is given by

$$K=NY^{-1}. \tag{17}$$

Proof An obvious application of the Schur's complement formula to (14) yields

$$\begin{bmatrix} -P & 0 & (A_0+B_0K)^T \\ 0 & -Q & (A_1+B_1K)^T \\ (A_0+B_0K) & (A_1+B_1K) & -(P+Q)^{-1} \end{bmatrix}<0.$$

Next, left and right multiply this last expression by $\mathrm{diag}\{I,I,(P+Q)\}$, multiply the result by $\mathrm{diag}\{(P+Q)^{-1},(P+Q)^{-1},(P+Q)^{-1}\}$, and finally make the substitutions

$$\begin{aligned} Z&=(P+Q)^{-1}P(P+Q)^{-1}>0\\ U&=(P+Q)^{-1}Q(P+Q)^{-1}>0\\ Y&=(P+Q)^{-1}>0 \end{aligned} \tag{18}$$

to obtain

$$\begin{bmatrix} -Z & (*) & (*) \\ 0 & -U & (*) \\ (A_0+B_0K)Y & (A_1+B_1K)Y & -Y \end{bmatrix}<0 \tag{19}$$

and it is immediate from (18) that $Z+U=Y$. Finally, use of (17) completes the proof. □

5. NUMERICAL EXAMPLE

As a numerical example to demonstrate the application of Theorem 6 consider the case when

$$A_0=\begin{bmatrix} -0.26 & -1.62 \\ 0.80 & 1.62 \end{bmatrix},\quad A_1=\begin{bmatrix} -1.90 & -1.69 \\ 0.38 & 0.95 \end{bmatrix},$$
$$B_0=\begin{bmatrix} -0.10 & -1.25 \\ -1.46 & -0.86 \end{bmatrix},\quad B_1=\begin{bmatrix} 1.22 & 0.02 \\ 0.92 & 1.96 \end{bmatrix}.$$

In this case (15) holds, as do (16) and (17) with

$$Y=\begin{bmatrix} 1352.207 & 1921.414 \\ 1921.414 & 2863.086 \end{bmatrix},\ Z=\begin{bmatrix} 967.6176 & 1338.597 \\ 1338.5975 & 1928.818 \end{bmatrix},$$
$$U=\begin{bmatrix} 384.5895 & 582.816 \\ 582.8165 & 934.268 \end{bmatrix},\ N=\begin{bmatrix} 4579.7440 & 6792.544 \\ -3582.4700 & -5191.444 \end{bmatrix},$$

and the resulting control law matrix is given by

$$K=\begin{bmatrix} 0.3390 & 2.1449 \\ -1.5697 & -0.7598 \end{bmatrix}.$$

The open loop pass profile sequence is shown in Fig. 3 and the closed loop Fig. 4 with zero input and reference vector respectively. The boundary conditions were $x_0(p)=1(p), 0\leq p\leq 25, x_k(\alpha)=0,\ k>0$.

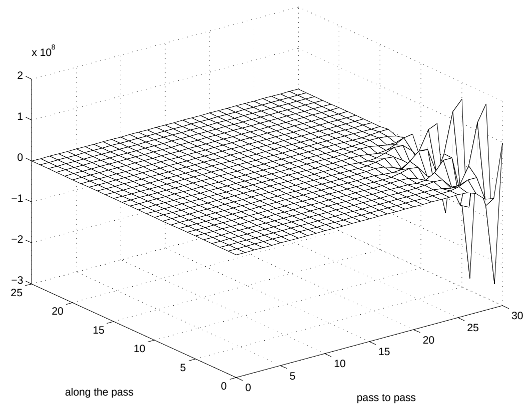

Fig. 3. Simulation for open loop process – $x_k^1(p)$.

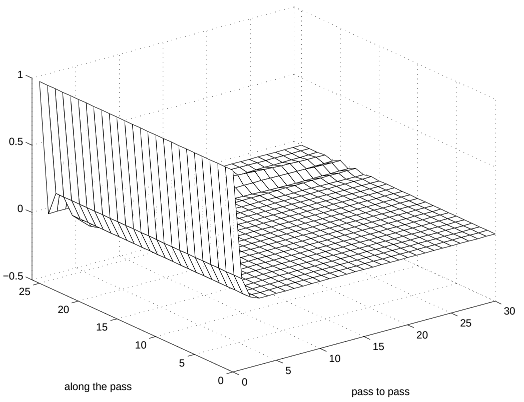

Fig. 4. Simulation for closed loop process – $x_k^1(p)$.

6. CONCLUSIONS

This paper has proposed a new model for discrete linear repetitive processes to include terms missing from current models but which can arise in applications. Some fundamental results on the systems theoretic properties of this new model have also been developed. These relate to controllability and quadratic stability open and closed loop.

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