THE INHERENT ROBUSTNESS OF CONSTRAINED LINEAR MODEL PREDICTIVE CONTROL

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Abstract: We show that a sufficient condition for the robust stability of constrained linear model predictive control is for the plant to be open-loop stable, for zero to be a feasible solution of the associated quadratic programme and for the input weighting be sufficiently high. The result can be applied equally to state feedback and output feedback controllers with arbitrary prediction horizon. If integral action is included a further condition on the steady state modelling error is required for stability.

Keywords: Predictive control, constraints, robust stability.

1. INTRODUCTION

Model predictive control (MPC) is a popular control strategy widely used in industry for plants with constraints (Qin and Badgwell, 2003). We are concerned with demonstrating the robustness of linear MPC to plant uncertainty with stable plants. Linear MPC has a linear state space model, linear equality and inequality constraints and a quadratic cost function with weights on both predicted states and inputs.

It might seem intuitively obvious that with sufficiently high weighting on the control input such a controller would be both nominally and robustly stable. However there are remarkably few results in the literature concerning constrained linear MPC's robustness to model uncertainty. A sufficient condition for robust stability of state feedback MPC is provided by (Zheng, 1999), while sufficient conditions for nominal stability of output feedback MPC are provided by (Zheng and Morari, 1995) and (Findeisen *et al.*, 2003). More generally, the majority of the literature is devoted to the further augmentation of the MPC cost or constraints to guarantee stability: see (Mayne *et al.*, 2000) for a survey of methodologies for guaranteeing nominal state feedback stability and more recently (Kerrigan and Maciejowski, 2004; Sakizlis *et al.*, 2004) and references therein for guaranteeing robustness. However we believe Zafiriou's critique of such approaches (Zafiriou, 1990) remains pertinent (briefly Zafiriou observed that such augmentation "dramatically increases the computational load" and recommended as an alternative the study of existant control structures for robustness).

We have recently shown that the multivariable circle criterion can be used to guarantee the closed-loop stability of certain MPC schemes (Heath *et al.*, 2004), provided the constraints allow zero as a feasible solution to the associated constrained optimisation problem. This is always true (for example) if the only constraints are simple bounds on the inputs.

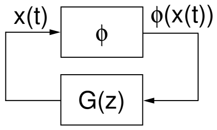

Fig. 1. Feedback around the nonlinearity.

In this paper we use the result to provide a sufficient condition for the robust stability of both state feedback and output feedback MPC, with and without integral action. In particular, if there is no integral action, it is sufficient that both plant and model are stable and the input weighting is sufficiently high. We also consider the case with integral action. In particular we consider the two-stage form corresponding to the scheme of (Muske and Rawlings, 1993) where the input and state steady state values are computed via a separate optimization at each control stage. We require an additional condition for stability that the steady state behaviour of the plant and model should be sufficiently close (in some sense).

Although the results are both conservative and limited to open-loop stable plants, we should note that the model and plant are not assumed to match, no terminal constraints are introduced and the results are independent of signal norms. Furthermore there is no requirement that the steady state should lie on the interior (as opposed to the boundary) of the constraint set.

The paper is structured as follows. In Section 2 we quote two sufficient conditions for closed-loop asymptotic stability. Each is derived from the discrete multivariable circle criterion. In Section 3 we introduce the MPC notation. Sections 4 and 5 contain the main contributions of the paper, where we provide a stability analysis of linear MPC with (Section 5) and without (Section 4) integral action.

For brevity proofs of the lemmas are omitted. They may be found in a longer version of this paper (Heath and Wills, 2004). This also includes further discussion, simulation examples and the application of the results to velocity form integral action, corresponding to the scheme of (Prett and García, 1988) where only input and output changes are weighted in the cost function.

2. PRELIMINARIES: SPR RESULTS

The discrete version of the multivariable circle criterion (Haddad and Bernstein, 1994) states that if ϕ is a continuous static map satisfying $\phi(x)^T(\phi(x)+x) \leq 0$ and if $I+G(z)$ is SPR (strongly positive real) then the closed loop system $x(t) = G(z)\phi(x(t))$ is stable (see Fig 1).

Simple multiplier theory (Khalil, 2002; Heath *et al.*, 2004) gives the following lemma as a corollary:

Lemma 1: Suppose ϕ is a continuous static map satisfying $\phi(x)^T H\phi(x) + \phi(x)^T x \leq 0$. If H is positive definite and $H+G(z)$ is SPR then the closed-loop system $x(t) = G(z)\phi(x(t))$ is stable. □

We have shown that certain quadratic programmes can be included in the class of such functions (Heath *et al.*, 2004). Hence the further lemma:

Lemma 2: Suppose we have the closed-loop equations

$$\begin{aligned} x(t) &= G(z)\phi(x(t)) \\ \phi(x(t)) &= \arg \min_{\nu} \nu^T H\nu + 2\nu^T x(t) \\ &\qquad \text{s. t. } A\nu \preceq b(t) \text{ and } C\nu = 0 \end{aligned} \tag{1}$$

with H positive definite, $G(z)$ strictly proper and stable and $\nu = 0$ always feasible. Then a sufficient condition for stability is that $H+G(z)$ be SPR. □

3. MPC NOTATION

Given a horizon N, let $J = J(X,U)$ describe the cost function

$$J = ||x_N - x_{ss}||_P^2 + \sum_{k=1}^{N-1} ||x_k - x_{ss}||_Q^2 + \sum_{k=0}^{N-1} ||u_k - u_{ss}||_R^2 \tag{2}$$

Here X and U are sequences of predicted states and inputs $X = (x_1, x_2, \ldots, x_N)$ with $x_k \in \mathbb{R}^{n_x}$ and $U = (u_0, u_1, \ldots, u_{N-1})$ with $u_k \in \mathbb{R}^{n_u}$. Where convenient we will consider X and U to be stacked vectors $X \in \mathbb{R}^{Nn_x}$ and $U \in \mathbb{R}^{Nn_u}$ without change of notation. The terms x_{ss} and u_{ss} correspond to desired steady state values. The weighting matrices P and Q are positive semi-definite while R is positive definite.

We will consider two choices for the terminal cost weighting matrix P. One possibility is simply to choose $P = Q$. The other possibility, which we will term LQR tuning, is to choose P to satisfy the discrete algebraic Riccati equation (DARE)

$$A^T PA - P - A^T PB(R + B^T PB)^{-1} B^T PA + Q = 0 \tag{3}$$

With LQR tuning, unconstrained MPC is equivalent to unconstrained LQR control with an infinite cost horizon (Bitmead *et al.*, 1990). Furthermore the corresponding state-feedback constrained MPC with LQR tuning is nominally optimal for open-loop stable plants provided the horizon N is sufficiently large and the set-point is away from boundaries (Muske and Rawlings, 1993; Chmielewski and Manousiouthakis, 1996). Consequently LQR tuning with fixed N has been proposed by (Muske and Rawlings, 1993) for output feedback constrained MPC with integral action. Its successful industrial application has been reported, including by the current authors (Wills and Heath, 2005).

Given a state evolution model $x_{i+1} = Ax_i + Bu_i$ and state and input constraint sets $\mathbb{X}$ and $\mathbb{U}$ we may define the MPC law to be:

MPC: Set $u(t)$ to $u(t) = \bar{E}U^*$ where $\bar{E} = \begin{bmatrix} I & 0 & \ldots & 0 \end{bmatrix}$ and

$$[X^*, U^*] = \arg \min_{X,U} J(X,U) \quad \text{s. t. } x_{i+1} = Ax_i + Bu_i, \; x_{i+1} \in \mathbb{X} \text{ and } u_i \in \mathbb{U} \text{ for } i = 0,\ldots,N-1 \tag{4}$$

We will consider the cases with and without integral action (or "offset free" action) separately. With integral action we will only consider output feedback MPC.

Without integral action, x_{ss} and u_{ss} are derived from external variables, and for stability analysis may be considered zero without loss of generality. In this case state feedback MPC defines a law $u(t) = \kappa(x(t))$ for some κ with $x_0 = x(t)$ where $x(t)$ is the plant state (Mayne *et al.*, 2000). Similarly output feedback MPC defines a law $u(t) = \kappa(\hat{x}(t))$ with $x_0 = \hat{x}(t)$ where $\hat{x}(t)$ is some observed state value.

For two-stage form integral action x_{ss} and u_{ss} depend on some disturbance term d_0. In this case output feedback MPC defines a law $u(t) = \kappa(\hat{x}(t), \hat{d}(t))$ for some κ with $x_0 = \hat{x}(t)$ as before and $d_0 = \hat{d}(t)$ for some disturbance estimate $\hat{d}(t)$.

It is standard to express MPC in implicit form by projecting onto the equality constraints defined by the model. Introduce the matrices

$$\bar{P} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & P \end{bmatrix}, \bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$$
$$\Phi = \begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \Lambda = \begin{bmatrix} A \\ \vdots \\ A^N \end{bmatrix} \tag{5}$$

Note that $\bar{P} = P$ when $N = 1$. Define $\bar{H} = \bar{R} + \Phi^T \bar{P} \Phi$ and $\bar{L} = \Phi^T \bar{P} \Lambda$. Also define $I_x = \begin{bmatrix} I & \cdots & I \end{bmatrix}^T$ with $I_x \in \mathbb{R}^{n_x, Nn_x}$ and $I_u = \begin{bmatrix} I & \cdots & I \end{bmatrix}^T$ with $I_u \in \mathbb{R}^{n_u, Nn_u}$.

Define the implicit cost

$$J_I(U) = U^T \bar{H} U + 2U^T \left(\bar{L} x_0 - \Phi^T \bar{P} I_x x_{ss} - \bar{R} I_u u_{ss} \right) \tag{6}$$

We can then replace (4) in the MPC law by expressing U^* as

$$U^* = \arg \min_{U} J_I(U) \quad \text{s. t. } U \in \bar{\mathbb{U}} \tag{7}$$

where $\bar{\mathbb{U}}$ is the natural generalisation of $\mathbb{X}$ and $\mathbb{U}$ to U.

4. STABILITY OF MPC WITHOUT INTEGRAL ACTION

4.1 State feedback

Consider the plant

$$x(t) = G_x(z) u(t) \tag{8}$$

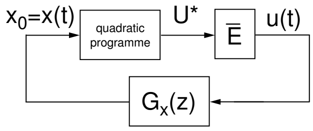

Fig. 2. State feedback MPC.

with $G_x(z)$ stable and strictly proper. We will model the plant with some

$$\hat{G}_x(z) = (zI - A)^{-1} B \tag{9}$$

Note that we do *not* necessarily assume the plant $G_x(z)$ and model $\hat{G}_x(z)$ to be equal.

We wish to establish the stability of the state feedback system comprising $G_x(z)$ with the MPC control law $u(t) = \kappa(x(t))$. As stated above we assume x_{ss} and u_{ss} to be zero without loss of generality. We further assume the constraints $U \in \bar{\mathbb{U}}$ can be written as a set of (possibly time varying) linear inequalities and equalities

$$A_U U \preceq b_U \text{ and } C_U U = 0 \tag{10}$$

with $U = 0$ always feasible. Since the control law comprises a quadratic programme and linear multiplication (see Fig 2) we may apply Lemma 2 to prove stability. Specifically we may say:

Result 1. Consider the closed-loop feedback system comprising the plant $x(t) = G_x(z)u(t)$ and MPC controller $u(t) = \kappa(x(t))$ with horizon N and with P chosen either as $P = Q$ or as the solution of the DARE (3). If $G_x(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if the constraints on U can be written in the form (10) with $U = 0$ feasible and if R is sufficiently large then the system is stable.

Proof: From Lemma 2 and the implicit form of MPC, it is sufficient that $T(z) = \bar{H} + \bar{L} G_x(z) \bar{E}$ be SPR. Suppose we put $R = \rho R_0$ for some positive definite R_0 and $\rho > 0$. If P is chosen as the solution of the DARE (3) then for A stable $P_\infty = \lim_{\rho \to \infty} P$ exists (Kwakernaak and Sivan, 1972) and is the solution to the discrete Lyapunov equation $A^T P_\infty A - P_\infty + Q = 0$. Hence, for either choice of P,

$$\lim_{\rho \to \infty} \frac{1}{\rho} T(z) = \bar{R}_0 \text{ for } |z| = 1 \text{ and } z = 0. \tag{11}$$

where $\bar{R}_0 = \text{diag}\left(\begin{bmatrix} R_0 & \cdots & R_0 \end{bmatrix} \right)$. Thus for sufficiently large ρ, $T(z)$ is SPR and the closed-loop system is stable. □

Result 1 is useful when the horizon N is small. But for large N it becomes somewhat unsatisfactory on two counts. Firstly the dimension of $T(z)$ increases with horizon N, and secondly we would like to find a ρ such that the closed-loop is guaranteed stable for any N.

To address the first issue, note that following (Heath *et al.*, 2004) it is sufficient to examine the eigenvalues of

$$M(z) = \begin{bmatrix} \bar{E} \\ G_x(z)^H \bar{L}^T \end{bmatrix} \bar{H}^{-1} \begin{bmatrix} \bar{L} G_x(z) & \bar{E}^T \end{bmatrix} \tag{12}$$

We find $M(z) \in \mathbb{C}^{2n_u, 2n_u}$ with dimension independent of horizon N.

In addressing the second issue we will consider only LQR tuning, where P is chosen as the solution of the DARE (3). Let $\mathrm{e}[X]$ denote the non-zero eigenvalues of matrix X. We have the following two lemmas:

Lemma 3: We have the identity

$$\mathrm{e}\left[M(z)\right] = \mathrm{e}\left[\bar{H}^{-\frac{1}{2}}(\bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T)\bar{H}^{-\frac{1}{2}}\right] \tag{13}$$

Furthermore with LQR tuning we may express $M(z)$ as

$$M(z) = \begin{bmatrix} KG_x(z) & H^{-1} \\ M_{2,1}(z) & G_x(z)^H K^T \end{bmatrix} \tag{14}$$

with

$$M_{2,1}(z) = G_x(z)^H \left(\sum_{i=1}^{N} (A^T)^i PBH^{-1}B^T PA^i \right) G_x(z) \tag{15}$$

We also have the identity $H = R + B^T PB$ and K is the LQR gain $K = H^{-1}B^T PA$. □

Lemma 4: For R sufficiently large and for all values of z on the unit circle, $2 + \min \mathrm{eig}\left[M(z)\right] > 0$ for all N. □

So we may say:

Result 2: Consider the closed-loop feedback system comprising the plant $x(t) = G_x(z)u(t)$ and MPC controller $u(t) = \kappa(x(t))$ with LQR tuning. If $G_x(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if the constraints on U can be written in the form (10) with $U = 0$ feasible and if R is sufficiently large then the system is stable for any horizon.

Proof: We require that $T(z)$ be SPR. Given that $G_x(z)$ is both stable and strictly proper, it is sufficient to show for all values of z on the unit circle that

$$\min \mathrm{eig}\left[2\bar{H} + \bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T\right] > 0 \tag{16}$$

Equivalently it is sufficient that for all values of z on the unit circle

$$2 + \min \mathrm{eig}\left[\bar{H}^{-\frac{1}{2}}(\bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T)\bar{H}^{-\frac{1}{2}}\right] > 0 \tag{17}$$

Hence Lemmas 3 and 4 give the result. □

4.2 *Output feedback*

A similar result for output feedback MPC follows immediately. Specifically, suppose the plant is given by

$$y(t) = G_y(z)u(t) \tag{18}$$

and we have an observer for the state

$$\hat{x}(t) = J_u(z)u(t) + J_y(z)y(t) \tag{19}$$

for some strictly proper stable transfer function matrix $J_u(z)$ and some stable transfer function matrix $J_y(z)$. Then we can combine the observer with the MPC law $u(t) = \kappa(\hat{x}(t))$; see Fig 3. We may say:

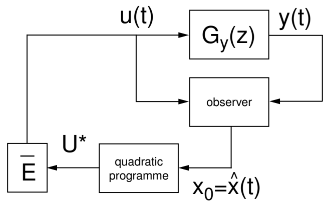

Fig. 3. Output feedback MPC with an observer.

Result 3: Consider the closed-loop feedback system comprising the plant $y(t) = G_y(z)u(t)$, the observer $\hat{x}(t) = J_u(z)u(t) + J_y(z)y(t)$ and MPC controller $u(t) = \kappa(\hat{x}(t))$ with either $P = Q$ or LQR tuning. If $G_y(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if $J_u(z)$ and $J_y(z)$ are stable (with $J_u(z)$ strictly proper), if the constraints on U can be written in the form (10) with $U = 0$ feasible and if R is sufficiently large then for given horizon N the system is stable. If furthermore we have LQR tuning and if R is sufficiently large then the system is stable for any horizon.

Proof: The is exactly the same form as the previous case if we write $G_x(z) = J_u(z) + J_y(z)G_y(z)$. Since there is no requirement for the plant $G_x(z)$ to match the model $(Iz - A)^{-1}B$, the result follows immediately from Results 1 and 2. □

5. INTEGRAL ACTION

Most practical applications of MPC require (when feasible) the rejection of constant disturbances. In this section we consider one scheme (Muske and Rawlings, 1993) for achieving this which we term two-stage form integral action; see also (Pannocchia and Rawlings, 2003) for a recent discussion.

5.1 *Controller structure*

We will consider output feedback MPC for the plant $y(t) = G_y(z)u(t)$. For integral action we let x_{ss} and u_{ss} be dependent on some disturbance estimate $\hat{d} = \hat{d}(t)$ so that the MPC law may be expressed as $u(t) = \kappa(\hat{x}(t), \hat{d}(t))$ for some κ. Specifically, given an output disturbance model (the idea can be straightforwardly generalised to an input disturbance)

$$x_{i+1} = Ax_i + Bu_i,\ y_i = Cx_i + d_i \tag{20}$$

we put $x_{ss} = (I - A)^{-1}Bu_{ss}$ with

$$u_{ss} = \arg \min_u \left|\left|C(I-A)^{-1}Bu + \hat{d} - r\right|\right|^2_{Q_{ss}}$$
$$\text{s. t.} \quad u \in \mathbb{U} \text{ and } (I-A)^{-1}Bu \in \mathbb{X} \tag{21}$$

Here r is the external set-point. We assume the weighting matrix Q_{ss} to be positive definite.

Given plant input $u(t)$ and output $y(t)$ the state and disturbance estimates are given by

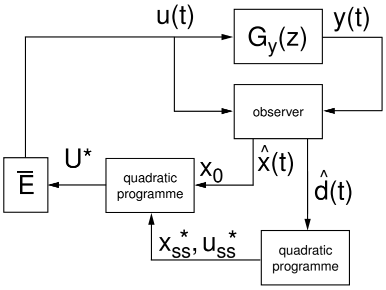

Fig. 4. Output feedback MPC with two-stage form integral action.

$$\begin{aligned}\hat{x}(t) &= J_u(z)u(t) + J_y(z)\left(y(t) - \hat{d}(t)\right)\\ \hat{d}(t) &= J_d(z)\left(y(t) - C\hat{x}(t)\right)\end{aligned} \quad (22)$$

with $J_u(z)$ stable and strictly proper, $J_y(z)$ stable and $J_d(z)$ stable with $J_d(1) = I$. See Fig 4.

5.2 Sector bound result

We now have two quadratic programmes in the closed-loop system, so can no longer apply Lemma 2 for stability analysis. Instead we will show that the mapping from a linear combination of $\hat{x}(t)$ and $\hat{d}(t)$ to a linear combination of U^* and u_{ss} takes the form of ϕ in Lemma 1. We will assume the conditions $U \in \bar{\mathbb{U}}$, $u \in \mathbb{U}$ and $(I-A)^{-1}Bu \in \mathbb{X}$ can be written as the (possibly time varying) linear inequality and equality constraints (10) with $U = 0$ feasible (and hence $u = 0$ also feasible). We will also define

$$\begin{aligned}\bar{F} &= -\frac{1}{2}\left(\Phi^T\bar{P}I_x(I-A)^{-1}B + \bar{R}I_u\right)\\ F_{ss} &= B^T(I-A)^{-T}C^TQ_{ss}\\ H_{ss} &= B^T(I-A)^{-T}C^TQ_{ss}C(I-A)^{-1}B\\ \bar{\bar{H}} &= \begin{bmatrix}\bar{H} & \bar{F}\\ \bar{F}^T & \mu H_{ss}\end{bmatrix}\end{aligned} \quad (23)$$

Then we may say:

Lemma 5: Let ϕ define the map

$$\begin{bmatrix}U^*\\ u_{ss}\end{bmatrix} = \phi\left(\begin{bmatrix}\bar{L}x_0\\ \mu F_{ss}(\hat{d}-r)\end{bmatrix}\right) \quad (24)$$

For any $\mu > 0$ we find $\phi(.)$ is a continuous function satisfying $\phi(x)^T\bar{\bar{H}}\phi(x) + \phi(x)^Tx \leq 0$. Also $\bar{\bar{H}}$ is positive definite provided $\mu > 0$ is chosen sufficiently big. □

5.3 Stability analysis

If we put $U^* = U^*(t)$ and $u_{ss} = u_{ss}(t)$ we have the dynamic relationship

$$\begin{bmatrix}\hat{x}(t)\\ \hat{d}(t)\end{bmatrix} = \begin{bmatrix}I & J_y(z)\\ J_d(z)C & I\end{bmatrix}^{-1}\begin{bmatrix}G_x(z)\\ J_d(z)G_y(z)\end{bmatrix} \times \begin{bmatrix}\bar{E} & 0\end{bmatrix}\begin{bmatrix}U^*(t)\\ u_{ss}(t)\end{bmatrix} \quad (25)$$

where, as before $G_x(z) = J_u(z) + J_y(z)G_y(z)$. It follows from Lemmas 1 and 5 that the system is closed-loop stable provided $T_\mu(z)$ is SPR with

$$\begin{aligned}T_\mu(z) = &\begin{bmatrix}\bar{H} & \bar{F}\\ \bar{F}^T & \mu H_{ss}\end{bmatrix}\\ &+\begin{bmatrix}\bar{L} & 0\\ 0 & \mu F_{ss}\end{bmatrix}\begin{bmatrix}I & J_y(z)\\ J_d(z)C & I\end{bmatrix}^{-1}\\ &\times\begin{bmatrix}G_x(z)\\ J_d(z)G_y(z)\end{bmatrix}\begin{bmatrix}\bar{E} & 0\end{bmatrix}\end{aligned} \quad (26)$$

Define the model and model error as

$$\begin{aligned}\hat{G}_y(z) &= C(zI-A)^{-1}B\\ \Delta G_y(z) &= G_y(z) - \hat{G}_y(z)\end{aligned} \quad (27)$$

Furthermore put

$$\hat{G}_x(z) = J_u(z) + J_y(z)\hat{G}_y(z) \quad (28)$$

We will assume $J_u(z)$ and $J_y(z)$ take the form

$$\begin{aligned}J_u(z) &= (zI-A+LC)^{-1}B\\ J_y(z) &= (zI-A+LC)^{-1}L\end{aligned} \quad (29)$$

so that (28) is consistent with (9).

Omitting the argument z for brevity, we may express $T_\mu = T_\mu(z)$ as:

Lemma 6:

$$T_\mu = \begin{bmatrix}\bar{H} & \bar{F}\\ \bar{F}^T & \mu H_{ss}\end{bmatrix} + \begin{bmatrix}\bar{L}W_1\\ \mu F_{ss}W_2\end{bmatrix}\begin{bmatrix}\bar{E} & 0\end{bmatrix} \quad (30)$$

with

$$\begin{aligned}W_1 &= \hat{G}_x + J_y\left[I - J_dCJ_y\right]^{-1}\left[I - J_d\right]\Delta G_y\\ W_2 &= \left[I - J_dCJ_y\right]^{-1}J_d\left[I - CJ_y\right]\Delta G_y\end{aligned} \quad (31)$$

□

The following results for special cases follow immediately:

When $J_d(z) = 0$, we have the relation

$$T_\mu(z) = \begin{bmatrix}\bar{H} + \bar{L}G_x(z)\bar{E} & \bar{F}\\ \bar{F}^T & \mu H_{ss}\end{bmatrix} \quad (32)$$

Thus $T_\mu(z)$ is SPR when $J_d(z) = 0$ and μ is sufficiently big.

If we put $J_d(z) = I$ we have

$$T_\mu(z) = \begin{bmatrix}\bar{H} + \bar{L}\hat{G}_x(z)\bar{E} & \bar{F}\\ \bar{F}^T & \mu H_{ss}\end{bmatrix} + \begin{bmatrix}0 & 0\\ \mu F_{ss}\Delta G_y(z)\bar{E} & 0\end{bmatrix} \quad (33)$$

Thus $T_\mu(z)$ is positive definite provided the model is sufficiently close to the plant and provided μ is sufficiently large. Note that we always have, at steady state, $J_d(1) = I$.

Thus, if there is sufficiently small uncertainty at low frequency, stability can be guaranteed by ensuring R is sufficiently large and $J_d(z)$ has sufficiently low bandwidth. To be specific:

Result 4: Consider the feedback system comprising the plant $y(t) = G_y(z)u(t)$, the state and disturbance observers (22) satisfying (29) and MPC controller with two stage form integral action $u(t) = \kappa(\hat{x}(t), \hat{d}(t))$ with horizon N. The weighting matrix P is chosen either as $P = Q$ or via LQR tuning. If $G_y(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if $J_u(z)$ and $J_y(z)$ are stable (with $J_u(z)$ strictly proper), if the constraints on U can be written in the form (10) with $U = 0$ feasible, if R is sufficiently large, if $J_d(z)$ has sufficiently low bandwidth, and if a μ can be found such that both $T_\mu(1) + T_\mu(1)^T$ is positive definite and $T_\mu(z)$ evaluated with $J_d(z) = 0$ is SPR then the system is stable in closed-loop. □

In a similar manner to before, it is sufficient to check $2 + \min \text{eig}[(M_{ts}(z)] > 0$, where the dimension of $M_{ts}(z)$ is independent of the prediction horizon N, and M_{ts} is given by

$$M_{ts} = \begin{bmatrix} \bar{E} & 0 \\ W_1^H \bar{L}^T & \mu W_2^H F_{ss}^T \end{bmatrix} \begin{bmatrix} \bar{H} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix}^{-1} \times \begin{bmatrix} \bar{L} W_1 & \bar{E}^T \\ \mu F_{ss} W_2 & 0 \end{bmatrix} \quad (34)$$

6. CONCLUSION

We have demonstrated the closed-loop asymptotic stability of constrained linear MPC for stable plants. Without integral action we simply require the input weighting to be sufficiently high. With integral action a further condition on the accuracy of the steady state model is required. The results are equally applicable to state feedback and output feedback MPC schemes.

The model and plant need not match, no terminal constraints are introduced, and the results are independent of signal norms. The steady state may lie on either the boundary or the interior of the constraint set.

Proofs of the lemmas as well as further discussion and some illustrative simulation examples may be found in a longer version of this paper (Heath and Wills, 2004).

REFERENCES

Bitmead, R. R., M. Gevers and V. Wertz (1990). *Adaptive Optimal Control -The Thinking Man's GPC*. Prentice Hall, Inc.. Englewood Cliffs, New Jersey.

Chmielewski, D. and M. Manousiouthakis (1996). On constrained infinite-time linear quadratic optimal control. *Systems & Control Letters* **29**, 121–129.

Findeisen, R., L. Imsland, F. Allgöwer and B. A. Foss (2003). State and output feedback nonlinear model predictive control: an overview. *European Journal of Control* **9**, 190–206.

Haddad, W. M. and D. S. Bernstein (1994). Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. Part II: discrete-time theory. *International Journal of Robust and Nonlinear Control* **4**, 249–265.

Heath, W. P., A. G. Wills and J. A. G. Akkermans (2004). A sufficient robustness condition for constrained model predictive control. UKACC Control 04, University of Bath, 6th-9th Sept.

Heath, W. P. and A. G. Wills (2004). The inherent robustness of constrained linear model predictive control. Report EE04005, University of Newcastle.

Kerrigan, E. C. and J. M. Maciejowski (2004). Feedback min-max model predictive control using a single linear program: robust stability and the explicit solution. *International Journal of Robust and Nonlinear Control* **14**, 395–413.

Khalil, H. K. (2002). *Nonlinear Systems (third edition)*. Prentice Hall. Upper Saddle River.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. John Wiley & Sons, Inc.. New York.

Mayne, D. Q., J. B. Rawlings, C. V. Rao and P. O. M. Scokaert (2000). Constrained model predictive control: Stability and optimality. *Automatica* **36**, 789–814.

Muske, K. R. and J. B. Rawlings (1993). Model predictive control with linear models. *AIChE Journal* **39**(2), 262–287.

Pannocchia, G. and J. B. Rawlings (2003). Disturbance models for offset-free model-predictive control. *AIChE Journal* **49**, 426–436.

Prett, D. M. and C. E. García (1988). *Fundamental Process Control*. Butterworth-Heinemann. Boston.

Qin, S. J. and T. A. Badgwell (2003). A survey of industrial model predictive control technology. *Control Engineering Practice* **11**, 733–764.

Sakizlis, V., N. M. P. Kakalis, V. Dua, J. D. Perkins and E. N. Pistikopoulos (2004). Design of robust model-based controllers via parammetric programming. *Automatica* **40**, 129–201.

Wills, A. G. and W. P. Heath (2005). Application of barrier function based model predictive control to an edible oil refining process. *Journal of Process Control* **15**, 183–200.

Zafiriou, E. (1990). Robust model predictive control of processes with hard constraints. *Computers & Chemical Engineering* **14**(4–5), 359–371.

Zheng, A. (1999). Robust stability analysis of constrained model predictive control. *Journal of Process Control* **9**, 271–278.

Zheng, A. and M. Morari (1995). Stability of model predictive control with mixed constraints. *IEEE Transactions on Automatic Control* **40**, 1818–1823.