THE H_2 CONTROL PROBLEM FOR DESCRIPTOR SYSTEMS

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Abstract: A solution of the H_2 control problem is presented for linear descriptor systems. The solution proceeds in two steps. Firstly, the set of all controllers that stabilize the control system is parametrized. The mathematical tool applied are doubly coprime, proper stable factorizations of rational matrices. The factors are expressed in terms of stabilizing descriptor feedback and output injection gains, which represent degrees of freedom that can be used in the subsequent optimization. In a coordinate system in which dynamic and non-dynamic modes are separated, the corresponding gains are used to regularize the problem and minimize the norm. Finally, a projection result is applied to obtain an optimal controller.

Keywords: descriptor systems, linear systems, optimal controllers, stabilizing controllers.

1. INTRODUCTION

The H_2 control problem consists of stabilizing the control system while minimizing the H_2 norm of its transfer function. Several solutions to this problem are available. For systems in *state space* form, and under the standard regularity assumptions, Doyle, *et al*. (1989) obtained an optimal regulator in observer form by solving two algebraic Riccati equations. A pole placement interpretation of this solution leads to an alternative construction (Kučera, 1999) in which the optimal regulator's transfer function is obtained through operations with polynomial matrices.

In the absence of the standard regularity assumptions, the H_2 control problem for systems in state space form was studied by Stoorvogel (1992), who established a condition for an H_2 optimal controller to exist. Chen and Saberi (1993) showed when such a controller is unique. Saberi, *et al*. (1996) then parametrized all H_2 optimal controllers and identified the fixed modes of the optimal control system.

For systems described by *transfer functions*, Park and Bongiorno (1989) employed Wiener-Hopf optimization to obtain an optimal regulator transfer function via spectral factorizations and stable projections. Under the standard assumptions, Meinsma (2000) obtained a solution using operations with proper stable rational matrices. Kučera (2004) derived a general solution in the sense that no assumptions on the plant are made other than those securing the existence of spectral factors.

The above approaches are not equivalent. Due to different mathematical tools applied, the H_2 control problem is solved at different levels of generality under different assumptions. The state space solution is streamlined and efficient but it is restricted to systems with *proper* rational transfer function. The transfer function solution allows for systems that are more general but the solution is more involved.

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The aim of this paper is to present a solution of the H_2 control problem for systems in *descriptor* form. Such a solution combines the elegance of state-space approach with the generality offered by the transfer function approach. The solution proceeds in two steps. Firstly, the set of all controllers that stabilize the control system is parametrized. The mathematical tool applied are doubly coprime, proper stable factorizations of rational transfer matrices. The factors are expressed in terms of stabilizing descriptor feedback and output injection gains. In a coordinate system in which dynamic and non-dynamic modes are separated, these gains are conveniently split. The gains that correspond to non-dynamic modes are used to regularize the problem while those corresponding to dynamic modes are used to manipulate the norm to make the optimizing choice of the parameter obvious.

2. DESCRIPTOR SYSTEMS

Consider a descriptor system of the form

$$E\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where x is the n-vector descriptor variable, u is the m-vector input, y is the p-vector output and A, B, C, D, and E are real matrices of appropriate sizes.

The pencil $sE - A$ is assumed *regular*, i.e., det $(sE - A)$ is a non-zero polynomial in s. Then the transfer matrix of the system exists and equals

$$F = C(sE - A)^{-1}B + D := \left[\begin{array}{c|c} sE - A & B \\ \hline C & D \end{array}\right].$$

Clearly, F is a real-rational matrix, not necessarily proper or stable.

Let r = rank E and q = deg det $(sE - A)$. Then the descriptor system has q exponential modes, which correspond to the (finite) eigenvalues of the pencil $sE - A$. The system also has $r - q$ impulsive modes and $n - r$ non-dynamic modes, which correspond to the infinite eigenvalues of the pencil $sE - A$. The infinite eigenvalues of $sE - A$ are the eigenvalues at $\lambda = 0$ of $E - \lambda A$.

The descriptor system is said to be *stable* if the pencil $sE - A$ is regular, has no impulsive modes, and its exponential modes are located within the open left-half complex plane. This is equivalent to $(sE - A)^{-1}$ being a proper and stable rational matrix.

The descriptor system is said to be *controllable* if the matrix

$$\begin{bmatrix} sE - A & B \end{bmatrix}$$

has full row rank for all finite complex s, *stabilizable* if the matrix has full row rank for all s in the open left-half complex plane, and *impulse controllable* if the matrix has full row rank for $s = \infty$. If and only if the system is controllable and impulse controllable, there exists a descriptor feedback gain matrix L such that the pencil $sE - (A+BL)$ is regular, the system $(A+BL, B, C+DL, D, E)$ has no impulsive modes, and its exponential modes are placed arbitrarily in the complex plane (Bender and Laub, 1987).

The descriptor system is said to be *observable* if the matrix

$$\begin{bmatrix} sE - A \\ C \end{bmatrix}$$

has full column rank for all finite complex s, *detectable* if the matrix has full column rank for all s in the open left-half complex plane, and *impulse observable* if the matrix has full column rank for $s = \infty$. If and only if the system is observable and impulse observable, there exists an output injection gain matrix K such that the pencil $sE - (A+KC)$ is regular, the system $(A+KC, B+KD, C, D, E)$ has no impulsive modes, and its exponential modes are placed arbitrarily in the complex plane (Bender and Laub, 1987).

3. H_2 NORM

The set of all real-rational matrix functions F of the complex variable s that are strictly proper and analytic on the imaginary axis is denoted by RL_2. The symbol RH_2 will be used to denote the set of strictly proper rational matrices that are analytic in the closed *right-half* complex plane, while $RH_2^{\perp}$ will denote the set of strictly proper rational matrices that are analytic in the closed *left-half* complex plane. Then RH_2 is a subspace of RL_2 and $RH_2^{\perp}$ is the orthogonal complement of RH_2 in RL_2. The *H_2 norm* of a function F from RL_2 is defined as

$$\| F \| := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace} F^{\mathrm{T}}(-j\omega) F(j\omega) \mathrm{d}\omega \right)^{\frac{1}{2}}.$$

4. PROBLEM FORMULATION

The control system is considered in the standard configuration, shown in Fig. 1, where G is the generalized plant and R is the controller to be designed. The plant has two sets of inputs – the exogenous inputs w and the control inputs u, and has two sets of outputs – the measured outputs y and the regulated outputs z.

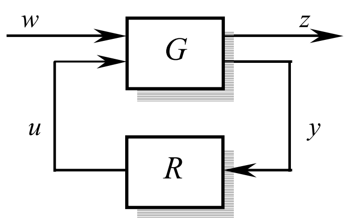

Fig. 1. Standard control system.

It is assumed that the plant is a linear and time-invariant system in *descriptor* form. In particular, the realization of G is taken to be

$$E\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{11} w + D_{12} u$$
$$y = C_2 x + D_{21} w + D_{22} u \, .$$

Using the standard notation, we have

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} := \left[\begin{array}{c|cc} sE - A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

The H_2 *control problem* is to find a controller R in descriptor form that stabilizes G in the standard control system and minimizes the H_2 norm of the transfer matrix H from w to z.

5. STABILIZING CONTROLLERS

The symbol RH_∞ will be used to denote the set of proper rational matrices that are analytic in the closed right-half complex plane.

Since stability is equivalent to system pencils being invertible in RH_∞, it is helpful to factorize rational transfer matrices into doubly coprime factors over RH_∞. Specifically, for the generalized plant, let

$$G = M^{-1}N = \bar{N}\bar{M}^{-1}$$

where M, N are *left coprime* RH_∞ matrices while $\bar{M}, \bar{N}$ are *right coprime* RH_∞ matrices.

Assumption 1: The subsystem (A, B_2, C_2, D_{22}, E) of G is stabilizable and impulse controllable.

Assumption 2: The subsystem (A, B_2, C_2, D_{22}, E) of G is detectable and impulse observable.

If Assumptions 1 and 2 hold, a doubly coprime factorization of G can be obtained by applying a descriptor feedback and an output injection. Let K and L be any two matrices such that the pencils $sE - (A+KC_2)$ and $sE - (A+B_2L)$ are regular and invertible in RH_∞. Then (Zhou, 1998)

$$M = \left[\begin{array}{c|cc} sE-(A+KC_2) & 0 & K \\ \hline C_1 & I & 0 \\ C_2 & 0 & I \end{array}\right] := \begin{bmatrix} I & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

$$N = \left[\begin{array}{c|cc} sE-(A+KC_2) & B_1+KD_{21} & B_2+KD_{22} \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$$

and

$$\bar{M} = \left[\begin{array}{c|cc} sE-(A+B_2L) & B_1 & B_2 \\ \hline 0 & I & 0 \\ L & 0 & I \end{array}\right] := \begin{bmatrix} I & 0 \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}$$

$$\bar{N} = \left[\begin{array}{c|cc} sE-(A+B_2L) & B_1 & B_2 \\ \hline C_1+D_{12}L & D_{11} & D_{12} \\ C_2+D_{22}L & D_{21} & D_{22} \end{array}\right] := \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix}.$$

To determine the set of all controllers that stabilize G, let X, Y and $\bar{X}, \bar{Y}$ be four RH_∞ matrices satisfying the Bézout identity

$$\begin{bmatrix} M_{22} & -N_{22} \\ -Y & X \end{bmatrix} \begin{bmatrix} \bar{X} & \bar{N}_{22} \\ \bar{Y} & \bar{M}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Specifically (Zhou, 1998),

$$X = \left[\begin{array}{c|c} sE-(A+KC_2) & B_2+KD_{22} \\ \hline -L & I \end{array}\right]$$

$$Y = \left[\begin{array}{c|c} sE-(A+KC_2) & K \\ \hline -L & 0 \end{array}\right]$$

and

$$\bar{X} = \left[\begin{array}{c|c} sE-(A+B_2L) & K \\ \hline -C_2-D_{22}L & I \end{array}\right]$$

$$\bar{Y} = \left[\begin{array}{c|c} sE-(A+B_2L) & K \\ \hline -L & 0 \end{array}\right].$$

Theorem 1. Let a plant G in Fig. 1 be given. Then,

(a) there exists a controller that stabilizes G if and only if Assumptions 1 and 2 hold;

(b) the set $R_S(W)$ of all controllers that stabilize G is given by

$$\begin{aligned} R_S(W) &= (X+WN_{22})^{-1}(Y+WM_{22}) \\ &= (\bar{Y}+\bar{M}_{22}W)(\bar{X}+\bar{N}_{22}W)^{-1} \end{aligned}$$

where W ranges over the set of RH_∞ matrices such that $X+WN_{22}$ (equivalently, $\bar{X}+\bar{N}_{22}W$) is non-singular.

Proof:

(a) Suppose that Assumptions 1 and 2 hold and consider the observer-based controller

$$R_0 = \left[\begin{array}{c|c} sE-(A+B_2L+KC_2+KD_{22}L) & K \\ \hline -L & 0 \end{array}\right].$$

Then the pencil of the closed-loop system is equivalent to

$$\begin{bmatrix} sE-(A+B_2L) & B_2L \\ 0 & sE-(A+KC_2) \end{bmatrix}.$$

This shows that the closed-loop system is stable.

On the other hand, if either Assumption 1 or Assumption 2 fails to hold, there are some modes of the closed-loop system that are impulsive or unstable exponential, no matter what the controller is.

(b) In view of (a), the stabilization of G is reduced to that of G_{22}. Applying the standard result (Vidyasagar, 1985) one obtains the parametrization of all stabilizing controllers for the control system configuration shown in Fig. 1.

6. STANDARD COORDINATE SYSTEM

Consider the descriptor model of the plant. Suppose that E is an $n \times n$ matrix of rank $r \le n$. When the need arises to separate the dynamic (exponential and impulsive) modes from the non-dynamic ones, one can define a *standard coordinate system* derived by performing the following transformation of E:

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where P and Q are nonsingular real matrices and I is the $r \times r$ identity matrix. In such a coordinate system, A, B_1, B_2, C_1, and C_2 take the form

$$PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; PB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \; PB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}$$

$$C_1Q = [C_{11} \; C_{12}], \; C_2Q = [C_{21} \; C_{22}].$$

An output injection gain K takes the form

$$PK = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

and a descriptor feedback gain L transforms to

$$LQ = [L_1 \; L_2].$$

The components K_1 and L_1 affect the dynamic modes while the components K_2 and L_2 operate on the non-dynamic modes.

The standard coordinate system can be used to concentrate the non-dynamic modes in the feedthrough matrices. In particular, for the doubly coprime factors of G, one obtains

$$\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} = \left[\begin{array}{cc|c} sI-(A_{11}+K_1C_{21}) & -(A_{12}+K_1C_{22}) & B_{11}+K_1D_{21} \\ -(A_{21}+K_2C_{21}) & -(A_{22}+K_2C_{22}) & B_{12}+K_2D_{21} \\ \hline C_{11} & C_{12} & D_{11} \\ C_{21} & C_{22} & D_{21} \end{array}\right]$$

Assumption 3. The matrix $\begin{bmatrix} A_{22} \\ C_{22} \end{bmatrix}$ has full column rank.

Then there exists a matrix K_2 such that $A_{22} + K_2C_{22}$ is nonsingular. Consequently, the pencil $sE - (A + KC_2)$ is regular and its inverse is an element of RH_∞. Applying the matrix inversion formula

$$\begin{bmatrix} sI-\Lambda & -\Phi \\ -\Psi & -\Omega \end{bmatrix}^{-1} = \left[\begin{array}{c|cc} sI-(\Lambda-\Phi\Omega^{-1}\Psi) & I & -\Phi\Omega^{-1} \\ \hline I & 0 & 0 \\ -\Omega^{-1}\Psi & 0 & -\Omega^{-1} \end{array}\right]$$

to the matrix

$$\begin{bmatrix} sI-(A_{11}+K_1C_{21}) & -(A_{12}+K_1C_{22}) \\ -(A_{21}+K_2C_{21}) & -(A_{22}+K_2C_{22}) \end{bmatrix}$$

one obtains

$$\begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} = \left[\begin{array}{c|c} sI-(A_K+K_1C_{2K}) & B_{1K}+K_1D_{21K} \\ \hline C_{1K} & D_{11K} \\ C_{2K} & D_{21K} \end{array}\right]$$

where

$$A_K = A_{11} - A_{12}(A_{22}+K_2C_{22})^{-1}(A_{21}+K_2C_{21})$$
$$B_{1K} = B_{11} - A_{12}(A_{22}+K_2C_{22})^{-1}(B_{12}+K_2D_{21})$$
$$C_{1K} = C_{11} - C_{12}(A_{22}+K_2C_{22})^{-1}(A_{21}+K_2C_{21})$$
$$C_{2K} = C_{21} - C_{22}(A_{22}+K_2C_{22})^{-1}(A_{21}+K_2C_{21})$$
$$D_{11K} = D_{11} - C_{12}(A_{22}+K_2C_{22})^{-1}(B_{12}+K_2D_{21})$$
$$D_{21K} = D_{21} - C_{22}(A_{22}+K_2C_{22})^{-1}(B_{12}+K_2D_{21}).$$

Similarly, for the dual doubly coprime factors, one has

$$\begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \end{bmatrix} = \left[\begin{array}{cc|cc} sI-(A_{11}+B_{21}L_1) & -(A_{12}+B_{21}L_2) & B_{11} & B_{21} \\ -(A_{21}+B_{22}L_1) & -(A_{22}+B_{22}L_2) & B_{12} & B_{22} \\ \hline C_{11}+D_{12}L_1 & C_{12}+D_{12}L_2 & D_{11} & D_{12} \end{array}\right].$$

Assumption 4. The matrix $\begin{bmatrix} A_{22} & B_{22} \end{bmatrix}$ has full row rank.

Then there exists a matrix L_2 such that $A_{22} + B_{22}L_2$ is nonsingular. Consequently, the pencil $sE - (A + B_2L)$ is regular and its inverse is an element of RH_∞. Applying the same matrix inversion formula to the matrix

$$\begin{bmatrix} sI-(A_{11}+B_{21}L_1) & -(A_{12}+B_{21}L_2) \\ -(A_{21}+B_{22}L_1) & -(A_{22}+B_{22}L_2) \end{bmatrix}$$

one obtains

$$\begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \end{bmatrix} = \left[\begin{array}{c|cc} sI-(A_L+B_{2L}L_1) & B_{1L} & B_{2L} \\ \hline C_{1L}+D_{12L}L_1 & D_{11L} & D_{12L} \end{array}\right]$$

where

$$A_L = A_{11} - (A_{12}+B_{21}L_2)(A_{22}+B_{22}L_2)^{-1}A_{21}$$
$$B_{1L} = B_{11} - (A_{12}+B_{21}L_2)(A_{22}+B_{22}L_2)^{-1}B_{12}$$
$$B_{2L} = B_{21} - (A_{12}+B_{21}L_2)(A_{22}+B_{22}L_2)^{-1}B_{22}$$
$$C_{1L} = C_{11} - (C_{12}+D_{12}L_2)(A_{22}+B_{22}L_2)^{-1}A_{21}$$
$$D_{11L} = D_{11} - (C_{12}+D_{12}L_2)(A_{22}+B_{22}L_2)^{-1}B_{12}$$
$$D_{12L} = D_{12} - (C_{12}+D_{12}L_2)(A_{22}+B_{22}L_2)^{-1}B_{22}.$$

7. NORM MINIMIZATION

For any rational matrix F, we shall use the shorthand notation $F^*(s) := F^{\mathrm{T}}(-s)$.

A matrix $F \in RH_\infty$ is said to be *inner* if $F^*F = I$ and *co-inner* if $FF^* = I$. Left multiplication by an inner matrix preserves H_2 norms. So does right multiplication by a co-inner matrix.

In order to minimize the H_2 norm of a transfer function, a standard projection result will be used in the form presented by Meinsma (2000).

Lemma 1. Let Γ and Δ be RH_∞ matrices with equally many rows. Suppose that Γ is strictly proper, $\Delta^*\Gamma$ is in $RH_2^\perp$ and Δ is inner. Then for any RH_2 matrix T,

$$\|\Gamma - \Delta T\|^2 = \|\Gamma\|^2 + \|T\|^2.$$

Lemma 2. Let Γ and Δ be RH_∞ matrices with equally many columns. Suppose that Γ is strictly proper, $\Gamma\Delta^*$ is in $RH_2^\perp$ and Δ is co-inner. Then for any RH_2 matrix T,

$$\|\Gamma - T\Delta\|^2 = \|\Gamma\|^2 + \|T\|^2.$$

8. FURTHER ASSUMPTIONS AND LEMMAS

It is convenient to summarize the additional assumptions and lemmas needed in the sequel.

Assumption 5. The matrix

$$\begin{bmatrix} A - j\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full column rank for all finite ω.

Assumption 6. The matrix

$$\begin{bmatrix} A - j\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has full row rank for all finite ω.

Assumption 7. The matrix

$$\begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12} \end{bmatrix}$$

has full column rank.

Assumption 8. The matrix

$$\begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21} \end{bmatrix}$$

has full row rank.

Assumption 9. It holds

$$\operatorname{rank}\begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21} \\ C_{12} & D_{11} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21} \end{bmatrix}$$

and if S_1, S_2 are such that

$$\begin{bmatrix} S_1 & S_2 \end{bmatrix}\begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21} \end{bmatrix} = \begin{bmatrix} C_{12} & D_{11} \end{bmatrix}$$

then S_1 is assumed to have full column rank and

$$\operatorname{rank}\begin{bmatrix} S_1 & S_2 \end{bmatrix} = \operatorname{rank} S_1.$$

Assumption 10. It holds

$$\operatorname{rank}\begin{bmatrix} A_{22} & B_{22} & B_{12} \\ C_{12} & D_{12} & D_{11} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12} \end{bmatrix}$$

and if T_1, T_2 are such that

$$\begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12}\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix}=\begin{bmatrix}B_{12}\\ D_{11}\end{bmatrix}$$

then

$$\operatorname{rank}\begin{bmatrix}T_1\\ T_2\end{bmatrix}=\operatorname{rank}T_1 .$$

Lemma 3. Suppose that Assumptions 1, 5, and 7 hold. Then the Riccati equation

$$A_L^{\mathrm T} X_L + X_L A_L + C_{1L}^{\mathrm T} C_{1L} = (B_{2L}^{\mathrm T} X_L + D_{12L}^{\mathrm T} C_{1L})^{\mathrm T} (D_{12L}^{\mathrm T} D_{12L})^{-1}(B_{2L}^{\mathrm T} X_L + D_{12L}^{\mathrm T} C_{1L})$$

has a unique non-negative definite solution X_L such that the matrix $A_L + B_{2L}L_1$ is stable, where

$$L_1 = -(D_{12L}^T D_{12L})^{-1}(B_{2L}^T X_L + D_{12L}^T C_{1L}) .$$

Proof: Observe that D_{12L} is the Schur complement of $A_{22}+B_{22}L_2$ in the matrix

$$\begin{bmatrix} A_{22}+B_{22}L_2 & B_{22} \\ C_{12}+D_{12}L_2 & D_{12}\end{bmatrix}=\begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12}\end{bmatrix}\begin{bmatrix} I & 0 \\ L_2 & I\end{bmatrix}.$$

In view of Assumption 7, the preceding matrix has full column rank. Thus, D_{12L} has full column rank and $D_{12L}^{\mathrm T} D_{12L} > 0$. The rest of the proof is standard (Zhou, 1998).

Lemma 4. (Zhou, 1998). Apply the matrix L_1 of Lemma 3 in the matrices $\bar N_{11}$ and $\bar N_{12}$ and denote

$$\bar U := (D_{12L}^{\mathrm T} D_{12L}) .$$

Then $\bar N_{12}\bar U^{-\frac{1}{2}}$ is inner and $(\bar N_{12}\bar U^{-\frac{1}{2}})^*\bar N_{11}$ is in $RH_2^{\perp}$.

Lemma 5. Suppose that Assumptions 2, 6, and 8 hold. Then the Riccati equation

$$A_K X_K + X_K A_K^{\mathrm T} + B_{1K}B_{1K}^{\mathrm T} = (B_{1K}D_{21K}^{\mathrm T} + X_K C_{2K}^{\mathrm T})(D_{21K}D_{21K}^{\mathrm T})^{-1}(B_{1K}D_{21K}^{\mathrm T} + X_K C_{2K}^{\mathrm T})^{\mathrm T}$$

has a unique non-negative definite solution X_K such that the matrix $A_K + K_1C_{2K}$ is stable, where

$$K_1 = -(B_{1K}D_{21K}^{\mathrm T} + X_K C_{2K}^{\mathrm T})(D_{21K}D_{21K}^{\mathrm T})^{-1} .$$

Proof: Observe that D_{21K} is the Schur complement of $A_{22}+K_2C_{22}$ in the matrix

$$\begin{bmatrix} A_{22}+K_2C_{22} & B_{12}+K_2D_{21} \\ C_{22} & D_{21}\end{bmatrix}=\begin{bmatrix} I & K_2 \\ 0 & I\end{bmatrix}\begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21}\end{bmatrix}.$$

In view of Assumption 8, the preceding matrix has full row rank. Thus, D_{21K} has full column rank and $D_{21K}D_{21K}^{\mathrm T} > 0$. The rest of the proof is standard (Zhou, 1998).

Lemma 6. (Zhou, 1998). Apply the matrix K_1 of Lemma 5 in the matrices N_{11} and N_{21} and denote $U = D_{21K}D_{21K}^{\mathrm T}$. Then the matrix $U^{-\frac{1}{2}}N_{21}$ is co-inner and the matrix $N_{11}(U^{-\frac{1}{2}}N_{21})^*$ belongs to $RH_2^{\perp}$.

9. PRIMAL AND DUAL APPROACHES

The closed loop transfer function of the standard control system in Fig. 1 equals

$$\begin{aligned} H &= G_{11} + G_{12}(I - RG_{22})^{-1}RG_{21} \\ &= G_{11} + G_{12}R(I - G_{22}R)^{-1}G_{21} . \end{aligned}$$

The strategy adopted to solve the optimization problem is to note that, in a stable control system, H is an affine function of the free parameter W defined in Theorem 1(b) and then pick special factorizations of G so that the square of the H_2 norm of H (as a function of W) has no linear term. This is achieved by applying Lemmas 1 and 2.

The primal approach is based on the *first* expression for H. Let R_S be a stabilizing controller for G. Using doubly coprime factorizations over RH_∞ and, in particular, $R_S = X_S^{-1}Y_S$, one obtains

$$H = \bar N_{11} - \bar N_{12}(X_S\bar M_{21} - Y_S\bar N_{21}) .$$

Write

$$H = \bar N_{11} - (\bar N_{12}\bar U^{-\frac{1}{2}})(\bar U^{\frac{1}{2}}\bar V)$$

with $\bar U$ defined in Lemma 4 and

$$\bar V := (X_S\bar M_{21} - Y_S\bar N_{21}) .$$

Both $\bar N_{11}$ and $\bar N_{12}$ depend on L. In a standard coordinate system,

$$\begin{bmatrix}\bar N_{11} & \bar N_{12}\end{bmatrix} = \left[\begin{array}{c|cc} sI-(A_L+B_{2L}L_1) & B_{1L} & B_{2L} \\ \hline C_{1L}+D_{12L}L_1 & D_{11L} & D_{12L}\end{array}\right]$$

and the gains L_1 and L_2 will now be employed to make the hypotheses of Lemma 1 hold.

We begin by making $\bar N_{11}$ strictly proper. This is achieved by selecting L_2 in such a way as to make $D_{11L} = 0$. Observe that

$$D_{11L} = D_{11} - (C_{12}+D_{12}L_2)(A_{22}+B_{22}L_2)^{-1}B_{12}$$

is the Schur complement of $A_{22}+B_{22}L_2$ in the matrix

$$\begin{bmatrix} A_{22}+B_{22}L_2 & B_{12} \\ C_{12}+D_{12}L_2 & D_{11}\end{bmatrix}.$$

Thus $D_{11L} = 0$ if and only if the rank of the preceding matrix equals the rank of its first block column. In view of Assumption 10, D_{11L} is made zero by solving the equation

$$\begin{bmatrix} A_{22} & B_{22} \\ C_{12} & D_{12}\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix}=\begin{bmatrix}B_{12}\\ D_{11}\end{bmatrix}$$

for T_1 and T_2 and, subsequently, solving the equation $L_2T_1 = T_2$ for L_2.

We proceed by making $\bar N_{12}\bar U^{-\frac{1}{2}}$ inner and $(\bar N_{12}\bar U^{-\frac{1}{2}})^*\bar N_{11}$ an element of $RH_2^{\perp}$ by selecting an appropriate matrix L_1. This is done in Lemma 4.

Now the hypotheses of Lemma 1 are all satisfied. Applying Lemma 1 to H yields

$$\|H\|^2 = \|\bar N_{11}\|^2 + \left\|\bar U^{\frac{1}{2}}\bar V\right\|^2$$

provided $\bar V$ is strictly proper.

The remaining degrees of freedom are embodied in

$$\bar V = (X_S\bar M_{21} - Y_S\bar N_{21}) .$$

Theorem 1(b) implies that

$$X_S = X + WN_{22}, \quad Y_S = Y + WM_{22}$$

for a parameter $W \in RH_\infty$. Hence

$$\begin{aligned} \bar U^{\frac{1}{2}}\bar V &= \bar U^{\frac{1}{2}}(X\bar M_{21} - Y\bar N_{21}) - \bar U^{\frac{1}{2}}W(M_{22}\bar N_{21} - N_{22}\bar M_{21}) \\ &= \bar U^{\frac{1}{2}}(X\bar M_{21} - Y\bar N_{21}) - \bar U^{\frac{1}{2}}WN_{21} \\ &= \bar U^{\frac{1}{2}}N_{11L} - (\bar U^{\frac{1}{2}}WU^{\frac{1}{2}})(U^{-\frac{1}{2}}N_{21}) \end{aligned}$$

with U defined in Lemma 6. In particular, the identity

$$N_{21} = M_{22}\overline{N}_{21} - N_{22}\overline{M}_{21}$$

follows from simple manipulations with the doubly coprime factors of G. The matrix

$$N_{11L} := X\overline{M}_{21} - Y\overline{N}_{21}$$

can be obtained from N_{11} by replacing C_1 with L and D_{11} with 0.

Both N_{11L} and N_{21} depend on K. In a standard coordinate system,

$$\begin{bmatrix} N_{11L} \\ N_{21} \end{bmatrix} = \left[\begin{array}{c|c} sI - (A_K + K_1 C_{2K}) & B_{1K} + K_1 D_{21K} \\ \hline L_K & \Theta_K \\ C_{2K} & D_{21K} \end{array}\right]$$

where

$$L_K = L_1 - L_2(A_{22} + K_2 C_{22})^{-1}(A_{21} + K_2 C_{21})$$
$$\Theta_K = -L_2(A_{22} + K_2 C_{22})^{-1}(B_{12} + K_2 D_{21})$$

duly replace the matrices C_{1K} and D_{11K} that appear in N_{11}. The gains K_1 and K_2 will now be employed to make the hypotheses of Lemma 2 hold.

We begin by making N_{11L} strictly proper. This is achieved by selecting K_2 in such a way as to make $\Theta_K = 0$. Observe that Θ_K is the Schur complement of $A_{22} + K_2 C_{22}$ in the matrix

$$\begin{bmatrix} A_{22} + K_2 C_{22} & B_{12} + K_2 D_{21} \\ L_2 & 0 \end{bmatrix}.$$

Thus $\Theta_K = 0$ if and only if the rank of the preceding matrix equals the rank of its first block row. The following assumption is now made:

Assumption 11.

$$\operatorname{rank}\begin{bmatrix} C_{12} & D_{11} \\ L_2 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} C_{12} & D_{11} \end{bmatrix}.$$

In view of Assumption 9, Θ_K is made zero by solving the equation

$$\begin{bmatrix} S_1 & S_2 \end{bmatrix}\begin{bmatrix} A_{22} & B_{12} \\ C_{22} & D_{21} \end{bmatrix} = \begin{bmatrix} C_{12} & D_{11} \end{bmatrix}$$

for S_1 and S_2 and, subsequently, solving the equation $S_1 K_2 = S_2$ for K_2. Assumption 11 then guarantees that these equations have the same solution K_2 when C_{12} is replaced by L_2 and D_{11} by 0.

We proceed by making $U^{-\frac{1}{2}}N_{21}$ co-inner and $(\overline{U}^{\frac{1}{2}}N_{11L})(U^{-\frac{1}{2}}N_{21})^*$ an element of $RH_2^{\perp}$ through a choice of K_1. This follows from Lemma 6 on replacing N_{11} with N_{11L} and noting that

$$(\overline{U}^{\frac{1}{2}}N_{11L})(U^{-\frac{1}{2}}N_{21})^* = \overline{U}^{\frac{1}{2}}(N_{11L}N_{21}^*)U^{-\frac{1}{2}\mathrm{T}}.$$

Now the hypotheses of Lemma 2 are all satisfied. Applying Lemma 2 to $\overline{U}^{\frac{1}{2}}\overline{V}$ yields

$$\left\| \overline{U}^{\frac{1}{2}}\overline{V} \right\|^2 = \left\| \overline{U}^{\frac{1}{2}}N_{11L} \right\|^2 + \left\| \overline{U}^{\frac{1}{2}}WU^{\frac{1}{2}} \right\|^2$$

provided W is strictly proper.

The dual approach starts with the *second* expression for H, employs the dual doubly coprime factors, makes an assumption dual to Assumption 11, namely

Assumption 12.

$$\operatorname{rank}\begin{bmatrix} B_{12} & K_2 \\ D_{11} & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_{12} \\ D_{11} \end{bmatrix},$$

and results in an equivalent formula for the norm.

10. OPTIMAL CONTROLLER

The resulting expression for the H_2 norm of H is

$$\|H\|^2 = \|\overline{N}_{11}\|^2 + \left\| \overline{U}^{\frac{1}{2}}N_{11L} \right\|^2 + \left\| \overline{U}^{\frac{1}{2}}WU^{\frac{1}{2}} \right\|^2.$$

where W is a parameter that ranges over RH_2. This expression makes the optimizing choice of W obvious.

Theorem 2. Suppose that Assumptions 1 – 12 hold. Then there exists a unique optimal controller

$$R_0 = R_S(0) = X^{-1}Y = \overline{Y}\overline{X}^{-1}$$
$$= \left[\begin{array}{c|c} sE - (A + B_2 L + KC_2 + KD_{22}L) & K \\ \hline -L & 0 \end{array}\right],$$

where

$$K = P^{-1}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}Q^{-1}$$

and where K_1, K_2 and L_1, L_2 are specified in Section 9. Moreover,

$$\min \|H\|^2 = \|\overline{N}_{11}\|^2 + \left\| \overline{U}^{\frac{1}{2}}N_{11L} \right\|^2.$$

Proof: Clearly $\|H\|$ achieves minimum for $W = 0$.

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