STABILIZATION OF CONTINUOUS-TIME SWITCHED SYSTEMS

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Abstract: This paper addresses two strategies for stabilization of continuous time linear switched system. The first one, is of open loop nature (trajectory independent) and is based on the determination of a minimum dwell time by means of a family of quadratic Lyapunov functions. Interestingly, the proposed stability condition does not require the Lyapunov function be uniformly decreasing at every switching time. The second one, is of closed loop nature (trajectory dependent) and is designed from the solution of what we call Lyapunov-Metzler inequalities. Being non-convex, a more conservative version of the Lyapunov-Metzler inequalities, expressed in terms of linear matrix inequalities is given.

Keywords: Switched systems, Continuous-time systems, Linear matrix inequality

1. INTRODUCTION

This paper aims to provide new results on stability analysis and stabilizing control synthesis for a continuous time switched linear system of the following general form

$$\dot{x}(t) = A_{\sigma(t)}x(t)\ ,\ x(0) = x_0 \qquad (1)$$

defined for all $t \geq 0$ where $x(t) \in \mathbb{R}^n$ is the state, $\sigma(t)$ is the switching rule and x_0 is the initial condition. We consider the class of switched systems characterized by the fact that the switching rule, for each $t \geq 0$, is such that

$$A_{\sigma(t)} \in \{A_1, A_2, \cdots, A_N\} \qquad (2)$$

The model (2) naturally imposes a discontinuity on $A_{\sigma(t)}$ since this matrix must jump instantaneously from A_i to A_j for some $i \neq j = 1, \cdots, N$ once switching occurs. In other words, $A_{\sigma(t)}$ is constrained to jump among the N vertices of the matrix polytope $\{A_1, A_2, \cdots, A_N\}$.

In the last years, stability of continuous time linear switched systems have been addressed by many authors, (Branicky, 1998), (Hockerman *et al.*, 1998), (Johansson *et al.*, 1998) and (Ye *et al.*, 1998). The recent paper (Hespanha, 2004), dealing with extensions of LaSalle's Invariance Principle provides an interesting discussion on a collection of results on uniform stability of switched systems. Generally speaking, when $\sigma(t)$ is state independent, that is, when it is a piecewise constant signal, the reported stability conditions are obtained using a family of symmetric and positive definite matrices $\{P_1, \cdots, P_N\}$ each one associated to each matrix of the set $\{A_1, \cdots, A_N\}$ and such that the Lyapunov function $v(x(t))$ is non increasing with respect to $\sigma(t)$ at every switching time. In this paper, for minimum dwell time design preserving global stability the last condition is relaxed. It is replaced by the weaker condition that at every switching time t_k the sequence $v(x(t_k))$, for $k = 0, \cdots, \infty$, converges uniformly to zero.

For switched systems with $\sigma(\cdot)$ being state dependent, the stability condition is expressed as a set of inequalities that we call "Lyapunov-Metzler inequalities" because the variables involved are a set of symmetric and positive matrices $\{P_1, \cdots, P_N\}$ and a Metzler matrix Π. The point to be noticed is that our asymptotical stability condition does not require that the set $\{A_1, \cdots, A_N\}$ be composed only by asymptotically stable matrices. The price to be paid, however, is the non-convex nature of the Lyapunov-Metzler inequalities being thus difficult to solve numerically. Therefore, a more conservative but easier to solve asymptotical stability condition is proposed.

The notation used throughout is standard. Capital letters denote matrices, small letters denote vectors and small Greek letters denote scalars. For matrices or vectors $(')$ indicates transpose. For symmetric matrices, $X > 0$ (≥ 0) indicates that X is positive definite (nonnegative definite). The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$ respectively. The $\mathcal{L}_2$ norm of $x(t) \in \mathbb{R}^n$ defined for all $t \geq 0$ equals $\|x(t)\|_2^2 = \int_0^\infty x(t)'x(t)dt$.

2. TIME SWITCHING CONTROL

This section is entirely dedicated to design a time switching control law for the switched system defined by the model (1) and (2). The problem under consideration can be stated as follows : Determine a minimum dwell time $T > 0$ such that the equilibrium point $x = 0$ of the system (1) is globally asymptotically stable with the time switching control

$$\sigma(t) = i \in \{1, \cdots, N\}\ , \quad t \in [t_k, t_{k+1}) \tag{3}$$

where t_k and t_{k+1} are successive switching times satisfying $t_{k+1} - t_k \geq T$ for all $k \in \mathbb{N}$. It is interesting to observe that the index $i \in \{1, \cdots, N\}$ selected at each instant of time $t \geq 0$ is arbitrary. Hence, asymptotical stability is preserved whenever it remains unchanged for a period of time greater or equal to the minimum dwell time T. The next theorem provides the theoretical basis towards the solution of the proposed problem. It uses the concept of multiple Lyapunov function with the innovation that the classical assumption on its decreasing at switching times is no longer needed.

Theorem 1. Assume that, for some $T > 0$, there exists a collection of positive definite matrices $\{P_1, \cdots, P_N\}$ of compatible dimensions such that

$$A_i'P_i + P_iA_i < 0\ ,\ \forall\ i = 1, \cdots, N \tag{4}$$

and

$$e^{A_i'T}P_je^{A_iT} - P_i < 0\ ,\ \forall\ i \neq j = 1, \cdots, N \tag{5}$$

The time switching control (3) makes the equilibrium solution $x = 0$ of (1) globally asymptotically stable.

Proof: Consider, in accordance to (3), that $\sigma(t) = i \in \{1, \cdots, N\}$ for all $t \in [t_k, t_{k+1})$ where $t_{k+1} = t_k + T_k$ with $T_k \geq T > 0$ and that at $t = t_{k+1}$ the time switching control jumps to $\sigma(t) = j \in \{1, \cdots, N\}$. From (4), it is seen that, for all $t \in [t_k, t_{k+1})$, the time derivative of the Lyapunov function $v(x(t)) = x(t)'P_{\sigma(t)}x(t)$, along an arbitrary trajectory of (1) satisfies

$$\begin{aligned}\dot{v}(x(t)) &= x(t)'(A_i'P_i + P_iA_i)x(t) \\ &< 0\end{aligned} \tag{6}$$

which enables us to conclude that there exist scalars $\alpha > 0$ and $\beta > 0$ such that

$$\|x(t)\|_2^2 \leq \beta e^{-\alpha(t-t_k)}v(x(t_k))\ ,\ \forall t \in [t_k, t_{k+1}) \tag{7}$$

On the other hand, using the inequalities (5), we have

$$\begin{aligned}v(x(t_{k+1})) &= x(t_{k+1})'P_jx(t_{k+1}) \\ &= x(t_k)'e^{A_i'T_k}P_je^{A_iT_k}x(t_k) \\ &< x(t_k)'e^{A_i'(T_k-T)}P_ie^{A_i(T_k-T)}x(t_k) \\ &< x(t_k)P_ix(t_k) \\ &< v(x(t_k))\end{aligned} \tag{8}$$

where the second inequality holds from the fact that for every $\tau = T_k - T \geq 0$ it is true that $e^{A_i'\tau}P_ie^{A_i\tau} \leq P_i$. The consequence is that there exists $\mu \in (0\ , 1)$ such that

$$v(x(t_k)) \leq \mu^k v_0(x_0)\ ,\ \forall k \in \mathbb{N} \tag{9}$$

which together with (7) implies that the equilibrium solution $x = 0$ of (1) is globally asymptotically stable. □

This result deserves some comments. First, (4) imposes that all matrices of the set $\{A_1, \cdots, A_N\}$ must be asymptotically stable. In view of this fact, the constraints (5) are always satisfied whenever $T > 0$ is taken arbitrarily large. In this case, the arbitrary time invariant control law $\sigma(t) = i \in \{1, \cdots, N\}$, $\forall t \geq 0$, is stabilizing. Second, assume matrices $A_1, \cdots, A_N$ are quadratically stable, which is the same to say that they share an unique positive definite matrix P such that

$$A_i'P + PA_i < 0\ ,\ \forall i = 1, \cdots, N \tag{10}$$

In this case, the inequality (5) is satisfied for $P_1 = \cdots = P_N = P$ for all $T > 0$ meaning that the switching policy (3) may jump from i to j arbitrarily fast preserving once again asymptotical stability. Hence, Theorem 1 contains, as a particular case, the quadratic stability condition. Third, with $T > 0$ satisfying the constraints (4) and (5) it is always possible to define a time switching control strategy (3) such that $A_{\sigma(t)}$ is periodic. As a consequence, a necessary condition for the feasibility of those constraints is

$$\theta(T) := \max_{q=1,\cdots,n} \left| \lambda_q \left(\prod_{p=1}^{N} e^{B_p T} \right) \right| < 1 \qquad (11)$$

where $\lambda_q(\cdot)$ denotes a generic eigenvalue of $(\cdot)$ and $\{B_1, \cdots, B_N\}$ are matrices corresponding to any combination among those of the set $\{A_1, \cdots, A_N\}$. Since (3) may produce non-periodic policies, generally this necessary condition for solvability of conditions provided by Theorem 1 does not meet sufficiency. This aspect will be illustrated by an example.

The minimum value of the dwell time T_* can be calculated with no big difficulty from the optimal solution of the optimization problem

$$\min_{T>0, P_1>0, \cdots, P_N>0} \{T \ : \ (4)-(5)\} \qquad (12)$$

which, for each $T > 0$ fixed, reduces to a convex programming problem with linear matrix inequalities constraints that can be handled by any LMI solver available in the literature to date, see (Boyd *et al.*, 1994) for an important study on systems and LMIs. A line search procedure is then used to deal with the scalar variable $T > 0 \in \mathbb{R}$.

For illustration purpose of the theoretical results obtained so far, let us consider the following example characterized by $N = 2$ and matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} \ , \ \ A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix}$$

which are not quadratically stable. First we have calculated the minimum dwell time as being $T_* = 2.76$. To give and idea of its conservativeness we have calculated the value $T_{per} = 2.71$ corresponding to the necessary condition for stability arising from periodic systems (11). The value of T_{per} corresponds to the minimum value of T such that (11) holds for all $T \geq T_{per}$. Both being very close indicates, for this simple example, a good precision on the determination of the minimum dwell time.

3. STATE SWITCHING CONTROL

In this section we consider the system (1) where the switching rule satisfies (2). The main difference from the previous section is that, presently, it is assumed that the state vector $x(t)$ is available for feedback for all $t \geq 0$, that is our goal is to determine the function $u(\cdot) : \mathbb{R}^n \to \{1, \cdots, N\}$, such that

$$\sigma(t) = u(x(t)) \qquad (13)$$

makes the equilibrium point $x = 0$ of (1) asymptotically stable. In this case, we do not assume that the matrices of the set $\{A_1, \cdots, A_N\}$ are asymptotically stable. To this end, let us define the simplex

$$\Lambda := \left\{ \lambda \in \mathbb{R}^N \ : \ \sum_{i=1}^{N} \lambda_i = 1 \ , \ \lambda_i \geq 0 \right\} \qquad (14)$$

which together with the set of symmetric and positive definite matrices $\{P_1, \cdots, P_N\}$ enables us to introduce the following parameter dependent Lyapunov function

$$\begin{aligned} v(x) &:= \min_{i=1,\cdots,N} x' P_i x \\ &= \min_{\lambda \in \Lambda} \left(\sum_{i=1}^{N} \lambda_i x' P_i x \right) \end{aligned} \qquad (15)$$

As it will be clear in the sequel this Lyapunov function is crucial for our purposes. However, it presents some difficulties to be handled including the fact that it is not differentiable everywhere. To analyze this aspect the set $I(x) = \{i : v(x) = x' P_i x\}$ plays a central role since $v(x)$ fails to be differentiable on $x \in \mathbb{R}^n$ such that $I(x)$ is composed by more that one element or, in other words, when the result of the minimization indicated in (15) is not unique, (Rockafellar, 1970).

Before proceed, let us recall the class of Metzler matrices denoted by $\mathcal{M}$ and constituted by all matrices $\Pi \in \mathbb{R}^{N \times N}$ with elements π_{ij}, such that

$$\pi_{ij} \geq 0 \ \forall i \neq j \ , \quad \sum_{i=1}^{N} \pi_{ij} = 0 \ \forall j \qquad (16)$$

It is clear that any $\Pi \in \mathcal{M}$ presents an eigenvalue at the origin of the complex plane since $c'\Pi = 0$ where $c' = [1 \ \cdots \ 1]$. In addition, it is well known that the eigenvector associated to the null eigenvalue of Π is non-negative yielding the conclusion that there always exists $\lambda_\infty \in \Lambda$ such that $\Pi \lambda_\infty = 0$. The next theorem summarizes the main result of this section.

Theorem 2. Assume there exist a set of positive definite matrices $\{P_1, \cdots, P_N\}$ and $\Pi \in \mathcal{M}$ satisfying the Lyapunov-Metzler inequalities

$$A_i' P_i + P_i A_i + \sum_{j=1}^{N} \pi_{ji} P_j < 0 \ , \ i = 1, \cdots, N \qquad (17)$$

The state switching control (13) with

$$u(x(t)) = \arg \min_{i=1,\cdots,N} x(t)' P_i x(t) \qquad (18)$$

makes the equilibrium solution $x = 0$ of (1) globally asymptotically stable.

Proof: It follows from the Lyapunov function (15) which is not differentiable for all $t \geq 0$. For this reason we need to deal with the Dini derivative (see (Garg, 1998))

$$D^+ v(x(t)) = \lim_{h \to 0^+} \sup \frac{v(x(t+h)) - v(x(t))}{h} \tag{19}$$

Assume, in accordance to (18), that at an arbitrary $t \geq 0$, the state switching control is given by $\sigma(t) = u(x(t)) = i$ for some $i \in I(x(t))$. Hence, from (19) and the system dynamic equation (1) we have

$$\begin{aligned} D^+ v(x(t)) &= \min_{l \in I(x(t))} x(t)'(A_i' P_l + P_l A_i) x(t) \\ &\leq x(t)'(A_i' P_i + P_i A_i) x(t) \end{aligned} \tag{20}$$

where the inequality holds from the fact that $i \in I(x(t))$. Finally, remembering that (16) is valid for $\Pi \in \mathcal{M}$ and that $x(t)' P_j x(t) \geq x(t)' P_i x(t)$ for all $j \neq i = 1, \cdots, N$ once again due to the fact that $i \in I(x(t))$, using the Lyapunov-Metzler inequalities (17) one gets

$$\begin{aligned} D^+ v(x(t)) &< -x(t)' \left(\sum_{j=1}^{N} \pi_{ji} P_j \right) x(t) \\ &< -\left(\sum_{j=1}^{N} \pi_{ji} \right) x(t)' P_i x(t) \\ &< 0 \end{aligned} \tag{21}$$

which proves the proposed theorem since the Lyapunov function $v(x(t))$ defined in (15) is radially unbounded. □

It is important to observe that Theorem 2 does not require the set $\{A_1, \cdots, A_N\}$ be composed exclusively by asymptotically stable matrices. Indeed, with $\Pi \in \mathcal{M}$, a necessary condition for the Lyapunov-Metzler inequalities be feasible with respect to $\{P_1, \cdots, P_N\}$ is matrices $A_i + (\pi_{ii}/2)I$ for all $i = 1, \cdots, N$ be asymptotically stable. Since $\pi_{ii} \leq 0$ this condition does not imply on the asymptotical stability of A_i. However, an interesting case occurs when all matrices $\{A_1, \cdots, A_N\}$ are asymptotically stable for which the choice $\Pi = 0$ is possible and the proposed state switching strategy preserves stability. Furthermore, if the set $\{A_1, \cdots, A_N\}$ is quadratically stable then the Lyapunov-Metzler inequalities admit a solution $P_1 = \cdots = P_N = P$ and $I(x(t)) = \{1, \cdots, N\}$ for all $t \geq 0$. In this classical but particular case, at any $t \geq 0$, the control law $u(x(t))$ being any logic state $i \in \{1, \cdots, N\}$ preserves asymptotical stability. Hence, Theorem 2, contains as a particular (since the Lyapunov-Metzler inequalities do not depend on Π anymore) case the quadratic stability condition.

Remark 1. From the observation that any $\alpha \geq 0$ and $\Pi \in \mathcal{M}$ implies $\alpha\Pi \in \mathcal{M}$, standard Kronecker calculus shows that the existence of solutions to the Lyapunov-Metzler inequalities with Π replaced by $\alpha\Pi$ is equivalent to the asymptotic stability of the time-invariant, Nn^2 - dimensional, continuous-time system

$$\dot{\xi}(t) = (A_{big} + \alpha \Pi_{big}) \xi(t) \tag{22}$$

where the indicated matrices are given by

$$A_{big} = \begin{bmatrix} A_1' \oplus A_1' & 0 & \cdots & 0 \\ 0 & A_2' \oplus A_2' & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & A_N' \oplus A_N' \end{bmatrix}$$

and

$$\Pi_{big} = \Pi' \otimes I_{n^2}$$

Here the symbols $\oplus$ and $\otimes$ denote the Kronecker sum and Kronecker product, respectively [1]. It can be proven that, as α goes to infinity, n^2 eigenvalues of the matrix

$$\hat{A}_{big}(\alpha) = A_{big} + \alpha \Pi_{big} \tag{23}$$

tends to the eigenvalues of

$$\left(\sum_{i=1}^{N} \lambda_{\infty i} A_i' \right) \oplus \left(\sum_{i=1}^{N} \lambda_{\infty i} A_i' \right) \tag{24}$$

where $\lambda_\infty \in \Lambda$ is such that $\Pi \lambda_\infty = 0$. These eigenvalues are located in the left hand side of the complex plane if and only if the *average* matrix, defined as

$$A_{ave} := \sum_{i=1}^{N} \lambda_{\infty i} A_i \tag{25}$$

is asymptotically stable. Moreover, as α goes to infinity, it follows that

$$\lim_{\alpha \to +\infty} P_i(\alpha) = P_{ave} \ , \ i = 1, \cdots, N \tag{26}$$

where $P_{ave} > 0$ satisfies the following Lyapunov inequality, associated to the average matrix

$$A_{ave}' P_{ave} + P_{ave} A_{ave} < 0 \tag{27}$$

For illustration purpose, let us consider a pair of unstable matrices given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix} \ , \ \ A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 8 \end{bmatrix}$$

[1] While the Kronecker product is more or less standard, the sum requires a formal definition. In this respect we define the Kronecker sum of two matrices D and E as $D \oplus E = D \otimes I + I \otimes E$. It is important to recall that the eigenvalues of the Kronecker sum $D \oplus E$ are given by all sums of all eigenvalues of D and E.

and take

$$\Pi = \begin{bmatrix} -0.51 & 0.49 \\ 0.51 & -0.49 \end{bmatrix}$$

Clearly, the eigenvector associated to the null eigenvalue of Π is given by $\lambda'_\infty = [0.49 \quad 0.51]$ and it can be determined numerically that the Lyapunov-Metzler inequalities, with Π replaced by $\alpha\Pi$, have a solution for all $\alpha \in \mathbb{R}$ such that $\alpha \geq 615.7374$. In addition, thanks to the factorization of matrix $\Pi_{big} = B_{big}C_{big}$, the zeros of $(A_{big}, B_{big}, C_{big})$ being equal to $-0.33, -0.33, -0.33 \pm j0.226$ are also obtained from the eigenvalues of the asymptotically stable average matrix $A_{ave} = 0.49A_1 + 0.51A_2$, taking all sums. □

The next lemma introduces a guaranteed cost associated to the proposed state switching control law (18).

Lemma 1. Let $Q \geq 0$ be given. Assume there exist a set of positive definite matrices $\{P_1, \cdots, P_N\}$ and $\Pi \in \mathcal{M}$ satisfying the Lyapunov-Metzler inequalities

$$A_i'P_i + P_iA_i + \sum_{j=1}^{N} \pi_{ji}P_j + Q < 0 \ , \ i = 1, \cdots, N \tag{28}$$

The state switching control (13) with $u(x(t))$ given by (18) makes the equilibrium solution $x = 0$ of (1) globally asymptotically stable and

$$\int_0^\infty x(t)'Qx(t)dt \leq \min_{i=1,\cdots,N} x_0'P_ix_0 \tag{29}$$

The numerical determination, if any, of a solution of the Lyapunov-Metzler inequalities with respect to the variables $(\Pi, \{P_1, \cdots, P_N\})$ is not a simple task and certainly deserves additional attention. The main source of difficulty stems from its non-convex nature due the products of variables and so LMI solvers do not apply. Perhaps, a point to be further investigated is that its particular structure with π_{ji} being scalars may help on the design of an interactive method based on relaxation.

In this paper we pursue an alternative route. The main idea is to get a simpler, although certainly more conservative stability condition that can be expressed by means of LMIs being thus solvable by the machinery available in the literature to date. The next theorem shows that working with a subclass of Metzler matrices, characterized by having the same diagonal elements, this goal is accomplished.

Theorem 3. Let $Q \geq 0$ be given. Assume there exist a set of positive definite matrices $\{P_1, \cdots, P_N\}$ and a scalar $\gamma > 0$ satisfying the modified Lyapunov-Metzler inequalities

$$A_i'P_i + P_iA_i + \gamma(P_j - P_i) + Q < 0 \ , \ j \neq i = 1, \cdots, N \tag{30}$$

The state switching control (13) with $u(x(t))$ given by (18) makes the equilibrium solution $x = 0$ of (1) globally asymptotically stable and

$$\int_0^\infty x(t)'Qx(t)dt \leq \sum_{i=1}^{N} x_0'P_ix_0 \tag{31}$$

Proof: The proof follows from the choice of $\Pi \in \mathcal{M}$ such that $\pi_{ii} = -\gamma$ and the remaining elements satisfying

$$\gamma^{-1} \sum_{j\neq i=1}^{N} \pi_{ji} = 1 \tag{32}$$

for all $i = 1, \cdots, N$. Taking into account that $\pi_{ji} \geq 0$ for all $j \neq i = 1, \cdots, N$ and multiplying (30) by π_{ji}, summing up for all $j \neq i = 1, \cdots N$ and finally multiplying the result by $\gamma^{-1} > 0$ we get

$$\begin{aligned} A_i'P_i + P_iA_i + Q &< -\sum_{j\neq i=1}^{N} \pi_{ji}(P_j - P_i) \\ &< -\sum_{j=1}^{N} \pi_{ji}P_j \end{aligned} \tag{33}$$

which being valid for all $i = 1, \cdots, N$ are the Lyapunov-Metzler inequalities (28). From Lemma 1, the upper bound (29) holds which trivially implies that (31) is verified. The proposed theorem is thus proved. □

The basic theoretical features of Theorem 2 and Lemma 1 are still present in Theorem 3. The most important point is that the asymptotical stability of the set of matrices $\{A_1, \cdots, A_N\}$ still is not required. In addition, notice that the guaranteed cost (31) is clearly worse than the one provide d by Lemma 1 but the former being convex makes possible to solve the problem

$$\min_{\gamma>0, P_1>0, \cdots, P_N>0} \left\{ \sum_{i=1}^{N} x_0'P_ix_0 \ : \ (30) \right\} \tag{34}$$

by LMI solvers and line search. The next example illustrates some aspects of the theoretical results obtained so far.

Consider the system (1) with $N = 2$ and matrices $\{A_1, A_2\}$ given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ 2 & -9 \end{bmatrix} \ , \ A_2 = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$$

which, as it can be easily verified by inspection, are both unstable. Considering $Q = I$ and the initial condition $x_0 = [1 \ 1]'$, problem (34) has been solved by line search, fixing γ and minimizing its

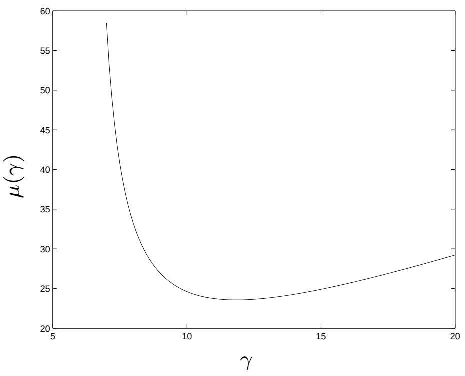

Fig. 1. Guaranteed cost as a function of γ

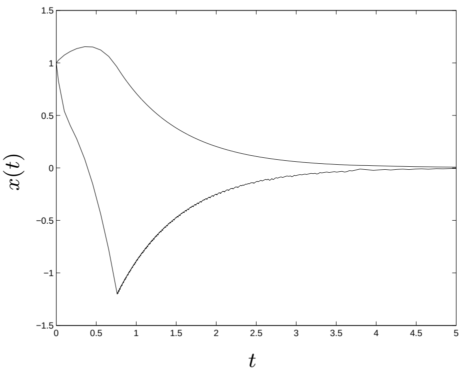

Fig. 2. Time simulation

objective function, denoted $\mu(\gamma)$, with respect to the remaining variables.

Figure 1 shows the behavior of the function $\mu(\gamma)$ which enables us to determine its minimum value $\mu^* = 23.56$, corresponding to $\gamma^* = 11.80$. It is important to stress that, in this particular example, the function $\mu(\gamma)$ has an unique minimum. However we do not have any evidence that this is a generic property valid in all cases.

Figure 2 shows the trajectories of the state variable $x(t) \in \mathbb{R}^2$ versus time for the system controlled by the state switching rule $\sigma(t) = u(x(t))$ given by (18) with the positive definite matrices

$$P_1 = \begin{bmatrix} 6.7196 & 1.6293 \\ 1.6293 & 1.0222 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 6.0825 & 2.1293 \\ 2.1293 & 2.2206 \end{bmatrix}$$

that have been obtained from the optimal solution of problem (34). As it can be seen, the proposed control strategy is very effective to stabilize the system under consideration.

4. CONCLUSION

In this paper we have introduced stability conditions for switched linear systems. They have been used for control synthesis of state independent (open loop) and state dependent (closed loop) switching rules. In the second case, the determination of a guaranteed cost associated to the proposed control strategy has been addressed. Special attention has been devoted towards the numerical solvability of the design problems by means of methods based on linear matrix inequalities.

Two issues deserve more attention. The first is related to the development of numerical algorithms for the solution of the Lyapunov-Metzler inequalities and together with the results of (Geromel *et al.*, 1991), (Geromel *et al.*, 1998) the development of new stability conditions for time varying polytopic linear systems. The second one is the possible generalization of the stability conditions to cope with linear control design and quadratic cost. Taking into account the nonlinear nature of the involved stability conditions, this point constitutes a real theoretical challenge.

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