FORMAL LINEARIZATION OF NONLINEAR TIME-VARYING DYNAMIC SYSTEMS USING CHEBYSHEV AND LAGUERRE POLYNOMIALS

Hitoshi Takata * Kazuo Komatsu ** Hideki Sano *

* *Kagoshima University*
** *Kumamoto National College of Technology*

Abstract: This paper is concerned with a formal linearization problem for a general class of nonlinear time-varying dynamic systems. To a given system, a linearization function is made up of Chebyshev polynomials about its state variables. The nonlinear time-varying system is transformed into a linear time-varying system in terms of the linearization function using Chebyshev interpolation to state variables and Laguerre expansion to time variable. An error bound formula of this linearization which is derived in this paper explains that the accuracy of this algorithm is improved as the order of Chebyshev and Laguerre polynomials increases. As its application, a nonlinear observer is designed to demonstrate the usefulness of this formal linearization approach.

Keywords: Nonlinear system, Linearization, Time-varying system, Polynomial transforms, Observers

1. INTRODUCTION

It has been received wide recognition to use linearization method as an important tool in analysis and synthesis of nonlinear dynamic systems. One of the most popular and practical approaches is the linearization by Taylor expansion truncated at the first order (Yu, *et al.*, 1970). This is powerful but limited to implement in small regions or almost linear systems. To relax this limitation and improve the accuracy, various studies of linearization problems have been made since the early works of Poincaré and Sternberg (Sternberg, 1959). For the last few decades, this problem has been explored from the viewpoint of differential geometry (Brockett, 1978; Krener, 1984). Though many interesting results have been developed, they are generally not so easily applicable to practical systems. Therefore, it is eager to develop a linearization approach of easy implementation with the aide of computers (Kadiyala, 1993). Authors have been studied computer algorithms of formal linearization for some kinds of nonlinear systems (Takata, 1979; Komatsu and Takata, 1996). In this paper we present a formal linearization approach for a general class of nonlinear time-varying dynamic system. This approach introduces a linearization function which is made up of a finite number of Chebyshev polynomials about state variables. A given nonlinear time-varying system is transformed into a linear time-varying system with respect to the linearization function by applying Chebyshev interpolation to state variables and Laguerre expansion to time variable. A computer algorithm of this formal linearization approach is presented, and then its error bound formula is derived. As an application of this approach, a nonlinear time-varying observer is well designed. With the aid of computers, we easily carry out the numerical computation of this formal linearization and the nonlinear observer. Numerical experiments show that the accuracy of this approach is improved as both the orders of Chebyshev and Laguerre polynomials increase.

2. FORMAL LINEARIZATION

Consider a time-varying nonlinear system described by

$$\Sigma_1 : \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}), \boldsymbol{x}(0) = \boldsymbol{x}_0 \in D \tag{1}$$

where $\dot{}$ $= d/dt$ and D is a compact domain denoted by the Cartesian product : $D = \prod_{i=1}^{n}[m_i - p_i, m_i + p_i] \subset R^n$ where $m_i (m_i \in R)$ is the middle of the domain and $p_i (p_i > 0)$ is half of the domain $(i = 1, \cdots, n)$. $\boldsymbol{x} = [x_1, \cdots, x_n]^T \in D$ is n state vector and T denotes transpose. Let the time domain be $D_t = [0, \infty)$. Assume that $\boldsymbol{f}$ is a sufficiently smooth nonlinear function on $D_t \times D$ such as $\boldsymbol{f}(t, \boldsymbol{x}) = [f_1(t, \boldsymbol{x}), \cdots, f_n(t, \boldsymbol{x})]^T \in C^{N+1}(D_t \times D; R^n)$, $\int_0^\infty \boldsymbol{f}(t, \boldsymbol{x})^T \boldsymbol{f}(t, \boldsymbol{x}) e^{-\alpha t} dt < \infty$ for each $\boldsymbol{x} \in D$. Here $\alpha > 0$ and N is the maximal order defined in Eq. (20) below. In order to apply Chebyshev interpolation (Hildebrand, 1956), state variable $\boldsymbol{x}$ is changed into $\boldsymbol{y}$ so that $\boldsymbol{y}$ has the basic domain of Chebyshev polynomials: $D_0 = \prod_{i=1}^{n}[-1, 1]$ and $\boldsymbol{y}$ is rewritten by

$$\boldsymbol{y} = P^{-1}(\boldsymbol{x} - M) \tag{2}$$

where

$$M = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}, P = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_n \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

The dynamics of $\boldsymbol{y}$ becomes

$$\dot{\boldsymbol{y}}(t) = P^{-1}\boldsymbol{f}(t, P\boldsymbol{y} + M) \equiv F(t, \boldsymbol{y}) \tag{3}$$

where $\equiv$ is the definition notation. The Chebyshev polynomials $\{T_r(\cdot)\}$ are defined by

$$T_r(y_i) \equiv \frac{(-2)^r r!}{(2r)!}(1 - y_i^2)^{\frac{1}{2}} \frac{d^r}{dy_i^r}(1 - y_i^2)^{r - \frac{1}{2}} \tag{4}$$

or, $T_0(y_i) = 1$, $T_1(y_i) = y_i$, $T_2(y_i) = 2y_i^2 - 1$, $T_3(y_i) = 4y_i^3 - 3y_i$, $T_4(y_i) = 8y_i^4 - 8y_i^2 + 1, \cdots$. The recurrence formula of Chebyshev polynomials is described by $T_{r+1}(y_i) = 2y_i T_r(y_i) - T_{r-1}(y_i)$, $(r \geq 1)$, $T_0(y_i) = 1$, $T_1(y_i) = y_i$. The derivative of Chebyshev polynomials $S_r(y_i) \equiv \frac{dT_r(y_i)}{dy_i}$ has the recurrence formula $S_{r+1}(y_i) = 2T_r(y_i) + 2y_i S_r(y_i) - S_{r-1}(y_i)$, $(r \geq 1)$, $S_0(y_i) = 0$, $S_1(y_i) = 1$. The orthogonal condition of Chebyshev polynomials under summation over the zeros of $T_{N_i+1}(y_i)$ is

$$\sum_{\ell_i=0}^{N_i} T_{(q)}(y_{i\ell_i}) T_{(r)}(y_{i\ell_i}) = \begin{cases} 0 & (q \neq r) \\ \dfrac{N_i + 1}{2} & (q = r \neq 0) \\ N_i + 1 & (q = r = 0) \end{cases}$$

where $y_{i\ell_i} = \cos \frac{2\ell_i + 1}{2N_i + 2}\pi$, $(\ell_i = 0, 1, \cdots, N_i)$. Using these Chebyshev polynomials of N_i-th order $(i = 1, \cdots, n)$, we introduce a linearizing function as

$$\begin{aligned} \phi(\boldsymbol{y}) = [&T_{(10\cdots0)}(\boldsymbol{y}), T_{(01\cdots0)}(\boldsymbol{y}), \cdots, T_{(0\cdots01)}(\boldsymbol{y}), \\ &T_{(11\cdots0)}(\boldsymbol{y}), T_{(101\cdots0)}(\boldsymbol{y}), \cdots, T_{(10\cdots1)}(\boldsymbol{y}), \\ &T_{(20\cdots0)}(\boldsymbol{y}), T_{(21\cdots0)}(\boldsymbol{y}), \cdots, T_{(r_1\cdots r_n)}(\boldsymbol{y}), \\ &\cdots, T_{(N_1\cdots N_n)}(\boldsymbol{y})]^T \end{aligned} \tag{5}$$

where $T_{(r_1\cdots r_n)}(\boldsymbol{y}) = \prod_{i=1}^{n} T_{r_i}(y_i)$. We derive the dynamics of each element of ϕ :

$$\begin{aligned} \dot{T}_{(r_1\cdots r_n)}(\boldsymbol{y}) &= \frac{\partial T_{(r_1\cdots r_n)}(\boldsymbol{y})}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}} = [S_{r_1}(y_1) \\ &T_{r_2}(y_2)\cdots T_{r_n}(y_n), T_{r_1}(y_1)S_{r_2}(y_2)\cdots T_{r_n}(y_n), \\ &T_{r_1}(y_1)T_{r_2}(y_2)\cdots S_{r_n}(y_n)]P^{-1}\boldsymbol{f}(t, P\boldsymbol{y} + M) \\ &= \sum_{i=1}^{n} S_{(r_1\cdots r_n)}^{(i)}(\boldsymbol{y})\frac{1}{p_i}f_i(t, P\boldsymbol{y} + M) \end{aligned}$$

where

$$S_{(r_1\cdots r_n)}^{(i)}(\boldsymbol{y}) \equiv T_{r_1}(y_1)T_{r_2}(y_2)\cdots S_{r_i}(y_i)\cdots T_{r_n}(y_n).$$

Thus, $\dot{\phi}(\boldsymbol{y})$ becomes

$$\begin{aligned} \dot{\phi}(\boldsymbol{y}) &= \big[\dot{T}_{(10\cdots0)}(\boldsymbol{y}), \cdots, \dot{T}_{(r_1\cdots r_n)}(\boldsymbol{y}), \cdots, \\ &\dot{T}_{(N_1\cdots N_n)}(\boldsymbol{y})\big]^T = \frac{\partial \phi(\boldsymbol{y})}{\partial \boldsymbol{y}^T}F(t, \boldsymbol{y}) \equiv G(t, \boldsymbol{y}) \\ &= \big[G_{(10\cdots0)}(t, \boldsymbol{y}), \cdots, G_{(r_1\cdots r_n)}(t, \boldsymbol{y}), \\ &\quad \cdots, G_{(N_1\cdots N_n)}(t, \boldsymbol{y})\big]^T \end{aligned} \tag{6}$$

where

$$G_{(r_1\cdots r_n)}(t, \boldsymbol{y}) \equiv \sum_{i=1}^{n} S_{(r_1\cdots r_n)}^{(i)}(\boldsymbol{y})\frac{1}{p_i}f_i(t, P\boldsymbol{y} + M).$$

To this $G(t, \boldsymbol{y})$, we exploit Chebyshev interpolation of N_i-th order with respect to the state variable y_i and Laguerre expansion of N_L-th order to the time variable t. Laguerre polynomials are defined on the domain $D_t = [0, \infty)$ by

$$L_r(t) \equiv e^t \frac{d^r}{dt^r}(t^r e^{-t}),\ (r = 0, 1, 2, \cdots),$$

whose general form for $\alpha > 0$ is

$$L_r(\alpha t) \equiv e^{\alpha t}\frac{d^r}{dt^r}(t^r e^{-\alpha t}). \tag{7}$$

The polynomials are in the form : $L_0(t) = 1,\ L_1(t) = 1-t,\ L_2(t) = 2-4t+t^2,\ L_3(t) = 6-18t+9t^2-t^3,\ L_4(t) = 24-96t+72t^2-16t^3+t^4, \cdots$. The orthogonal condition of the generalized Laguerre polynomials is

$$\int_0^\infty e^{-\alpha t} L_K(\alpha t) L_r(\alpha t) dt = \begin{cases} 0 & (K \neq r) \\ \dfrac{(K!)^2}{\alpha} & (K = r) \end{cases} .$$

Applying Chebyshev and Laguerre expansions, $G_{(r_1 \cdots r_n)}(t, \boldsymbol{y})$ is approximated by

$$\hat{G}_{(r_1 \cdots r_n)}(t, \boldsymbol{y}) \approx \sum_{K=0}^{N_L} \sum_{q_1=0}^{N_1} \cdots \sum_{q_n=0}^{N_n} C_{(Kq_1 \cdots q_n)}^{(r_1 \cdots r_n)} L_K(\alpha t) T_{(q_1 \cdots q_n)}(\boldsymbol{y}),$$

where $C_{(Kq_1 \cdots q_n)}^{(r_1 \cdots r_n)} \equiv \dfrac{\alpha}{(K!)^2} 2^{n-\gamma} (\prod_{i=1}^{n} \dfrac{1}{N_i+1}) \int_0^\infty \sum_{\ell_1=0}^{N_1} \cdots \sum_{\ell_n=0}^{N_n} e^{-\alpha t} G_{(r_1 \cdots r_n)}(t, y_{1\ell_1}, \cdots, y_{n\ell_n}) \times L_K(\alpha t) T_{(q_1, \cdots, q_n)}(y_{1\ell_1}, \cdots, y_{n\ell_n}) dt$, $\gamma = \{\text{the number of } q_i = 0 : 1 \leq i \leq n\}$. Substituting this $\hat{G}$ into Eq.(6) yields

$$\dot{T}_{(r_1 \cdots r_n)}(y) \approx \sum_{K=0}^{N_L} \sum_{q_1=0}^{N_1} \cdots \sum_{q_n=0}^{N_n} C_{(Kq_1 \cdots q_n)}^{(r_1 \cdots r_n)} L_K(\alpha t) T_{(q_1 \cdots q_n)}(\boldsymbol{y}).$$

Thus, $\dot{\phi}(\boldsymbol{y})$ is approximated by

$$\dot{\boldsymbol{\phi}}(y) \approx A(t)\boldsymbol{\phi}(y) + B(t) \tag{8}$$

where $\eta = \eta(r_1, \cdots, r_n),\ \varsigma = \varsigma(q_1, \cdots, q_n)$,

$$[A_{\eta\varsigma}(t)] = [\sum_{K=0}^{N_L} C_{(Kq_1 \cdots q_n)}^{(r_1 \cdots r_n)} L_K(\alpha t)],$$

$$[B_\varsigma(t)] = [\sum_{K=0}^{N_L} C_{(Kq_1 \cdots q_n)}^{(0 \cdots 0)} L_K(\alpha t)],$$

$$\eta, \varsigma \in \{1, \cdots, (N_1+1)(N_2+1)\cdots(N_n+1)-1\}.$$

Using the same coefficients as in Eq.(8), we design a formal linear system by

$$\Sigma_2 : \dot{\boldsymbol{z}}(t) = A(t)\boldsymbol{z}(t) + B(t), \tag{9}$$
$$\boldsymbol{z}(0) = \phi(\boldsymbol{y}(0)) = \phi(P^{-1}(\boldsymbol{x}_0 - M)).$$

The inversion is simply obtained by Eqs.(2) and (5) as

$$\boldsymbol{x}(t) = P\boldsymbol{y}(t) + M = PH\phi(\boldsymbol{y}(t)) + M \tag{10}$$

where $H = [I : 0]$ and I is an $n \times n$ unit matrix. As a result, an approximate $\hat{\boldsymbol{x}}(t)$ of the state $\boldsymbol{x}(t)$ becomes

$$\hat{\boldsymbol{x}}(t) = PH\boldsymbol{z}(t) + M \tag{11}$$

by a solution of Eq. (9).

3. ERROR BOUNDS

Let $\|\cdot\|$ denote the norm $\|X\| = \sqrt{X^T X}$ to a vector X and the corresponding induced matrix norm to a matrix.

Theorem 1: An error bound of the formal linearization is

$$\|\boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)\| \leq p_{max} \Big\{ e^{\mu t} \|\phi(\boldsymbol{y}(0)) - \boldsymbol{z}(0)\| + \sum_{K=1}^{n} \frac{e^{\mu t}-1}{\mu} \frac{\bar{M}_K}{2^{N_K}(N_K+1)!} \Big\} + \sqrt{\frac{e^{\alpha t}-1}{\alpha}} \lambda_t \epsilon_{N_L}$$
$$\equiv E_b(t, N_1, \cdots, N_n, N_L) \tag{12}$$

where $p_{max} = \max\{p_i :\ 1 \leq i \leq n\}$, $\mu = \sup\big\{\|A(t) + A^T(t)\|/2 :\ t \in D_t\big\}$,

$$\bar{M}_K = \sup\Big\{\Big\|\frac{\partial^{N_K+1}}{\partial y_K^{N_K+1}} \omega^{[K-1]} \bar{G}(t, \boldsymbol{y})\Big\| :\ \boldsymbol{y} \in D_0, t \in D_t\Big\},\ \lambda_t = \sup\Big\{\Big\|PH\psi(t,\tau)\frac{\partial\phi}{\partial \boldsymbol{y}^T}P^{-1}\Big\| : 0 \leq \tau \leq t, \boldsymbol{y} \in D_0\Big\},\ \epsilon_{N_L} = \sup\Big\{\Big[\int_0^\infty \|\boldsymbol{f}(\tau, \boldsymbol{x})\|^2 e^{-\alpha\tau} d\tau - \sum_{K=0}^{N_L} \|C_K^*(\boldsymbol{x})\|^2\Big]^{\frac{1}{2}} : \boldsymbol{x} \in D\Big\}.$$

Proof : A difference between Eq.(6) and Eq.(9) is

$$\frac{d}{dt}\big(\phi(\boldsymbol{y}(t)) - \boldsymbol{z}(t)\big) = \frac{\partial\phi(\boldsymbol{y})}{\partial \boldsymbol{y}^T} F(t, \boldsymbol{y}) - A(t)\boldsymbol{z} - B(t) = A(t)\big(\phi(\boldsymbol{y}) - \boldsymbol{z}(t)\big) + \Big(\frac{\partial\phi(\boldsymbol{y})}{\partial \boldsymbol{y}^T} F(t, \boldsymbol{y}) - A(t)\phi(\boldsymbol{y}) - B(t)\Big).$$

The solution of this differential equation is

$$\phi\big(\boldsymbol{y}(t)\big) - \boldsymbol{z}(t) = \psi(t, 0)\big\{\phi\big(\boldsymbol{y}(0)\big) - \boldsymbol{z}(0)\big\} + \int_0^t \psi(t, \tau)\epsilon\big(\tau, \boldsymbol{y}(\tau)\big) d\tau \tag{13}$$

where the state transition matrix $\psi(\cdot, \cdot)$ satisfies the matrix differential equation $\dfrac{d}{dt}\psi(t, \tau) = A(t)\psi(t, \tau)$, $\psi(\tau, \tau) = I$ and the linearization error is

$$\epsilon\big(t, \boldsymbol{y}(t)\big) = \frac{\partial\phi(\boldsymbol{y})}{\partial \boldsymbol{y}^T} F(t, \boldsymbol{y}) - A(t)\phi(\boldsymbol{y}) - B(t). \tag{14}$$

Note $\frac{d}{dt}\|\psi(t,\tau)\|^2 = 2\psi^T(t,\tau)\dot{\psi}(t,\tau) = \psi^T(t,\tau)\big(A(t)+A^T(t)\big)\psi(t,\tau) \leq 2\mu\|\psi(t,\tau)\|^2$, whose solution is $\|\psi(t,\tau)\|^2 \leq \|\psi(\tau,\tau)\|^2 e^{2\mu(t-\tau)} = e^{2\mu(t-\tau)}$, where $\mu = \sup\big\{\|A(t)+A^T(t)\|/2 :\ t \in D_t\big\}$. Thus we have $\|\psi(t,\tau)\| \leq e^{\mu(t-\tau)}$, and

$$\int_0^t \|\psi(t,\tau)\|d\tau \leq \int_0^t e^{\mu(t-\tau)}d\tau = \frac{(e^{\mu t}-1)}{\mu}. \quad (15)$$

From now on we shall consider Laguerre expansion about t. First, let us introduce a Hilbert space $Z_\alpha(D_t;R^n) \equiv \{\boldsymbol{u} : D_t \to R^n; \int_0^\infty e^{-\alpha t}\|\boldsymbol{u}(t)\|^2 dt < \infty\}$ with inner product $\langle \boldsymbol{u},\boldsymbol{v}\rangle_\alpha \equiv \int_0^\infty e^{-\alpha t}\boldsymbol{u}(t)^T\boldsymbol{v}(t)dt$ for $\boldsymbol{u},\boldsymbol{v} \in Z_\alpha(D_t;R^n)$. It should be noticed that, since $\{\varphi_K(t) = \frac{1}{K!}e^{-\frac{t}{2}}L_K(t) : K = 0,1,2,\cdots\}$ forms a complete orthonormal system in $L^2(D_t)$ (Nakamura, 1981), $\{\frac{\sqrt{\alpha}}{K!}L_K(\alpha t); K = 0,1,2,\cdots\}$ forms a complete orthonormal system in $Z_\alpha(D_t) \equiv Z_\alpha(D_t;R)$. Therefore, it holds that

$$\boldsymbol{f}(t,\boldsymbol{x}) = \sum_{K=0}^{\infty}\frac{\sqrt{\alpha}}{K!}C_K^*(\boldsymbol{x})L_K(\alpha t)$$

where $C_K^*(\boldsymbol{x}) = \frac{\sqrt{\alpha}}{K!}\int_0^\infty e^{-\alpha t}\boldsymbol{f}(t,\boldsymbol{x})L_K(\alpha t)dt$ for each $\boldsymbol{f}(t,\boldsymbol{x}) \in Z_\alpha(D_t;R^n)$ at any fixed $\boldsymbol{x}$. Thus we have

$$\epsilon_{N_L}(\boldsymbol{x}) \equiv \Big\{\int_0^\infty \|\epsilon_{N_L}^{(t)}(t,\boldsymbol{x})\|^2 e^{-\alpha t}dt\Big\}^{\frac{1}{2}}$$

$$= \Big\{\int_0^\infty \|\boldsymbol{f}(t,\boldsymbol{x})\|^2 e^{-\alpha t}dt - \sum_{K=0}^{N_L}\|C_K^*(\boldsymbol{x})\|^2\Big\}^{\frac{1}{2}} \quad (16)$$

where $\boldsymbol{f}(t,\boldsymbol{x}) = \sum_{K=0}^{N_L}\frac{\sqrt{\alpha}}{K!}C_K^*(x)L_K(\alpha t)+\epsilon_{N_L}^{(t)}(t,\boldsymbol{x})$, which indicates that $\epsilon_{N_L}(\boldsymbol{x}) \to 0$ as $N_L \to \infty$. Note that the Hölder inequality indicates

$$\int_0^t \|\epsilon_{N_L}^{(t)}(t,\boldsymbol{x})\|dt \leq \Big[\int_0^t e^{\alpha\tau}d\tau\Big]^{\frac{1}{2}}\Big[\int_0^t \|\epsilon_{N_L}^{(t)}(\tau,\boldsymbol{x})\|^2 e^{-\alpha\tau}d\tau\Big]^{\frac{1}{2}} \leq \sqrt{\frac{e^{\alpha t}-1}{\alpha}}\,\epsilon_{N_L}(\boldsymbol{x}). \quad (17)$$

We here introduce an operator $\Im$ which approximates a vector function $\boldsymbol{f}(t,\boldsymbol{x})$ by Laguerre expansion with respect to t:

$$\Im : \boldsymbol{f}(t,\boldsymbol{x}) \to \sum_{K=0}^{N_L}\frac{\sqrt{\alpha}}{K!}C_{(K)}^*(\boldsymbol{x})L_K(\alpha t) \quad (18)$$

so that $\epsilon_{N_L}(\boldsymbol{x}) = \big\{\int_0^\infty \|\boldsymbol{f}(t,\boldsymbol{x}) - \Im\boldsymbol{f}(t,\boldsymbol{x})\|^2 e^{-\alpha t}dt\big\}^{\frac{1}{2}}$. Next we shall consider Chebyshev interpolation about $\boldsymbol{y}$. Define $\pi(y_i) \equiv (y_i - y_{i0})(y_i - y_{i1})\cdots(y_i - y_{iN_i})$ using the zeros $\{y_{ij} : j = 0,\cdots,N_i\}$ of $T_{N_i+1}(y_i)$ for each y_i $(i = 1,\cdots,n)$. Then it holds that

$$\min_{\{y_{ij}\}}\max_{y_i\in D_0}|\pi(y_i)| = 2^{-N_i}. \quad (19)$$

Note that $\boldsymbol{f}(t,\boldsymbol{x}) \in C^{N+1}(D_t \times D;R^n)$ or $G_{(r_1\cdots r_n)}(t,\boldsymbol{y}) \in C^{N+1}(D_t \times D_0)$ where

$$N = \max\{N_i : i = 1,\cdots,n\}. \quad (20)$$

It should be noticed that, the function $\bar{G}(t,\boldsymbol{y})$ defined by $\bar{G}(t,\boldsymbol{y}) \equiv \Im G(t,\boldsymbol{y})$ in Eq.(6) can be expressed by Chebyshev interpolation as

$$\bar{G}(t,\boldsymbol{y}) = \sum_{q_i=0}^{N_i}C_{(q_i)}(t,\bar{\boldsymbol{y}}_i)T_{q_i}(y_i) + \epsilon_{N_i}^{(y_i)}(t,\boldsymbol{y}) \quad (21)$$

where $\bar{\boldsymbol{y}}_i \equiv [y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_n]^T$,

$$C_{(q_i)}(t,\bar{\boldsymbol{y}}_i) = \frac{2^{1-\gamma}}{N_i+1}\sum_{\ell_i=0}^{N_i}\bar{G}(t,y_1,\cdots,y_{i\ell_i},\cdots,y_n)T_{q_i}(y_{i\ell_i}),\ \gamma = \begin{cases}0, & (q_i = 0)\\ 1, & (q_i \neq 0)\end{cases},$$

$$\epsilon_{N_i}^{(y_i)}(t,\boldsymbol{y}) = \frac{1}{(N_i+1)!}\frac{\partial^{N_i+1}}{\partial y_i^{N_i+1}}\bar{G}(t,\boldsymbol{y})\big|_{y_i=\xi_i}\pi(y_i),$$

$$(\xi_i \in [-1,1]). \quad (22)$$

$\bar{G}(t,\boldsymbol{y})$ is in C^{N_i+1}-class with respect to y_i at fixed $(t,\bar{\boldsymbol{y}}_i)$. We here introduce an operator ω_i which approximates a vector function $\bar{G}(t,\boldsymbol{y})$ by Chebyshev interpolation in Eq.(21):

$$\omega_i : \bar{G}(t,\boldsymbol{y}) \to \sum_{q_i=0}^{N_i}C_{(q_i)}(t,\bar{\boldsymbol{y}}_i)T_{q_i}(y_i),$$

so that

$$\bar{G}(t,\boldsymbol{y}) - \omega_i\bar{G}(t,\boldsymbol{y}) = \epsilon_{N_i}^{(y_i)}(t,\boldsymbol{y}). \quad (23)$$

Define a product of the operator by $\omega^{[i]} \equiv \omega_i \circ \cdots \circ \omega_2 \circ \omega_1$, where $\omega^{[0]}\bar{G}(t,\boldsymbol{y}) = \bar{G}(t,\boldsymbol{y})$. Using this operator, we have $\omega^{[n]}\Im\frac{\partial\phi(\boldsymbol{y})}{\partial\boldsymbol{y}^T}F(t,\boldsymbol{y}) = \omega^{[n]}\bar{G}(t,\boldsymbol{y}) = A(t)\phi(\boldsymbol{y})+B(t)$ from Eq.(6) through Eq.(8). Therefore, Eq.(14) becomes

$$\epsilon(t,\boldsymbol{y}) = \{\frac{\partial\phi}{\partial\boldsymbol{y}^T}F(t,\boldsymbol{y}) - \Im\frac{\partial\phi}{\partial\boldsymbol{y}^T}F(t,\boldsymbol{y})\} + \{\bar{G}(t,\boldsymbol{y}) - \omega^{[1]}\bar{G}(t,\boldsymbol{y})\} + \{\omega^{[1]}\bar{G}(t,\boldsymbol{y}) - \omega^{[2]}\bar{G}(t,\boldsymbol{y})\}$$

$$+ \cdots + \big\{\omega^{[n-1]}\bar{G}(t,\boldsymbol{y}) - (A(t)\phi(\boldsymbol{y}) + B(t))\big\}$$

$$= \frac{\partial\phi}{\partial\boldsymbol{y}^T}P^{-1}\big\{\boldsymbol{f}(t,P\boldsymbol{y}+M) - \Im\boldsymbol{f}(t,P\boldsymbol{y}+M)\big\}$$

$$+\sum_{K=1}^{n}\{\omega^{[K-1]}\bar{G}(t,\boldsymbol{y})-\omega^{[K]}\bar{G}(t,\boldsymbol{y})\}. \tag{24}$$

From Eqs.(23), (22) and (19),

$$\|\omega^{[K-1]}\bar{G}(t,\boldsymbol{y})-\omega^{[K]}\bar{G}(t,\boldsymbol{y})\|$$
$$=\big\|(\omega^{[K-1]}\bar{G}(t,\boldsymbol{y}))-\omega_K(\omega^{[K-1]}\bar{G}(t,\boldsymbol{y}))\big\| =$$
$$\Big\|\frac{1}{(N_K+1)!}\frac{\partial^{N_K+1}}{\partial y_K^{N_K+1}}\omega^{[K-1]}\bar{G}(t,\boldsymbol{y})\mid_{y_K=\xi_K}\pi(y_K)\Big\|$$
$$\le \frac{\sup\{\|\frac{\partial^{N_K+1}}{\partial y_K^{N_K+1}}\omega^{[K-1]}\bar{G}(t,\boldsymbol{y})\|\|\pi(y_K)\|\}}{(N_K+1)!}$$
$$\le \frac{\bar{M}_K}{2^{N_K}(N_K+1)!} \tag{25}$$

where $\sup\{\|\pi(y_K)\| : y_K \in D_0\} = 2^{-N_K}, \bar{M}_k = \sup\Big\{\Big\|\frac{\partial^{N_K+1}}{\partial y_K^{N_K+1}}\omega^{[K-1]}\bar{G}(t,\boldsymbol{y})\Big\| :\ \boldsymbol{y}\in D_0,\ t\in D_t\Big\}$.
From Eqs.(10) and (11), it follows that

$$\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)=PH\Big(\phi\big(\boldsymbol{y}(t)\big)-\boldsymbol{z}(t)\Big). \tag{26}$$

Substituting Eq.(24) into Eq.(13) and then to Eq.(26) yields

$$\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)=PH\psi(t,0)\big(\phi(\boldsymbol{y}(0)-\boldsymbol{z}(0)\big)+\int_0^t PH$$
$$\psi(t,\tau)\frac{\partial\phi}{\partial\boldsymbol{y}^T}P^{-1}(\boldsymbol{f}(\tau,\boldsymbol{x})-\Im\boldsymbol{f}(\tau,\boldsymbol{x}))d\tau+\int_0^t PH$$
$$\psi(t,\tau)\Big[\sum_{K=1}^{n}\{\omega^{[K-1]}\bar{G}(\tau,\boldsymbol{y})-\omega^{[K]}\bar{G}(\tau,\boldsymbol{y})\}\Big]d\tau.$$

Thus it follows that

$$\|\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)\|\le\|PH\|\|\psi(t,0)\|\|\phi(\boldsymbol{y}(0))-\boldsymbol{z}(0)\|$$
$$+\sup\Big\{\|PH\psi(t,\tau)\frac{\partial\phi}{\partial\boldsymbol{y}^T}P^{-1}\|\Big\}\int_0^t\|(\boldsymbol{f}(\tau,\boldsymbol{x})$$
$$-\Im\boldsymbol{f}(\tau,\boldsymbol{x}))\|d\tau+\|PH\|\sum_{K=1}^{n}\|\omega^{[K-1]}\bar{G}(\tau,\boldsymbol{y})$$
$$-\omega_K\big(\omega^{[K-1]}\bar{G}(\tau,\boldsymbol{y})\big)\|\int_0^t\|\psi(t,\tau)\|d\tau. \tag{27}$$

Note that $\|PH\|\le\|P\|\|H\|\le p_{max},\ p_{max}=\max\{p_i>0:\ 1\le i\le n\},\ \lambda_t\equiv\sup\{\|PH$

$$\psi(t,\tau)\frac{\partial\phi}{\partial\boldsymbol{y}^T}P^{-1}\|:\ 0\le\tau\le t,\boldsymbol{y}\in D_0\}. \tag{28}$$

From Eqs.(15), (17), (25) and (28), Eq.(27) becomes

$$\|\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)\|\le p_{max}e^{\mu t}\big\|\phi\big(\boldsymbol{y}(0)\big)-\boldsymbol{z}(0)\big\|$$
$$+\lambda_t\sqrt{\frac{e^{\alpha t}-1}{\alpha}}\epsilon_{N_L}$$
$$+p_{max}\sum_{K=1}^{n}\frac{e^{\mu t}-1}{\mu}\frac{\bar{M}_K}{2^{N_K}(N_K+1)!}$$

which is reduced to Eq.(12). □
This theorem indicates that the 1st, 2nd and 3rd terms of Eq.(12) come from the errors of initial state, Chebyshev interpolation, and Laguerre expansion, respectively. In this formal linearization, the error becomes $\|\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)\|\to 0$ for any $t\in D_t$, as $\boldsymbol{x}(0)-\hat{\boldsymbol{x}}(0)\to 0,\ N_K\to\infty$ $(K=1,2,\cdots,n)$ and $N_L\to\infty$.

4. NONLINEAR OBSERVER

We synthesize a time-varying nonlinear observer as an application of the above linearization. Assume that a nonlinear dynamic system is the same as Eq.(1) : $\dot{\boldsymbol{x}}(t)=\boldsymbol{f}(t,\boldsymbol{x})$, and a measurement equation is

$$\boldsymbol{\eta}(t)=\boldsymbol{h}(t,\boldsymbol{x})\in R^m \tag{29}$$

where $\boldsymbol{\eta}$ is measurement data. The $\boldsymbol{h}(t,\boldsymbol{x})$ is a sufficiently smooth nonlinear function which holds the same condition as $\boldsymbol{f}(t,\boldsymbol{x})$. The dynamic system (Eq.(1)) is transformed into a time-varying linear system (Eq.(9)) by the formal linearization mentioned above. From Eq.(2), Eq.(29) is rewritten as $\boldsymbol{\eta}(t)=\boldsymbol{h}(t,P\boldsymbol{y}+M)$. Substituting $\boldsymbol{h}(t,P\boldsymbol{y}+M)$ instead of $G_{(\cdot)}(t,\boldsymbol{y})$ into Eq.(6) and applying Chebyshev interpolation and Laguerre expansion yields $\boldsymbol{\eta}(t)=C(t)\boldsymbol{z}(t)+d(t)$ in a similar manner to that described in Section 2. To this linearized system, we apply the linear observer theory (Luenberger, 1971) so that the identity observer is obtained as

$$\dot{\hat{\boldsymbol{z}}}(t)=A(t)\hat{\boldsymbol{z}}(t)+B(t)+$$
$$K(t)\big(\boldsymbol{\eta}(t)-C(t)\hat{\boldsymbol{z}}(t)-d(t)\big) \tag{30}$$

where $K(t)=\frac{1}{2}\Sigma(t)C(t)^TR(t), \dot{\Sigma}(t)=A(t)\Sigma(t)+\Sigma(t)A(t)^T+Q(t)-\Sigma(t)C(t)^TR(t)C(t)\Sigma(t)$. Here $Q(t)$, $R(t)$ and $\Sigma(0)$ are arbitrary real symmetric positive definite matrices. From Eq.(11), the estimate becomes

$$\hat{\hat{\boldsymbol{x}}}(t)=PH\hat{\boldsymbol{z}}(t)+M. \tag{31}$$

5. NUMERICAL EXPERIMENTS

We illustrate numerical experiments of the above formal linearization and observer.

5.1 *Formal Linearization*

Consider the following system :

$$\dot{x} = -e^{-2t} \log_e(x+1),\ x(0) = 1.$$

We linearize it by the formal linearization approach. The parameters of Eq.(2) are set as $M = 0.85,\ P = 0.16$. Figure 1 shows the errors defined by $J(t, N, N_L) \equiv \|x(\tau) - \hat{x}(\tau)\|$ whose $\hat{x}$ is the solution of Eq.(1) and the error bounds $Eb(t, N, N_L)$ of Eq.(12), when the order of Chebyshev and Laguerre polynomials are $N = N_L = 2$ and $N = N_L = 3$.

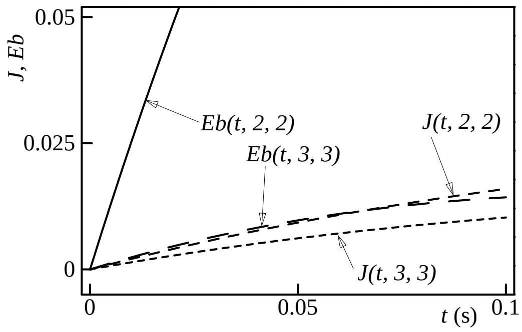

Fig. 1. Aprroximation errors and error bounds

5.2 *Nonlinear Observer*

Consider the following system:

$$\dot{x}_1(t) = -\frac{3x_1^2}{1 + x_2 + \sin t},\ \dot{x}_2(t) = -x_1^2,$$
$$\eta(t) = x_1^2 + x_2,\ D = [0, 1] \times [0.3, 1] \subset R^2.$$

Set the parameters for the formal linearization as

$$M = \begin{pmatrix} 0.5 \\ 0.68 \end{pmatrix}, P = \begin{pmatrix} 0.51 & 0 \\ 0 & 0.33 \end{pmatrix},$$

and the unknown initial value is $x(0) = [1, 1]^T$. Put $Q(t) = I,\ M(t) = 50,\ \Sigma(0) = I,\ \hat{\hat{x}}(0) = [0.3, 0.3]^T$, in the nonlinear observer of Eq.(30). Figure 2 shows the true value $\boldsymbol{x}$, the estimates $\hat{\hat{\boldsymbol{x}}}$ when $N_1 = N_2 = 3, N_L = 3$, and the observer estimate ($Taylor$) by the 1st order Taylor expansion as a conventional method .

6. CONCLUSIONS

This paper has developed the formal linearization approach for a general class of time-varying nonlinear systems using Chebyshev interpolation and Laguerre expansion. The error bound explains that the accuracy of this linearization is improved as the order of Chebyshev and Laguerre polynomials increases. The nonlinear observer has been synthesized as its application. Simulation results have been presented which are very encouraging.

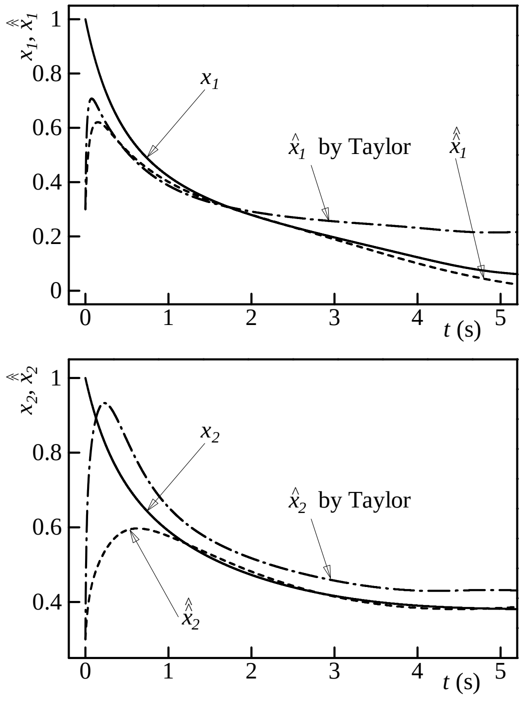

Fig. 2. Estimates x_1 and x_2 by nonlinear observers

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