ESTIMATION OF THE MEAN ESCAPE TIME IN LAGRANGIAN SYSTEMS WITH WEAK NOISE

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Abstract. In a number of applications, an admissible domain of the perturbed motion is associated with the domain of attraction of a stable equilibrium of the unperturbed system. If noise is weak, escape from the reference domain is a rare event associated with large deviations in the system. Despite the well developed large deviations theory, estimation of the statistical parameters for the multidimensional nonlinear systems remains difficult. This paper develops an asymptotic approximation of the mean escape time for a weakly perturbed Lagrangian system. The estimate is found explicitly, as a function of the kinetic and potential energy and the dissipation function of the system.

Key words: stochastic systems, large deviation.

1. INTRODUCTION

Stationary statistics of the Lagrangian dynamics can be found as a stationary solution of a relevant Kolmogorov equation, see, e.g., (Roberts and Spanos, 1990). However, an analysis of the nonstationary dynamics associated with the calculation of the mean escape time and escape probability encounters serious technical difficulties. This paper develops a procedure of the asymptotic approximation of the mean time until escape from a given domain for a weakly perturbed Lagrangian system.

Let $X(t)$ be a trajectory of the system, $X(0) = 0$. In the absence of perturbation the system is presumed to have an asymptotically stable equilibrium $X = 0$, with the domain of attraction G_a. An admissible domain of motion is usually associated with a bounded open set $G \subseteq G_a$. Whatever small noise may be, it induces escape of the process $X(t)$ from the reference domain with a non-zero probability.

A performance criterion of interest is the mean time $\mathbf{E}T^{\varepsilon}$, where $T^{\varepsilon} = \inf\,(t\colon X(t) \notin G)$ is the first moment the process $X(t)$ escapes the domain G. The parameter $\varepsilon > 0$ indicates the intensity of noise. The direct calculation of $\mathbf{E}T^{\varepsilon}$ in the small noise limit requires consideration of a singular Dirichlet problem for a multidimensional system. Both analytic and computational approaches to the asymptotic solution of this equation are prohibitively difficult.

Large deviations theory provides an alternative approach to the analysis of the weakly perturbed dynamics, see Varadhan (1984), Kushner (1987), Dupuis and Ellis (1997), Freidlin and Wentzell, (1998), and references therein. This theory gives a rough asymptotic of the mean escape time as $\mathbf{E}T^{\varepsilon} \sim \exp(K/\varepsilon)$, as $\varepsilon \to 0$. The parameter K is associated with the minimum of the action functional of the system. The derivation and calculation of this functional is the central point of large deviation theory.

Freidlin and Wentzell (1998) considered in details the nondegenerate diffusion and derived the Hamilton-Jacobi equation associated with the minimization of the action functional. The mean escape time and escape probability can be computed directly from this equation. Despite the well developed theory, explicit solutions for multidimensional systems are few in number. Meerkov (1988) calculated the logarithmic asymptotic of the mean escape time for a multidimensional linear system. Freidlin and Wentzell (1998) estimated the mean escape time and escape probability for a class of nonlinear systems with the "quasipotential" nonlinearity. The quasipotential structure is typical for a one-dimensional system, the recent advances in this field can be found in (Olivieri and Vares, 2005). However, the Lagrangian or Hamiltonian systems do not allow separation of the quasipotential nonlinearity. Then, the degenerate systems are more natural for applications.

Kovaleva (2003) applied the stochastic averaging approach to a problem of controlling large deviations in a Hamiltonian system in the plane. The averaging procedure results in the decomposition of the slow and fast variables. This reduces the problem to the analysis of a slow component of the process as a one-dimensional nondegenerate diffusion.

In this paper we consider a multidimensional Lagrangian system as a system with the degenerate noise. Kushner (1987) and Dupuis and Kushner (1989) developed an extension of large deviation theory to systems with degenerate noise and derived a control procedure associated with minimization of the action functional. Yet the problem of calculating the mean escape time for a multidimensional system has not been resolved explicitly. The objective of this paper is to extend Kushner's approach to the multi-degree-of-freedom (MDF) Lagrangian systems and to show that, in the small noise limit, the Hamilton-Jacobi equation for the associated variational problem can be resolved explicitly.

The paper is organized as follows. In Section 2 we remind the main issues of large deviation theory requisite for the further analysis. We make use of Kushner's definition of the action functional for a degenerate system and transform the mean escape time problem into a deterministic variational problem for the action functional. We present the Hamilton-Jacobi equation for the variational problem associated with estimation of the mean escape time for a degenerate diffusion.

For the sake of brevity, we employ a commonly used small white noise model as an approximation. However, small Gaussian (Kushner, 1987) or Markov perturbations (Freidlin and Wentzell, 1998, Kushner, 1987) can be investigated as well.

Section 3 discusses the basic result of the paper. We derive an explicit solution of the Hamilton-Jacobi equation associated with the mean escape time problem for Lagrangian systems. We show that, under some non-restrictive assumptions, the solution can be expressed through the parameters of the Lagrangian function and the dissipation function of the system.

In Section 4 we consider some physically meaningful examples.

2. BASIC METHODOLOGY

Consider the non-perturbed system in the form

$$\dot{X} = f(X),\ X(0) = 0 \in G \qquad (1)$$

where G is an open bounded set in R^m with a piecewise differentiable boundary Γ.

The perturbed system satisfies the equation

$$\dot{X} = f(X) + \varepsilon\Delta\,\dot{w}\,(t),\ X(0) = 0 \in G \qquad (2)$$

where $w(t) \in R^r$ is standard Wiener process, Δ is an $m\times r$ - matrix.

We assume that

(i) $f(X)$ is continuous and satisfies the Lipschitz condition in $G \cup \Gamma$;

(ii) system (1) has a unique asymptotically stable point $X = 0$ in G and all trajectories originating in the domain $G \cup \Gamma$ tend to $X = 0$.

(iii) system (2) can be written in the form

$$\dot{X}_1 = f_1(X)$$

$$\dot{X}_2 = f_2(X) + \varepsilon\Delta_2\,\dot{w}_2(t) \qquad (2a)$$

where the partition $X = (X_1, X_2)$ is of the same dimension as $f = (f_1, f_2)$, dimensionalities of the vectors f_2, w_2 and the matrix Δ_2 are compatible, the matrix $\Sigma_{22} = \Delta_2'\Delta_2$ is positive definite. Here and below the prime denotes transpose of a vector or a matrix.

Following (Kushner, 1987), we introduce a stochastic counterpart of the action functional. At the first stage, we consider the transformation

$$\Phi = \lim_{\varepsilon \to 0} \varepsilon^2 \ln\{\mathbf{E} \exp \int_0^{T/\varepsilon^2} \alpha'(\varepsilon^2 t)[f(X) + \Delta \dot{w}(t)]dt\} = \int_0^T F(X, \alpha(s))\, ds \qquad (3)$$

Since for a Gaussian zero-mean process $G(t)$ we have $\mathbf{E}\exp G = \exp\mathbf{E}G^2$, the function F can easily be calculated and written as

$$F(X, \alpha) = \alpha_1 f_1(X)\alpha_1 + \alpha_2 f_2(X)\alpha_2 + \frac{1}{2}\alpha_2' \Sigma_{22}\alpha_2 \qquad (4)$$

where the partition $\alpha = (\alpha_1, \alpha_2)$ is of the same dimension as $f = (f_1, f_2)$.

Integrand in the left hand side of relation (3) can be interpreted as the Hamiltonian function associated with system (2). In the deterministic case ($\Sigma_{22} = 0$) function (4) coincides with the Hamiltonian function constructed for system (1).

Let the function F and its first derivative be continuous in X. Introduce the dual Cramer transformation (also called the Legendre transformation) C

$$C(X, \beta) = \sup_{\alpha} [\beta' \alpha - F(X, \alpha)] \qquad (5)$$

Consider the partition $\beta = (\beta_1, \beta_2)$, where β_1, β_2 have the same dimension as f_1, f_2. Then

$$C(X,\beta) = (\beta_2 - f_2(x))' \Sigma_{22}^{-1}(\beta_2 - f_2(x))$$

$$\text{if } \beta_1 = f_1(x) \qquad (6a)$$

$$C(X, \beta) = \infty \text{ otherwise} \qquad (6b)$$

with an admissible set $U_2(X) = \{\beta_2: C(X,\beta_2) < \infty\}$.

If $\varphi(t)$ is an absolutely continuous function, then introduce the action functional as

$$S(T, \varphi) = \int_0^T C(\varphi, \dot{\varphi})\, dt \qquad (7)$$

where $C(\varphi, \dot{\varphi})$ is defined by relation (6a). If $\varphi(t)$ is not absolutely continuous, then we set $S(T,\varphi) = \infty$.

Let $\varphi(t)$ be an extremal of the functional (7) leading from an origin 0 to a point X, that is

$$S(X) = \inf \{S(T,\varphi): \varphi(0) = 0,\ \varphi(T) = X\} \qquad (8)$$

In a deterministic system the extremal depicts an orbit leading from an origin to a given point X (Arnold, 1989). Following large deviation theory, this extremal approximates the exit orbit of system (2) with probability close to 1.

Under these assumptions, the mean time needed to reach the boundary Γ of the domain G from an initial point $X = 0$ obeys the estimate (Kushner, 1987, Freidlin and Wentzell, 1998)

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln(\mathbf{E}T^{\varepsilon}) = \inf_{X \in \Gamma} S(X) = S_0 \qquad (9)$$

The minimum condition in equality (9) implies that an exit orbit intersects the boundary Γ at the minimal distance from the attracting point $X = 0$.

Estimation of the escape time is thus reduced to minimization of functional (7). The variational problem (6a) – (8) can be resolved in a standard way. Freidlin and Wentzell (1998) constructed the functional S and a relevant Hamilton – Jacobi equation in case the matrix Δ is nondegenerate. If the matrix Δ is degenerate and allows partition (2a), the Hamilton–Jacobi equation for the deterministic variational problem (6a) – (8) takes the form

$$(f_1(X), \frac{\partial S}{\partial X_1}) + (f_2(X), \frac{\partial S}{\partial X_2}) + \frac{1}{2}(\Delta_2 \frac{\partial S}{\partial X_2}, \Delta_2 \frac{\partial S}{\partial X_2}) = 0, \quad S(0) = 0 \qquad (10)$$

Suppose that

(iv) equation (10) has a unique continuous and continuously differentiable solution $S(X) > 0$.

Then, under assumptions (i) – (iv), condition (9) holds.

3. LARGE DEVIATIONS FOR LAGRANGIAN SYSTEMS

We employ Eq (10) in order to estimate the mean time of escape from a given domain for a Lagrangian system. In the system description we make use of the notations and definitions accepted in the classic mechanics, see, for example (Arnold, 1989).

The equation of motion for the Lagrangian system with dissipation is written in the form

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L}{\partial q} + B\dot{q} = \varepsilon\sigma\dot{w}(t) \qquad (11)$$

where $L(q, \dot{q})$ is the Lagrangian of the system, $q, \dot{q} \in G \subset R^{2n}$, G is an open admissible domain with boundary Γ, $w(t) \in R^r$ is standard Wiener process, σ is a symmetric non-degenerate square matrix. We suppose that the matrix $A = \sigma'\sigma$ is positive definite and the dimensionalities of the matrices B and σ are compatible. The parameter $\varepsilon << 1$ is the small parameter of the system.

The Lagrangian $L(q, \dot{q})$ takes the form

$$L(q, \dot{q}) = T(q, \dot{q}) - \Pi(q) \tag{12}$$

where $T(q, \dot{q}) = \frac{1}{2}\dot{q}' M(q)\dot{q}$ and $\Pi(q)$ are kinetic and potential energy of the system, respectively.

The matrices $M(q)$ and B are symmetric positive definite square matrix. The point $q = 0$ is assumed to be a unique minimum of the function $\Pi(q)$ in G such that $\Pi(0) = 0$, $\partial\Pi/\partial q_i > 0$ for all components q_i of the vector q. Under these assumptions, the point $q = \dot{q} = 0$ is the asymptotically stable equilibrium position of the unperturbed system ($\sigma = 0$). The domain of attraction of the equilibrium point is denoted as G_a, an admissible region is $G \subseteq G_a$.

We reduce system (11) to the form (2a). Using the Routh transform, we define the impulse

$$p = \frac{\partial L}{\partial \dot{q}} \tag{13}$$

and construct the Hamiltonian function

$$H(q, p) = (\dot{q}, p) - L(q, \dot{q}) \tag{14}$$

where, in view of (12), (13),

$$H(q, p) = T(q, \dot{q}) + \Pi(q)$$

$$\dot{q}(q, p) = M^{-1}(q)p \tag{15}$$

Substituting relations (13), (14) into Eq. (11), we obtain the system

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} - B\frac{\partial H}{\partial p} + \varepsilon\sigma\dot{w}(t) \end{aligned} \tag{16}$$

We suppose that function (12) and the coefficients in the right hand-side of system (15) are smooth enough, and the assumptions of Section 2 hold.

Using the notations $X_1 = q$, $X_2 = p$, $f_1(X) = \partial H/\partial p$, $f_2(X) = -\partial H/\partial q - B\partial H/\partial p$, we reduce system (16) to the form (2a). This implies the Hamilton Jacobi equation in the form

$$[H,S] - (B\frac{\partial H}{\partial p}, \frac{\partial S}{\partial p}) + \frac{1}{2}(\sigma\frac{\partial S}{\partial p}, \sigma\frac{\partial S}{\partial p}) = 0 \tag{17}$$

with the initial condition $S(0,0) = 0$. Here $[H, S]$ denotes the Poisson bracket, that is

$$[H, S] = (\frac{\partial H}{\partial p}, \frac{\partial S}{\partial q}) - (\frac{\partial H}{\partial q}, \frac{\partial S}{\partial p}) \tag{18}$$

It follows from relations (17), (18) that Eq (17) is satisfied if

$$\frac{\partial S}{\partial q} = K\frac{\partial H}{\partial q}, \quad \frac{\partial S}{\partial p} = K\frac{\partial H}{\partial p} \tag{19}$$

where the matrices $K = 2A^{-1}B$ and $A = \sigma'\sigma$.

Substitution of relations (14), (15) into Eqs (19) yields

$$\frac{\partial S}{\partial p} = KM^{-1}(q)p$$

$$S(q, p) = (p, KM^{-1}(q)p) + Q(q) \tag{20}$$

Inserting function (20) into Eqs (19) and considering definition (15) we find that the function $Q(q)$ satisfies the conditions

$$\frac{\partial Q}{\partial q} = K\frac{\partial \Pi}{\partial q}, \; Q(0) = 0 \tag{21}$$

Equations (20) and (21) determine the solution of Eq (17) and, therefore, estimate (9).

Consider some special cases allowing the precise solution.

1. The matrix $K = kI_n$, where I_n is the identity matrix of the *n-th* order, k is a scalar. Thus we obtain

$$S(q,p) = kH(q,p) \tag{22}$$

2. The matrix $M(q) = M = Const$, the matrix K is represented as $K = \mathrm{diag}\{k_1,\ldots,k_n\} = I_n k$, where the vector $k = (k_1,\ldots,k_n)$. Let the potential function allow decomposition

$$\Pi(q) = \sum_{i=1}^{n} f_i(q_i), \; f_i(0) = 0 \tag{23}$$

Thus we have

$$Q(q) = \sum_{i=1}^{n} k_i f_i(q_i) = (k, f(q)) \tag{24}$$

where the vector $f = (f_1,\ldots, f_n)$. It follows from relations (20), (21) and (24) that

$$S(q,p) = (p, KM^{-1}p) + (k, f(q)) \qquad (25)$$

Remarks. An asymptotic estimate, similar to formula (9), can be obtained if the matrix $B = \varepsilon\beta$. In this case the estimate of the mean escape time takes the form

$$\lim_{\varepsilon\to 0} \varepsilon \ln(\mathbf{E}T^{\varepsilon}) = \min_{X\in\Gamma} S(X) \qquad (26)$$

4. EXAMPLES

1. *Motion of a mass point in the plane*. There arise a number of problems where a system should be kept in a given domain G until some particular job is finished. For example, in the problem of pointing a telescope on a satellite, the domain G and the duration of the process are determined by the object to be photographed and the time required. Meerkov (1988) considered the pointing problem assuming the system to be nondegenerate and the boundary conditions to be fixed for all variables. Akulenko and Kovaleva (2003) discussed a problem of the controlled pointing for a system in the plane. The system was presumed to be degenerate and weakly dissipated, and the boundary conditions allowed separation. These assumptions allowed reduction of a four-dimensional system to a pair of the single-degree of freedom systems. The averaging approach was used to simplify the equations of motion and the solution. We obtain the solution of the mean escape time problem without the simplifying assumptions.

We specify an admissible domain of motion in the plane (q_1, q_2) as a circle of radius γ^2

$$G_q\text{: } \{q_1^2 + q_2^2 < \gamma^2\} \qquad (27)$$

The periphery of the circle is

$$\Gamma_q\text{: } \{q_1^2 + q_2^2 = \gamma^2\} \qquad (28)$$

Thus the boundary condition depends on the coordinates q but is independent of the velocities $\dot{q}$. There are no particular restrictions to the velocities $p_i = \dot{q}_i$. However, the boundedness of the admissible domain G is presumed. This implies the admissible domain in the four-dimensional phase space as $G : G_q \times P^2$, with the boundary $\Gamma : \Gamma_q \times P^2$, where $P^2 = \{p_1^2 + p_2^2 = p^2 < \infty\}$.

The angle between the vector of perturbation and the axe q_1 is assumed uniformly distributed within the interval $[0, 2\pi]$. This makes the projections of the perturbation force to the axes q_1, q_2 non-correlated.

The linear regulator is used to increase the time until escape from the admissible domain. Following Akulenko and Kovaleva (2003), we reduce the equations of motion to the form

$$\ddot{q}_i + b\,\dot{q}_i + \frac{\partial U(q_1,q_2)}{\partial q_i} = \varepsilon\sigma(t)\,\dot{w}(t)\,,\ i = 1, 2$$

$$U(q_1,q_2) = \frac{1}{2}\, c(q_1^2 + q_2^2) \qquad (29)$$

The equations are independent and identical but the variables q_1, q_2 are interconnected through the boundary condition (28).

Define the vector $p = \dot{q}$ and write the Hamiltonian function

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}\,[p_1^2 + p_2^2 + c(q_1^2 + q_2^2)] \qquad (30)$$

Thus we have

$$S(q, p) = \frac{k}{2}\,[p_1^2 + p_2^2 + c(q_1^2 + q_2^2)] \qquad (31)$$

where the coefficient $k = 2b/\sigma^2$.

Substitution of formula (31) into relation (9) yields the estimate

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln(\mathbf{E}T^{\varepsilon}) = \min_{\Gamma_q\times P^2} S(q, p)$$

$$= \min_{q_1^2+q_2^2=\gamma^2} \kappa U(q_1, q_1) = \frac{bc\gamma^2}{\sigma^2} \qquad (32)$$

It follows from the last equality that the stiffness and the dissipation coefficients of the regulator play an identical role in the increase of the mean escape time but dissipation in the system is requisite to make the fixed point $q = p = 0$ asymptotically stable.

2. *Motion of the particle in the Henon-Heiles potential.* The Henon-Heiles potential was introduced to describe the motion of a star in a gravitational potential of a galaxy. Since the angular momentum of the star is constant, the motion is equivalent to the motion in a plane, and the potential is two-dimensional.

It has been shown (Lichtenberg and Lieberman, 1983, Alligood *et al.,* 1997) that the system with the Henon-Heiles potential is an example of a conservative system with chaotic behaviour. The passage through the potential barrier is associated with the passage from regular to irregular motion. From this viewpoint, estimation of the mean escape time can be interpreted as estimation of lifetime of the system.

The equations of motion take the form (29) with the potential

$$U(q_1, q_2) = \frac{1}{2} (q_1^2 + q_2^2 + 2q_1^2 q_2 - \frac{2}{3} q_2^3) \quad (33)$$

Potential (33) has the equipotential curves $U(q_1, q_2) = \text{const} < 1/6$, the equality $U(q_1, q_2) = 1/6$ corresponds to the separatrix (Lichtenberg and Lieberman, 1983). The passage through the separatrix is associated with the occurrence of chaos and should be avoided. This implies the admissible in the form

$$G_q: \{U(q_1, q_2) < 1/6\}, \Gamma_q: \{U(q_1, q_2) = 1/6\} \quad (34)$$

Arguing as above, we find

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln (\mathbf{E}\, T^{\varepsilon}) = \min_{\Gamma_q} \kappa U(q_1, q_2) = \frac{bc}{6\sigma^2} \quad (35)$$

Let the admissible domain be the circle (27) with boundary (28) provided $U(q_1, q_2) < 1/6$, $q_1, q_2 \in \Gamma_q$. In this case we obtain

$$\min_{\Gamma_q} kU(q_1, q_2) = \frac{b}{\sigma^2} \gamma^2 (1 + \tfrac{14}{15}\sqrt{\tfrac{2}{5}}\, \gamma) \quad (36)$$

CONCLUSIONS

Theory of large deviations is applied to the problem of escape from the reference domain for a weekly-perturbed Lagrangian system. Formally, the system is interpreted as a nonlinear degenerate diffusion. The use of the large deviation theory allows reduction of the mean escape problem to the deterministic variational problem for a relevant action functional. It is shown that, under non-restrictive assumptions, the solution of the variational problem can be found in an analytic form, and the mean escape time can be expressed explicitly through the kinetic and potential energy and the dissipation parameters of the systems. Several applied examples are considered. In particular, the problem of escape through the potential barrier and the passage from regular to irregular dynamics are discussed. The results are of potential use in related problems of the asymptotic analysis and control of Lagrangian systems.

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