ON THE STABILITY IN ALMOST PERIODIC DISCRETE SYSTEMS

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Abstract: In this paper, the discrete system which is described by difference equations

$$x_{n+1} = f_n(x_n), \quad f_n(0) = 0, \quad n = 0, 1, 2, \ldots.$$

is considered. This system has the trivial (zero) solution $x_n = 0$. Sufficient conditions of its asymptotic stability are obtained in the cases when functions $f_n(x)$ are almost periodic in n.

Keywords: difference equations, stability, Lyapunov's direct method

1. INTRODUCTION

Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincaré and Perron in the end of nineteenth and the beginning of twentieth centuries. The systematic description of the theory of difference equations one can find in (Agarwal, 1992; Elaydi, 1996; Lakshmikantham & Trigiante, 1998). Difference equations are a convenient model for discrete dynamic systems description and for mathematical simulation of systems with impulse effect (Gladilina & Ignatyev, 2003; Halanay & Wexler, 1971; Ignatyev, 2003; Lakshmikantham *et. al.*, 1989; Samoilenko & Perestyuk, 1995). One of directions arising from applications of difference equations is linked with qualitative investigation of their solutions (stability, boundedness, controllability, observability, oscillation, robustness...) (Abu-Saris *et. al.*, 2002; Agarwal *et. al.*, 2003; Bacciotti & Biglio, 2001; Corduneanu, 1995; Győri *et.al.*, 1991; Győri & Pituk, 2001; Hooker *et.al.*, 1987; Marzulli & Trigiante, 1995).

Consider the discrete system of the form

$$x_{n+1} = f_n(x_n), \qquad f_n(0) = 0 \tag{1.1}$$

where $n = 0, 1, 2, \ldots$ is the discrete time, $x_n = (x_n^1, x_n^2, \ldots, x_n^p) \in \mathcal{R}^p$, $f_n = (f_n^1, f_n^2, \ldots, f_n^p) \in \mathcal{R}^p$, f_n satisfy Lipschitz conditions uniformly in n : $\|f_n(x) - f_n(y)\| \le L_r \|x - y\|$ for $\|x\| \le r$, $\|y\| \le r$. System (1.1) has the trivial (zero) solution

$$x_n \equiv 0. \tag{1.2}$$

Denote $x_n(n_0, u)$ the solution of system (1.1) coinciding with u under $n = n_0$. We also denote $B_r = \{x \in \mathcal{R}^p : \|x\| \le r\}$. Suppose functions $f_n(x)$ to be defined in B_H where $H > 0$ is some fixed number. According to (Lakshmikantham & Trigiante, 1998) we denote $\mathcal{Z}_+$ the set of nonnegative integers.

The sufficient conditions of the stability of solution (1.2) of system (1.1) are obtained in this paper provided that sequences $\{f_n(x)\}$ are almost periodic for all $x \in B_H$.

2. MAIN DEFINITIONS AND PRELIMINARIES

By analogy to ordinary differential equations (Hahn, 1967; Rouche *et. al.*, 1977; Savchenko & Ignatyev, 1989), let us introduce the following definitions.

Definition 2.1. Solution (1.2) of system (1.1) is said to be stable if for any $\varepsilon > 0$, $n_0 \in \mathcal{Z}_+$ there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that $\|x_{n_0}\| \leq \delta$ implies $\|x_n\| \leq \varepsilon$ for each $n > n_0$.

Definition 2.2. Solution (1.2) of system (1.1) is called attractive if for every $n_0 \in \mathcal{Z}_+$ there exists $\eta = \eta(n_0) > 0$ and for every $\varepsilon > 0$ and $x_{n_0} \in B_\eta$ there exists $\sigma = \sigma(\varepsilon, n_0, x_{n_0}) \in \mathcal{N}$ such that $\|x_n\| < \varepsilon$ for any $n \geq n_0 + \sigma$. Here $\mathcal{N}$ is the set of natural numbers.

In other words, solution (1.2) of system (1.1) is attractive if

$$\lim_{n\to\infty} \|x_n(n_0, x_{n_0})\| = 0. \tag{2.1}$$

for all $n_0 \in Z_+, x_{n_0} \in B_\eta$.

Definition 2.3. The zero solution of system (1.1) is called equiattractive if for every $n_0 \in \mathcal{Z}_+$ there exists $\eta = \eta(n_0) > 0$, and for any $\varepsilon > 0$ there is $\sigma = \sigma(\varepsilon, n_0) \in \mathcal{N}$ such that $\|x_n(n_0, x_{n_0})\| < \varepsilon$ for all $x_{n_0} \in B_\eta$ and $n \geq n_0 + \sigma$.

Definition 2.4. The trivial solution (1.2) of system (1.1) is called:

– asymptotically stable if it is stable and attractive;

– equiasymptotically stable if it is stable and equiattractive;

Definition 2.5. (Hahn, 1967; Rouche *et.al.*, 1977) Function $r : \mathcal{R}_+ \to \mathcal{R}_+$ belongs to the class of Hahn functions $\mathcal{K}$ $(r \in \mathcal{K})$ if r is continuous increasing function, and $r(0) = 0$.

Definition 2.6. A sequence $\{u_n\}_{-\infty}^{+\infty}$ is said to be almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathcal{N}$ such that each segment $[sl, (s+1)l]$, $s \in \mathcal{Z}$ contains an integer m such that $\|u_{n+m} - u_n\| < \varepsilon$ for all $n \in \mathcal{Z}$. Here $\mathcal{Z}$ is the set of integers. Numbers m with such properties are called ε-almost periods of the sequence $\{u_n\}$.

Definition 2.7. A sequence of functions $\{f_n(x)\}$ is called uniformly almost periodic if for every $\varepsilon > 0$ there exists $l = l(\varepsilon, r) \in \mathcal{N}$ such that each segment of the form $[sl, (s+1)l]$, $s \in \mathcal{Z}$ contains an integer m such that $\|f_{n+m}(x) - f_n(x)\| < \varepsilon$ for all $n \in \mathcal{Z}$, $\|x\| < r$.

A.Halanay and D.Wexler (Halanay & Wexler, 1971) proved the following theorems.

Theorem 2.1. *Solution (1.2) of system (1.1) is uniformly stable if there exists a sequence of functions $\{V_n(x)\}$, with the next properties:*

$$a(\|x\|) \leq V_n(x) \leq b(\|x\|), \quad a \in \mathcal{K},\ b \in \mathcal{K},\ n \in \mathcal{Z}_+, \tag{2.2}$$

$$V_n(x_n) \geq V_{n+1}(x_{n+1}) \text{ for every solution } x_n. \tag{2.3}$$

Theorem 2.2. *Suppose that there exists a sequence of functions $\{V_n(x)\}$, with properties (2.2) and*

$$V_{n+1}(x_{n+1}) - V_n(x_n) \leq -c(\|x_n\|), \quad c \in \mathcal{K}, \tag{2.4}$$

$$|V_n(x) - V_n(y)| \leq L\|x - y\|, \\ n \in \mathcal{Z}_+,\ x \in B_H,\ y \in B_H,\ L > 0. \tag{2.5}$$

Then the zero solution of system (1.1) is uniformly asymptotically stable.

In particular case, when system (1.1) is autonomous, i.e. $f_n(x) = f(x)$, the following theorem is valid (Halanay & Wexler, 1971, p.34):

Theorem 2.3. *If there exists a continuous function $V(x)$ such that $a(\|x\|) \leq V(x) \leq b(\|x\|)$, $a \in \mathcal{K}$, $b \in \mathcal{K}$, and*

$$V(x_{n+1}) - V(x_n) \leq 0 \tag{2.6}$$

for every nonzero solution x_n of system (1.1), and equality sign in (2.6) holds in some set which does not contain entire semitrajectories, then solution (1.2) of system (1.1) is asymptotically stable.

The purpose of this paper is to obtain conditions of asymptotic stability of solution (1.2) of system (1.1) assuming that sequences $\{f_n(x)\}$ are almost periodic.

3. STABILITY IN ALMOST PERIODIC SYSTEMS

Definition 3.1. The sequence of numbers $\{u_k\}_{k=1}^{\infty}$ is called finally nonzero if for any natural number M there exists $k > M$ such that $u_k \neq 0$.

Lemma 3.1. (Halanay & Wexler, 1971, p.125) *Let sequences $\{u_n^1\}, \{u_n^2\}, ..., \{u_n^M\}$ be almost periodic. Then for every $\varepsilon > 0$ there exists $l = l(\varepsilon) \in \mathcal{N}$ such that each segment of the form $[sl, (s+1)l]$, $s \in \mathcal{Z}$ contains at least one ε-almost period, common for all these sequences.*

Lemma 3.2. *If for every $x \in B_H$, a sequence $\{F_n(x)\}$ is almost periodic, and each function $F_n(x)$ satisfies Lipschitz condition uniformly in $n \in \mathcal{Z}$, $x \in B_H$, then this sequence is uniformly almost periodic.*

Proof. Functions $F_n(x)$ satisfy Lipschitz condition, hence

$$\|F_n(x) - F_n(y)\| \leq L_1\|x - y\|, \tag{3.1}$$

where L_1 is Lipschitz constant. Let ε be any positive number. B_H is bounded and closed, therefore it is a compact. It means that there exists a finite set of points $z_1,...,z_M$ such that $z_j \in B_H$ $(j = 1,...,M)$, and for any $x \in B_H$ there exists a natural number i $(1 \le i \le M)$ such that

$$\|x - z_i\| < \frac{\varepsilon}{3L_1}. \qquad (3.2)$$

From Lemma 3.1 it follows that there exists $l = l(\varepsilon) \in \mathcal{N}$ such that every segment $[sl,(s+1)l],\ s \in \mathcal{Z}$ contains a number $m \in \mathcal{Z}$ such that

$$\|F_n(z_i) - F_{n+m}(z_i)\| < \frac{\varepsilon}{3} \qquad (3.3)$$

for all $1 \le i \le M,\ n \in \mathcal{Z}$.

Now let show that for every $x \in B_H$, any integer m satisfying inequality (3.3) is ε-almost period of the sequence $\{F_n(x)\}$. Let z_k be the same element of the set $z_1,...,z_M$, for which $\|x-z_k\| < \varepsilon/(3L_1)$. Then (3.1)-(3.3) imply

$$\|F_{n+m}(x) - F_n(x)\| \le \|F_{n+m}(x) - F_{n+m}(z_k)\|$$
$$+\|F_{n+m}(z_k) - F_n(z_k)\| + \|F_n(z_k) - F_n(x)\|$$
$$\le \frac{\varepsilon}{3} + 2L_1 \cdot \frac{\varepsilon}{3L_1} = \varepsilon. \qquad (3.4)$$

Inequality (3.4) completes the proof of Lemma 3.2.

Theorem 3.1. *Let a sequence of continuous functions $\{V_n(x)\}$ satisfy conditions*

$$a(\|x\|) \le V_n(x), \quad a \in \mathcal{K},\ x \in B_H,\ V_n(0) = 0, \qquad (3.5)$$

and for every $n_0 \in \mathcal{Z}_+$ there exists $\Delta(n_0) > 0$ such that $\|x_{n_0}\| < \Delta$ implies that the sequence $\{V_n(x_n(n_0,x_{n_0}))\}$ does not increase and tends to zero. Then the zero solution of system (1.1) is equiattractive.

Proof. Pick arbitrary $\delta = \delta(n_0) \in (0,\Delta)$. According to conditions of the theorem, for any $\varepsilon > 0,\ n_0 \in \mathcal{Z}_+$, and $x_{n_0} \in B_\delta$ there exists $\sigma = \sigma(\varepsilon, n_0, x_{n_0}) \in \mathcal{N}$ such that

$$V_{n_0+\sigma}(x_{n_0+\sigma}(n_0,x_{n_0})) < \frac{1}{2}\varepsilon.$$

Because of the continuity of functions $V_n(x)$ and continuous dependence of solutions on initial data, there exists a neighbouhood $Q(x_{n_0})$ of the point x_{n_0} in which the inequality

$$V_{n_0+\sigma}(x_{n_0+\sigma}(n_0,y)) < \varepsilon \ \text{ for } \ y \in Q(x_{n_0}) \qquad (3.6)$$

is valid. Since the sequence $\{V_n\}$ monotonically does not increase along solutions of system (1.1), from (3.6) it follows $V_n(x_n(n_0,y)) < \varepsilon$ for $n \ge n_0 + \sigma(\varepsilon,n_0,x_{n_0}),\ y \in Q(x_{n_0})$ So the compact set B_δ is covered by the system of neighbourhoods $\{Q(x_{n_0})\}$ from which, by Heine - Borel's Lemma, it is possible to select the finite subcovering $Q_1,...,Q_j$ with corresponding numbers $\sigma_1,\ldots,\sigma_j$. Let $\sigma(\varepsilon,n_0) = \max\{\sigma_1,\ldots,\sigma_j\}$ (σ depends only on ε and n_0). Then $V_n(x_n(n_0,x_{n_0})) < \varepsilon$ for all $n \ge n_0 + \sigma(\varepsilon,n_0)$ if $\|x_{n_0}\| \le \delta(n_0)$. This inequality proves that solution (1.2) of system (1.1) is equiattractive.

Later on throughout this section, we shall assume that the sequence $\{f_n(x)\}$ in the right-hand side of system (1.1) is almost periodic for every fixed $x \in B_H$, and functions $f_n(x)$ satisfy Lipschitz condition uniformly in n.

Lemma 3.3. *Consider the solution $x_n(n_0,x_{n_0})$ of system (1.1). We suppose that $x_n(n_0,x_{n_0})$ belongs to B_r $(0 < r < H)$ for $n \ge n_0$. Let $\{\varepsilon_k\}$ be a monotonically approaching zero sequence of positive numbers, and $\{m_k\}$ a sequence of ε_k-almost periods of $\{f_n(x)\}$ (for every ε_k there corresponds an ε_k-almost period m_k). Then the limit relation*

$$\lim_{k\to\infty} \|x_{n_*}(n_0,x^{(k)}) - x_{n_*+m_k}(n_0,x_{n_0})\| = 0, \qquad (3.7)$$

holds where $x^{(k)} = x_{n_0+m_k}(n_0,x_{n_0})$, and n_* is a fixed natural number which is more than n_0 $(n_* > n_0)$.

Proof. Consider solutions

$$x_n(n_0,x^{(k)}) \qquad (3.8)$$

and

$$x_n(n_0+m_k,x^{(k)}) \qquad (3.9)$$

of system (1.1). After $\Delta n = n_* - n_0$ steps the point $x^{(k)}$ passes to $x_{n_*}(n_0,x_{n_0})$ along solution (3.8), and $x^{(k)}$ passes to the point $x_{n_*+m_k}(n_0+m_k,x^{(k)}) = x_{n_*+m_k}(n_0,x_{n_0})$ along solution (3.9). Solution (3.9) of system (1.1) with initial condition $(n_0+m_k,x^{(k)})$ can be interpreted as the solution of the system

$$x_{n+1} = f_{n+m_k}(x_n) \qquad (3.10)$$

with initial data $(n_0,x^{(k)})$. The sequence $\{f_n(x)\}$ is almost periodic, and every function $f_n(x)$ satisfies Lipschitz condition, hence right-hand sides of (1.1) and (3.10) differ arbitrary small from each other for k large enough. This implies limit relation (3.7).

Theorem 3.2. *Suppose that there exists a sequence of functions $\{V_n(x)\}$ such that*

a) for every fixed $x \in B_H$, the sequence $\{V_n(x)\}$ is almost periodic;

b) each member $V_n(x)$ satisfies condition (3.5) and Lipschitz condition uniformly in n;

c) $V_n(x_n) \ge V_{n+1}(x_{n+1})$ along any solution of (1.1);

d) the sequence $\{V_n(x_n)\}$ is finally nonzero along any nonzero solution of (1.1).

Then the zero solution of system (1.1) is equiasymptotically stable.

Proof. First let us show that solution (1.2) of system (1.1) is stable. Pick arbitrary $\varepsilon \in (0, H)$ and $n_0 \in \mathcal{Z}_+$. Let $\delta = \delta(\varepsilon, n_0) > 0$ be such that $V_{n_0}(x) < a(\varepsilon)$ for $x \in B_\delta$. Then

$$a(\|x_n\|) \leq V_n(x_n) \leq V_{n_0}(x_{n_0}) < a(\varepsilon)$$

whence we have $\|x_n\| < \varepsilon$ for $n > n_0$.

Now let us show that solution (1.2) is equiattractive. Take arbitrary $x_{n_0} \in B_\delta$. The sequence $\{V_n(x_n(n_0, x_{n_0}))\}$ monotonically does not increase, therefore there is the limit

$$\lim_{n\to\infty} V_n(x_n(n_0, x_{n_0})) = \eta \geq 0,$$

and $V_n(x_n(n_0, x_{n_0})) \geq \eta$ for $n \geq n_0$. Let us show that $\eta = 0$. Suppose the opposite: $\eta > 0$. Consider a monotonically approaching zero sequence of positive numbers $\{\varepsilon_k\}$ where ε_1 is sufficiently small. By lemmas 3.2 and 3.1, for every ε_i there exists a sequence of ε_i-almost periods $m_{i,1}, m_{i,2}$, ...,$m_{i,k}, \ldots$ $(m_{i,k} < m_{i,k+1}, \ \lim_{k\to+\infty} m_{i,k} = +\infty)$ for sequences $\{f_n(x)\}$ and $\{V_n(x)\}$ such that inequalities

$$|V_{n+m_{i,k}}(x) - V_n(x)| < \varepsilon_i,$$
$$\|f_{n+m_{i,k}}(x) - f_n(x)\| < \varepsilon_i$$

hold for any $n \in \mathcal{Z}, \ x \in B_\varepsilon$. Without loss of generality one can suppose $m_{i,k} < m_{i+1,k}$ for all $i \in \mathcal{N}, \ k \in \mathcal{N}$. Designate $m_k = m_{k,k}$.

Consider the sequence $\{x^{(k)}\}$ where $x^{(k)} = x_{n_0+m_k}(n_0, x_{n_0})$ $(k = 1, 2, ...)$. This sequence is bounded, therefore there exists its subsequence which converges to some point x^*. Without loss of generality we suppose that the sequence $\{x^{(k)}\}$ itself converges to x^*. The sequence $\{V_n(x)\}$ is almost periodic for every fixed $x \in B_H$, and each function $V_n(x)$ is continuous, hence

$$V_{n_0}(x_*) = \lim_{n\to\infty} V_{n_0}(x_n)$$
$$= \lim_{k\to\infty}\lim_{n\to\infty} V_{n_0+m_k}(x_n) = \lim_{n\to\infty} V_{n_0+m_n}(x_n)$$
$$= \lim_{n\to\infty} V_{n_0+m_n}(x_{n_0+m_n}(n_0, x_{n_0})) = \eta.$$

Consider the sequence $\{x_n(n_0, x^*)\}$. From conditions of the theorem, it follows that there exists $n_* > n_0$ $(n_* \in \mathcal{N})$ such that the inequality

$$V_{n_*}(x_{n_*}(n_0, x^*)) = \eta_1 < \eta$$

holds. Functions $f_n(x)$ satisfy Lipschitz condition, hence

$$\lim_{k\to\infty} \|x_{n_*}(n_0, x^{(k)}) - x_{n_*}(n_0, x^*)\| = 0$$

because

$$\lim_{k\to\infty} \|x^{(k)} - x^*\| = 0.$$

This implies

$$\lim_{k\to\infty} V_{n_*}(x_{n_*}(n_0, x^{(k)})) = \eta_1. \quad (3.11)$$

The almost periodicity of the sequence $\{f_n(x)\}$ and limit relation (3.7) imply

$$\|x_{n_*}(n_0, x^{(k)}) - x_{n_*+m_k}(n_0, x_{n_0})\| \leq \gamma_k \quad (3.12)$$

where $\gamma_k \to 0$ as $k \to \infty$. Since the sequence $\{V_n\}$ is almost periodic, we have

$$|V_{n_*}(x) - V_{n_*+m_k}(x)| < \varepsilon_k \quad (3.13)$$

for every $x \in B_H$, and conditions (3.11), (3.12) imply

$$|V_{n_*}(x_{n_*+m_k}(n_0, x_{n_0})) - \eta_1| < \xi_k \quad (3.14)$$

where $\xi_k \to 0$ as $k \to \infty$. From (3.13) it follows

$$|V_{n_*}(x_{n_*+m_k}(n_0, x_{n_0}))$$
$$-V_{n_*+m_k}(x_{n_*+m_k}(n_0, x_{n_0}))| < \varepsilon_k. \quad (3.15)$$

Inequalities (3.14), (3.15) imply

$$|V_{n_*+m_k}(x_{n_*+m_k}(n_0, x_{n_0})) - \eta_1| < \xi_k + \varepsilon_k \quad (3.16)$$

where $\xi_k + \varepsilon_k \to 0$ as $k \to \infty$.

On the other hand

$$\lim_{k\to\infty} V_{n_*+m_k}(x_{n_*+m_k}(n_0, x_{n_0})) = \eta. \quad (3.17)$$

Inequality (3.16) and limit relation (3.17) are in contradiction to the inequality $\eta_1 < \eta$. This contradiction proves that $\eta = 0$, hence, according to Theorem 3.1, we derive that solution (1.2) of system (1.1) is equiasymptotically stable.

Theorem 3.3. *Suppose that there exists a sequence of functions $\{V_n(x)\}$ such that for every $x \in B_H$, the sequence $\{V_n(x)\}$ is almost periodic, and each function $V_n(x)$ satisfies Lipschitz condition uniformly in n and next conditions:*

$|V_n(x)| \leq b(\|x\|), \quad b \in \mathcal{K}, \quad n \in \mathcal{Z}_+$ for $x \in B_H$;

for any $n \in \mathcal{Z}_+$ and $\delta > 0$ there is $x \in B_\delta$ such that $V_n(x) > 0$;

$V_{n+1}(x_{n+1}) \geq V_n(x_n)$ along any solution x_n;

Then solution (1.2) of system (1.1) is unstable.

Proof. Let $\varepsilon \in (0, H)$ be an arbitrary number. Take any $n_0 \in \mathcal{Z}_+$ and sufficiently small $\delta > 0$. Choose $x_0 \in B_\delta$ by such way that $V_{n_0}(x_{n_0}) > 0$. From conditions of the theorem, it follows that there exists $\eta > 0$ such that $|V_n(x)| < V_{n_0}(x_{n_0})$ for every $x \in B_\eta$. Consider the sequence $\{V_n\}$ where $V_n = V_n(x_n(n_0, x_{n_0}))$. This sequence does not decrease i.e. $V_n(x_n(n_0, x_{n_0})) \geq V_{n_0}(x_{n_0})$ for $n \geq n_0$. This means that $\|x_n(n_0, x_{n_0})\| \geq \eta$ for every $n \geq n_0$. Let us show that there is $N_0 \in \mathcal{N}$ $(N_0 > n_0)$ such that $\|x_{N_0}(n_0, x_{n_0})\| > \varepsilon$. Assume the opposite:

$$\eta \leq \|x_n(n_0, x_{n_0})\| \leq \varepsilon \quad (3.18)$$

for all $n > n_0$. Using the conditions of the theorem and inequality (3.18), we obtain the contradiction by the same way as in the proof of Theorem 3.2. We pass the literal repetition of these reasonings. The contradiction shows that the solution $x_n(n_0, x_{n_0})$ leaves B_ε. The proof is complete.

4. EXAMPLES

Example 4.1. Consider the system

$$x_{n+1} = -y_n \sin(\sqrt{3}n), \quad y_{n+1} = x_n \sin n \quad (4.1)$$

and the function $V_n(x_n, y_n) = x_n^2 + y_n^2$.

$$V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n)$$
$$= -(\cos^2 n)\ x_n^2 - (\cos^2 \sqrt{3}n)\ y_n^2. \quad (4.2)$$

According to Corduneanu (Corduneanu, 1989) for any sufficiently small $\varepsilon > 0$ there exists a sequence $n_1, n_2, ..., n_k, ... \to \infty$ such that $0 < \cos^2 n_k < \varepsilon, \quad 0 < \cos^2(\sqrt{3}n_k) < \varepsilon \quad (k = 1, 2, ...)$. This means that there is not a function $c \in \mathcal{K}$ such that the left-hand side of (4.2) satisfies inequality (2.4), so Theorem 2.2 cannot be applied to this system. System (4.1) is not an autonomous one, therefore Theorem 2.3 cannot be applied to the study of the stability property of its zero solution. But this system is almost periodic, and right-hand side of (4.2) is negative for each nonzero solution of system (4.1). Hence, according to Theorem 3.2, the zero solution of system (4.1) is equiasymptotically stable.

Example 4.2. Consider the system

$$x_{n+1} = X_n(x_n, y_n), \quad y_{n+1} = Y_n(x_n, y_n) \quad (4.3)$$

where

$$X_n = y_n - x_n^2 y_n (2 - \sin^2 n - \cos^2 \sqrt{2}n),$$
$$Y_n = x_n + x_n y_n^2 (2 - \sin^2 n - \cos^2 \sqrt{2}n).$$

If we choose $V_n(x_n, y_n) = x_n^2 + y_n^2$, then

$$V_{n+1}(x_{n+1}, y_{n+1}) - V_n(x_n, y_n)$$
$$= x_n^2 y_n^2 (x_n^2 + y_n^2)(2 - \sin^2 n - \cos^2 \sqrt{2}n)^2.$$

By Theorem 3.3, we can state that the zero solution of system (4.3) is unstable.

5. CONCLUSION

In this paper, the discrete system which is described by difference equations

$$x_{n+1} = f_n(x_n), \quad f_n(0) = 0, \quad n = 0, 1, 2, \ldots.$$

is considered. This system has the trivial (zero) solution $x_n = 0$. Sufficient conditions of its asymptotic stability are obtained in the cases when functions $f_n(x)$ are almost periodic in n.

REFERENCES

Abu-Saris, R., Elaydi, S., Jang, S. (2002). Poincaré types solutions of systems of difference equations. *J. Math. Anal. Appl.*, 275, 69-83.

Agarwal, R.P. (1992). *Difference Equations and Inequalities.* New York: Marcel Dekker.

Agarwal, R.P., Li, W.-T., Pang, P.Y.H. (2003). Asymptotic behavior of nonlinear difference systems. *Applied Mathematics and Computation*, 140, 307-316.

Bacciotti, A., Biglio, A. (2001). Some remarks about stability of nonlinear discrete-time control systems. *Nonlinear Differential Equations and applications*, 8, 425-438.

Corduneanu, C. (1989). *Almost Periodic Functions, 2nd edition.* New York: Chelsea Publishing Company.

Corduneanu, C. (1995). Discrete qualitative inequalities and applications. *Nonlinear Analysis TMA*, 25, 933-939.

Elaydi, S. (1996). *An Introduction to Difference Equations.* New York: Springer.

Gladilina, R.I., Ignatyev, A.O. (2003). On necessary and sufficient conditions for the asymptotic stability of impulsive systems. *Ukrainian Math. J.*, 55, No. 8, 1254-1264.

Győri, I., Ladas, G., Vlahos, P.N. (1991). Global attractivity in a delay difference equation. *Nonlinear Analysis TMA*, 17, 473-479.

Győri, I. and Pituk, M. (2001). The converse of the theorem on stability by the first approximation for difference equations. *Nonlinear Analysis*, 47, 4635-4640.

Hahn, W. (1967).*Stability of Motion.* New York - Berlin - Heidelberg: Springer.

Halanay, A., Wexler, D. (1971). *Qualitative Theory of Impulsive Systems.* Moscow: Mir (in Russian).

Hooker, J.W., Kwong, M.K., Patula, W.T. (1987). Oscillatory second order linear difference equations and Riccati equations. *SIAM J. Math. Anal.*, 18, 54-63.

Ignatyev, A.O. (2003). Method of Lyapunov functions in problems of stability of solutions of systems of differential equations with impulse action. *Sbornik: Mathematics*, 194, 1543-1558.

Lakshmikantham, V., Bainov, D.D., Simeonov, P.S. (1989). *Theory of impulsive differential equations.* Singapure – New Jersey – London: World Scientific.

Lakshmikantham, V., Trigiante, D. (1998). *Theory of Difference Equations: Numerical Methods and Applications.* New York – London: World Scientific.

Marzulli, P., Trigiante, D. (1995). Stability and Convergence of Boundary Value Methods for Solving ODE. *J. of Difference Equations and Applications*, 1, 45-55.

Rouche, N., Habets, P., Laloy, M. (1997). *Stability Theory by Liapunov's Direct Method.* New York: Springer-Verlag.

Samoilenko, A.M., Perestyuk, N.A. (1995). *Impulsive Differential Equations.* Singapore: World Scientific.

Savchenko, A.Ya., Ignatyev, A.O. (1989). *Some Problems of the Stability Theory.* Kiev: Naukova Dumka (in Russian).