PARTIAL STABILITY ANALYSIS BY MEANS OF SEMIDEFINITE LYAPUNOV FUNCTIONS

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Abstract: A nonautonomous system of ordinary differential equations is considered. This system has the zero solution, and there exists a nonnegative Lyapunov function which derivative is nonpositive. Theorems of the partial uniform stability and partial uniform asymptotic stability are proved.

Keywords: Lyapunov functions; Partial stability; Nonlinear systems; Continuous time systems

1. INTRODUCTION

Let $x := (x^1, x^2, \dots, x^s) \in R^s$. Denote $y := (y^1, \dots, y^l) = (x^1, \dots, x^l) \in R^l$, $z := (z^1, \dots, z^m) = (x^{l+1}, \dots, x^s) \in R^m$, $(m + l = s)$, $\|y\| = [(y^1)^2 + \dots + (y^l)^2]^{1/2}$, $\|z\| = [(z^1)^2 + \dots + (z^m)^2]^{1/2}$, $\|x\| = (\|y\|^2 + \|z\|^2)^{1/2}$, $\Gamma_r = \{y \in R^l : \|y\| \le r\}$, $\Omega_r = \{z \in R^m : \|z\| \le r\}$, $B_r = \{x \in R^s : \|x\| \le r\}$.

Consider the system of ordinary differential equations

$$\frac{dx}{dt} = X(t,x) \tag{1.1}$$

where $t \in R_+ = [0, \infty)$ is time, $X = (X_1, \dots, X_n) : R_+ \times \Gamma_H \times R^m \to R^s$. Functions $X_i(t,x) \quad (i = 1, ..., s)$ are supposed to be continuous and satisfying conditions that solutions of system (1.1) exist and unique in the domain $R_+ \times \Gamma_H \times R^m$ where $H > 0$ is a constant; $X(t,0) \equiv 0$. Under these assumptions, system (1.1) has the trivial solution

$$x = 0. \tag{1.2}$$

To investigate partial stability of solution (1.2) with respect to variables $y^1, \dots, y^l$, researchers usually use Lyapunov's direct method. This method assumes the existence of y-positive definite Lyapunov function $V(t,x)$ such that its derivative dV/dt along solutions of system (1.1) is negative definite function of all variables or of the part of variables (Rumyantsev, 1957; Rumyantsev and Oziraner, 1987; Savchenko and Ignatyev, 1989; Vorotnikov, 1998). But there are no standard methods of construction of Lyapunov functions for general systems, so for applications it is interesting to obtain criteria of partial uniform asymptotic stability by means of functions V and $\frac{dV}{dt}$ with more weak properties. Such creteria were obtained in (Ignatyev, 1999; Ignatyev, 1989; Ignatyev, 2002; Rumyantsev and Oziraner, 1987). In these criteria, it was assumed that V is y-positive definite, $\frac{dV}{dt} \le 0$, and $\frac{dV}{dt}$ is not y-negative definite. In this paper, the demand on function V is made weaker: it does not assumed to be y-positive definite, but y-component of every solution of system (1.1), lying on the integral set $V(t,x) = 0$, tends to zero as $t \to \infty$.

2. MAIN DEFINITIONS AND PRELIMINARIES

Consider system of ordinary differential equations (1.1). Let us introduce definitions and notations which were used in (Grudo, 1983; Ignatyev, 1992; Ignatyev, 1993; Rouche *et al.*, 1977).

Definition 2.1. The set M of the space (t,x) is called to be integral if for each point $(t_0,x_0)\in M$, the inclusion $(t,x(t))\in M, t\geq t_0$ holds where $x(t)=x(t,t_0,x_0)=(y(t,t_0,x_0),\ z(t,t_0,x_0))$ is the solution of system (1.1) with the initial condition $x(t_0)=x_0$.

Let $M\subset R_+\times R^s$. Denote by M_q the intersection of M with hyperplane $t=q$, and by $\rho(x,M_q)$ – the distance between point x and the set M_q.

By analogy with (Gaishun, 1999; Grudo, 1983; Rouche *et al.*, 1977), let us introduce next definitions.

Definition 2.2. Solution (1.2) of system (1.1) is called to be y-stable with respect to integral set M if for any $\varepsilon>0$ and $t_0\in R_+$ there exists $\delta=\delta(\varepsilon,t_0)>0$ such that $(t_0,x_0)\in M, \|x_0\|<\delta$ imply $\|y(t,t_0,x_0)\|<\varepsilon$ for $t\geq t_0$.

Definition 2.3. If in last definition δ can be chosen independent of t_0 (i.e. $\delta=\delta(\varepsilon)$), then solution (1.2) is called uniformly y-stable with respect to M.

Definition 2.4. The zero solution of system (1.1) is called to be y-attractive with respect to integral set M if for any $t_0\in R_+$ there exists $\eta=\eta(t_0)>0$, and for any $\varepsilon>0$ and $x_0\in M_{t_0}\bigcap(\Gamma_\eta\times R^m)$ there exists $\sigma=\sigma(\varepsilon,t_0,x_0)>0$ such that $\|y(t,t_0,x_0)\|<\varepsilon$ for all $t\geq t_0+\sigma$.

Definition 2.5. The trivial solution of equations (1.1) is called to be uniformly y-attractive with respect to M if for some $\eta>0$ and any $\varepsilon>0$ there exists $\sigma=\sigma(\varepsilon)>0$ such that $\|y(t,t_0,x_0)\|<\varepsilon$ for all $t_0\in R_+,\ x_0\in M_{t_0}\bigcap(\Gamma_\eta\times R^m)$ and $t\geq t_0+\sigma$.

Definition 2.6. Solution (1.2) of system of ordinary differential equations (1.1) is called to be:

– asymptotically y-stable with respect to integral set M if it is y-stable with respect to M and y-attractive with respect to M;

– uniformly asymptotically y-stable with respect to integral set M if it is uniformly y-stable with respect to M and uniformly y-attractive with respect to M.

Definition 2.7. We shall say that function $g: R_+\to R_+$ is Hahn function ($g\in\mathscr{K}$) if it is continuous, monotonically increasing and $g(0)=0$.

Definition 2.8. We shall say that every solution of system (1.1) is z-bounded if for any $\zeta>0$ there exists $N_\zeta>0$ such that $\|z(t,t_0,x_0)\|\leq N_\zeta$ for $t\geq t_0\geq 0,\ x_0\in B_\zeta$.

Denote $X:=(Y,Z)$ where $Y\in R^l,\ Z\in R^m$. We suppose Y to satisfy the condition

$$\|Y(t,x_1)-Y(t,x_2)\|\leq L\|y_1-y_2\|, \tag{2.1}$$

$x_1=(y_1,z_1)\in\Gamma_H\times\Omega_N,\ x_2=(y_2,z_2)\in\Gamma_H\times\Omega_N$, where the constant L, in general, depends on N.

If $x(t,t_0,x_1)\in\Gamma_H\times\Omega_N,\ x(t,t_0,x_2)\in\Gamma_H\times\Omega_N$, then

$$y(t,t_0,x_1)=y_1+\int_{t_0}^{t}Y(u,y(u,t_0,x_1),z(u,t_0,x_1))du,$$

$$y(t,t_0,x_2)=y_2+\int_{t_0}^{t}Y(u,y(u,t_0,x_2),z(u,t_0,x_2))du,$$

$$\|y(t,t_0,x_1)-y(t,t_0,x_2)\|\leq\|y_1-y_2\|+ \\ +\int_{t_0}^{t}\|Y(u,y(u,t_0,x_1),z(u,t_0,x_1)) \\ -Y(u,y(u,t_0,x_2),z(u,t_0,x_2))\|du.$$

Gronwall-Bellman Lemma and inequality (2.1) imply

$$\|y(t,t_0,x_1)-y(t,t_0,x_2)\|\leq\delta e^{L(t-t_0)}, \tag{2.2}$$

if $\|y_1-y_2\|\leq\delta$.

3. THEOREM ON THE UNIFORM Y-STABILITY

Theorem 3.1. *Let system (1.1) be such that its any solution is z-bounded, and there exists a continuous function $V(t,x): R_+\times\Gamma_H\times R^m\to R$ such that:*

i) *V is periodic in t with period ω;*

ii) $V(t,x)\geq 0,\quad V(t,0)\equiv 0$;

iii) *V does not increase along solutions of system (1.1);*

iv) *solution (1.2) of system (1.1) is uniformly asymptotically y-stable with respect to integral set M where*

$$M:=\{(t,x): t\in R_+,\ x\in\Gamma_H\times R^m,\quad V(t,x)=0\}.$$

Then the zero solution of system (1.1) is uniformly y-stable.

Proof. Denote

$$S(\alpha):=\{(t,x)\in R_+\times\Gamma_\alpha\times R^m:\ V(t,x)=0\},$$
$$S_t(\alpha):=\{x\in\Gamma_\alpha\times R^m:\ V(t,x)=0\}.$$

We shall prove this theorem by contradiction: suppose that solution (1.2) of system (1.1) is not uniformly y-stable. This means that there exists a positive number h and a sequence of initial conditions $\{(t_{0n},x_{0n})\}_{n=1}^{\infty}$ such that $t_{0n}\geq 0, \lim\limits_{n\to\infty}\|x_{0n}\|=0,\ \|y(t_{0n}+T_n^*,t_{0n},x_{0n})\|\geq h$ for some $T_n^*>0$. Let ε be an arbitrary positive number satisfying the condition $4\varepsilon<h$. In view of assumption iv) of the theorem, the trivial solution of system (1.1) is uniformly y-attractive with respect to M. Hence for some $\eta>0$ there exists $\sigma=\sigma(\varepsilon)>0$ such that $\|y(t,t_0,x_0)\|<\varepsilon$ for all $t_0\in R_+,\ x_0\in S_{t_0}(\eta)\bigcap B_\eta,\ t\geq t_0+\sigma$. Later on, we shall assume that ε is such that the inequality $4\varepsilon<\eta$ holds. Let numbers $T_n>0$ denote moments of time, such that

$$\|y(t_{0n}+T_n,t_{0n},x_{0n})\|=4\varepsilon, \\ \|y(t_{0n}+t,t_{0n},x_{0n})\|<4\varepsilon \texttt{ for } t<T_n. \tag{3.1}$$

In view of inequality (2.2), we derive $\lim\limits_{n\to\infty}T_n=+\infty$. Consider a sequence $\{t_n\}_{n=1}^{\infty}$ such that the inequalities

$$0<t_n<T_n,\quad \sigma\leq T_n-t_n\leq Q, \tag{3.2}$$

are valid for $n \geq n_0 \in \mathscr{N}$ where $\mathscr{N}$ is the set of natural numbers, Q is a constant satisfying the condition $Q > \sigma$ (for instance, we can choose $Q := 2\sigma$). Consider also a sequence $\{\tau_n\}_{n=1}^{\infty}$ which terms satisfy the conditions

$$\tau_n \in [0, \omega), \qquad \tau_n = t_{0n} + t_n - p_n\omega \tag{3.3}$$

where $p_n \in Z_+$ (Z_+ is the set of nonnegative integers).

Consider now a sequence $\{x_n\}$ in the phase space where

$$x_n = x(t_{0n} + t_n, t_{0n}, x_{0n}) = (y_n, z_n). \tag{3.4}$$

By virtue of assumptions (3.1) and (3.2), one can conclude that $y_n \in \Gamma_{4\varepsilon}$ for any $n \geq n_0 \in \mathscr{N}$. The sequence $\{z_n\}$ is bounded; $\Gamma_{4\varepsilon}$ is bounded and closed set in R^l, hence, the sequence $\{x_n\}$ includes a subsequence, convergent to some element $x_* \in \Gamma_{4\varepsilon} \times R^m$. Similarly, the sequence $\{\tau_n\}$ has a subsequence which converges to $\tau_* \in [0, \omega]$. Without loss of generality, we shall assume that these sequences themself converge to corresponding elements: $x_n \to x_*,\ \tau_n \to \tau_*$ as $n \to \infty$.

Taking into account properties of function V and notations (3.3), (3.4), we obtain

$$\begin{aligned} 0 \leq V(\tau_*, x_*) &= \lim_{n\to\infty} V(\tau_n, x_n) \\ &= \lim_{n\to\infty} V(t_{0n} + t_n, x_n) \\ &= \lim_{n\to\infty} V(t_{0n} + t_n, x(t_{0n} + t_n, t_{0n}, x_{0n})) \\ &\leq \lim_{n\to\infty} V(t_{0n}, x_{0n}) = 0. \end{aligned}$$

Therefore $V(\tau_*, x_*) = 0$, and $(\tau_*, x_*) \in S(4\varepsilon)$. Denote k a natural number, such that conditions

$$k \geq n_0,\ \ t_k \geq \sigma(\varepsilon),\ \ |\delta_k| < \max\{1, \frac{\varepsilon}{Lh} e^{-LQ}\},$$
$$\|y_k - y_*\| < \varepsilon e^{-QL}, \tag{3.5}$$

hold where $\delta_k = t_{0k} + t_k - p_k\omega - \tau_* = \tau_k - \tau_*$. It is clear that we can choose such k because $\delta_n \to 0$, $\|y_n - y_*\| \to 0$, $t_n \to +\infty$ for $n \to \infty$.

We have the estimate

$$\|y(t_{0k} + T_k, t_{0k}, x_{0k})\| \leq J_1 + J_2 + J_3$$

where

$$J_1 = \|y(t_{0k} + T_k, t_{0k}, x_{0k}) - y(t_{0k} + T_k, t_{0k} + t_k, x_*)\|,$$
$$\begin{aligned} J_2 = \|&y(t_{0k} + T_k, t_{0k} + t_k, x_*) \\ &- y(t_{0k} + T_k, t_{0k} + t_k - \delta_k, x_*)\|, \end{aligned}$$
$$\begin{aligned} J_3 &= \|y(t_{0k} + T_k, \tau_* + p_k\omega, x_*)\| \\ &= \|y(t_{0k} + T_k, t_{0k} + t_k - \delta_k, x_*)\|. \end{aligned}$$

By virtue of uniqueness of solutions, $y(t_{0k} + T_k, t_{0k}, x_{0k}) = y(t_{0k} + T_k, t_{0k} + t_k, x(t_{0k} + t_k, t_{0k}, x_{0k})) = y(t_{0k} + T_k, t_{0k} + t_k, x_k)$, hence in view of (2.2) and (3.5) we get

$$\begin{aligned} J_1 &= \|y(t_{0k} + T_k, t_{0k} + t_k, x_k) - y(t_{0k} + T_k, t_{0k} + t_k, x_*)\| \\ &\leq \|y_* - y_k\| e^{(T_k - t_k)L} \leq \|y_* - y_k\| e^{QL} < \varepsilon. \end{aligned} \tag{3.6}$$

Let us estimate now J_2 and J_3 taking into account estimates (2.2) and conditions (3.5).

$$\begin{aligned} J_2 &= \|y(t_{0k} + T_k, t_{0k} + t_k, x_*) \\ &\quad - y(t_{0k} + T_k, t_{0k} + t_k, x(t_{0k} + t_k, t_{0k} + t_k - \delta_k, x_*))\| \\ &\leq \|y_* - y(t_{0k} + t_k, t_{0k} + t_k - \delta_k, x_*)\| e^{LQ} \\ &\leq |\delta_k| Lh e^{LQ} < \varepsilon; \end{aligned} \tag{3.7}$$

$J_3 < \varepsilon$ because $(\tau_* + p_k\omega, x_*) \in S(4\varepsilon)$ and $T_k - t_k \geq \sigma(\varepsilon)$.

Note that estimate (3.7) was made under assumption $\delta_k \geq 0$, but one can show analogously that it is also true for $\delta_k \leq 0$.

Taking advantage of obtained estimates, we have $\|y(t_{0k} + T_k, t_{0k}, x_{0k})\| < 3\varepsilon$. But this is in contradiction to assumption (3.1). Obtained contradiction proves uniform y-stability of the zero solution of equations (1.1). The proof is complete.

4. THEOREM ON THE UNIFORM ASYMPTOTIC Y-STABILITY

Theorem 4.1. *Let system (1.1) be such that its any solution is z-bounded and there exists a function $V(t,x)$ such that:*

i) *V is differentiable and bounded on $R_+ \times \Gamma_H \times \Omega_\lambda$ for every $\lambda > 0$;*

ii) *V is periodic in t with period ω;*

iii) $V(t,x) \geq 0, \quad V(t,0) \equiv 0$;

iv) $\dfrac{dV}{dt} \leq -c(V(t,x)),\ c \in \mathscr{K}$;

v) *solution (1.2) of system (1.1) is uniformly asymptotically y-stable with respect to the integral set $S(H)$.*

Then the zero solution of system (1.1) is uniformly asymptotically y-stable.

Proof. According to Theorem 3.1, solution (1.2) of system (1.1) is uniformly y-stable. In addition, any solution of system (1.1) is z-bounded. Therefore for any $h > 0 \quad (h < H)$ there exist $\zeta > 0$ and $\lambda > 0$, such that $\|y(t, t_0, x_0)\| < h$, $\|z(t, t_0, x_0)\| < \lambda$ for all

$$x_0 \in B_\zeta, \quad t_0 \in R_+, \quad t \geq t_0. \tag{4.1}$$

Let us show now that the zero solution is uniformly y-attractive. Pick arbitrary ε ($\varepsilon < h$). From uniform y-stability of the trivial solution of system (1.1) with respect to integral set $S(h)$, it follows the existence of $\eta > 0$ such that

$$(\forall \xi > 0)\ (\exists \sigma_1 = \sigma_1(\xi) > 0)\ (\forall t_0 \in R_+)$$
$$(\forall x_1 \in S_{t_0}(\eta))\ (\forall t \geq t_0 + \sigma_1)\ \|y(t, t_0, x_1)\| < \xi. \tag{4.2}$$

Without loss of generality, further we shall suppose $\varepsilon < \eta < h$. Using (4.2), we obtain

$$\big(\exists \sigma_2 = \sigma_1(\varepsilon/3) > 0\big)(\forall t_0 \in R_+)\ (\forall x_1 \in S_{t_0}(\eta))$$

$$(\forall t \geq t_0+\sigma_2)\ \|y(t,t_0,x_1)\| < \frac{\varepsilon}{3}. \qquad (4.3)$$

Inequality (2.2) implies that there exists

$$\gamma = \gamma(\varepsilon) = \frac{1}{3}\varepsilon e^{-L\sigma_2(\varepsilon)} > 0 \qquad (4.4)$$

such that $t_0 \in R_+,\ \|x_0 - x_1\| < \gamma$ imply

$$\|y(t_0+\sigma_2,t_0,x_0) - y(t_0+\sigma_2,t_0,x_1)\| < \frac{\varepsilon}{3}. \qquad (4.5)$$

Let us show that

$$\lim_{t\to\infty} \rho(x(t,t_0,x_0),S_t(\eta)) = 0 \qquad (4.6)$$

is valid for every solution $x(t,t_0,x_0)$ satisfying conditions (4.1), and limit relation (4.6) holds uniformly with respect to $t_0 \in R_+,\ x_0 \in B_\zeta$. First let us show that there exists Hahn function a such that

$$V(t,x(t,t_0,x_0)) \geq a(\rho(x(t,t_0,x_0),S_t(\eta))). \qquad (4.7)$$

Denote $\psi(r) := \inf V(t,x)\ \texttt{for}\ \rho(x,S_t(\eta)) = r,\ t \in [0,\omega],\ x \in \Gamma_\eta \times \Omega_\lambda$. $\psi(r)$ is a scalar continuous function, such that $\psi(0) = 0,\ \psi(r) > 0$ for $r > 0$. Denote $a(r)$ continuous monotonically increasing function, such that $a(0) = 0,\ a(r) \leq \psi(r)$ for $0 < r < \eta$. It is clear that this function satisfies property (4.7). Let us show now that

$$(\forall \alpha > 0)\ (\exists T = T(\alpha))\ (\forall t_0 \in R_+)\ (\forall x_0 \in B_\zeta)$$
$$(\forall t \geq t_0 + T)\ V(t,x(t,t_0,x_0)) < \alpha. \qquad (4.8)$$

Denote $\sup\limits_{\substack{t\in[0,\omega]\\ x\in\Gamma_\eta\times\Omega_\lambda}} V(t,x) := P$. Let us estimate a time segment during which the inequality $V(t,x(t,t_0,x_0)) \geq \alpha$ holds. In this case

$$V(t,x(t)) = V(t_0,x_0) + \int_{t_0}^{t} \dot{V} dt \leq P - c(\alpha)(t-t_0);$$

$$c(\alpha)(t-t_0) \leq P - V(t,x(t)) \leq P - \alpha;$$

$t - t_0 \leq \dfrac{P-\alpha}{c(\alpha)}$. Therefore (4.8) is valid if $T = \dfrac{P-\alpha}{c(\alpha)}$.

According to properties (4.8) and (4.7), we get that limit relation (4.6) is true:

$$(\forall \gamma > 0)\ (\exists T_1 = T_1(\gamma) > 0)\ (\forall t_0 \in R_+)\ (\forall x_0 \in B_\zeta)$$
$$(\forall t \geq t_0 + T_1)\ \rho(x(t,t_0,x_0),S_t(\eta)) < \gamma. \qquad (4.9)$$

Choose $\sigma_* = \sigma_*(\varepsilon) = T_1(\varepsilon) + \sigma_2(\varepsilon)$. Let us show that

$$\|y(t_0+\sigma_*+t,t_0,x_0)\| < \varepsilon \quad \text{for any} \quad t_0 \in R_+,$$
$$x_0 \in B_\zeta,\ t \geq 0. \qquad (4.10)$$

From (4.9) it follows that

$$(\forall t_0 \in R_+)\ (\forall x_0 \in B_\zeta)\ (\exists x_1 \in S_{t_0+T_1+t}(\eta))$$
$$\|x(t_0+T_1+t,t_0,x_0) - x_1)\| < \gamma \qquad (4.11)$$

where γ is chosen according to (4.4). From (4.11) and (4.5) we obtain

$$\|y(t_0+T_1+\sigma_2+t,t_0+T_1+t,x(t_0+T_1+t,t_0,x_0))$$
$$-y(t_0+T_1+\sigma_2+t,t_0+T_1+t,x_1)\| < \frac{\varepsilon}{3}.$$

This inequality can be rewritten in the next form

$$\|y(t_0+T_1+\sigma_2+t,t_0,x_0)$$
$$-y(t_0+T_1+\sigma_2+t,t_0+T_1+t,x_1)\| < \frac{\varepsilon}{3}. \qquad (4.12)$$

From (4.3) we get

$$\|y(t_0+T_1+\sigma_2+t,t_0+T_1+t,x_1)\| < \frac{\varepsilon}{3}. \qquad (4.13)$$

From (4.12) and (4.13) it follows $\|y(t_0+T_1+\sigma_2+t,t_0,x_0\| < \frac{2}{3}\varepsilon < \varepsilon$.

Thus it has been proved that there exists $\zeta > 0$, such that for every $\varepsilon > 0$ there exists $\sigma_* = \sigma_*(\varepsilon) = T_1(\varepsilon) + \sigma_2(\varepsilon)$, such that (4.10) is true. This means that the zero solution of system (1.1) is uniformly y-attractive, and B_ζ is its domain of y-attraction. The proof is complete.

Remark. Note that the problem of stability with respect to all variables by means of semidefinite Lyapunov functions was studied in (Bulgakov and Kalitin, 1978; Kalitin, 1995; Kalitin, 2002; Kosov, 1986).

5. EXAMPLE

Consider the following system of ordinary differential equations

$$\frac{dy_1}{dt} = Y_1, \quad \frac{dy_2}{dt} = Y_2, \quad \frac{dz}{dt} = Z \qquad (5.1)$$

where $Y_1 = e^{-t}y_2 - y_1^3,\ Y_2 = y_1(1+\sin^2 z)\sin t - y_2 + (e^{-t}y_2 - y_1^3)(1+\sin^2 z)\sin t + y_1 \sin 2z \sin t[\sin(y_1 + y_2) + z - 2z^3] + y_1(1+\sin^2 z)\cos t,\ Z = \sin(y_1+y_2) + z - 2z^3$,

and study the stability of its trivial solution

$$y_1 = 0, \quad y_2 = 0, \quad z = 0 \qquad (5.2)$$

in variables y_1, y_2. System (5.1) has the integral set M, given by the equality

$$y_2 - y_1(1+\sin^2 z)\sin t = 0. \qquad (5.3)$$

Any solution of system (5.1) is z-bounded. To verify this, it is sufficient to choose $\max\{\|\zeta\|, 1\}$ as a N_ζ.

Let us take a Lyapunov function in the form $V = [y_2 - y_1(1+\sin^2 z)\sin t]^2$. Its derivative along solutions of system (5.1) is equal to

$$\frac{dV}{dt} = -2[y_2 - y_1(1+\sin^2 z)\sin t]^2 = -2V.$$

The first equation of system (5.1) has the form

$$\frac{dy_1}{dt} = y_1 e^{-t}(1+\sin^2 z)\sin t - y_1^3.$$

on the integral set M. According to results of (Ignatyev, 1987), the zero solution of this equation is uniformly asymptotically stable. The relation (5.3) implies that solution (5.2) of system (5.1) is also uniformly asymptotically y_2-stable with respect to integral set M. Hence all conditions of Theorem 4.1 are satisfied for system (5.1), and we can conclude that the zero solution of equations (5.1) is uniformly asymptotically y-stable.

6. CONCLUSION

A nonautonomous system of ordinary differential equations is considered. This system has the zero solution, and there exists a nonnegative Lyapunov function which derivative is nonpositive. Theorems on the partial uniform stability and partial uniform asymptotic stability are proved.

REFERENCES

Bulgakov N.V., Kalitin B.S. (1978). Generalization of theorems of Lyapunov's second method. *Vestsi of AS BSSR, Series in Physics and Mathematics* **3**, 32–36.

Gaishun I.V. (1999). *Introduction to the theory of linear nonautomous systems*. Institute of Mathematics of Byelorussian AS. Minsk.

Grudo E.I. (1983). On the theory of stability of systems of ordinary differential equations and Pfaff systems. *Differential Equations* **19**, No.5, 782–789.

Ignatyev A.A. (1999). On equiasymptotic stability with respect to part of the variables. *Journal of Applied Mathematics and Mechanics* **63**, No.5, 821–824.

Ignatyev A.O. (1987). On the asymptotic stability of nonautonomous systems. *Differentsial'nye Uravneniya* **23**, No.12, 2161–2163.

Ignatyev A.O. (1989). On the stability of almost periodic systems with respect to part of variables. *Differentsial'nye Uravneniya* **25**, No.8, 1446–1448.

Ignatyev A.O. (1992). Application of Lyapunov's direct method to the investigation of integral sets. *Ukrainian Math. Journal* **44**, No.10, 1342–1349.

Ignatyev A.O. (1993). On existence of Lyapunov function in problems of stability of integral sets. *Ukr. Math. J.* **45**, No.7, 932–941.

Ignatyev A.O. (2002). On the partial equiasymptotic stability in functional differential equations. *J. of Math. Analysis and Appl.* **268**, 615–628.

Kalitin B.S. (1995). On the method of semidefinite Lyapunov functions for nonautonomous differential systems. *Differentsial'nye Uravneniya* **34**, No.4, 583–590.

Kalitin B.S. (2002). Stability of closed invariant sets of semidynamical systems. The method of sign definite Lyapunov functions. *Differential Equations* **38**, No.11, 1662–1664.

Kosov A.A. (1986). On the method of vector Lyapunov functions. *Lyapunov functions and their applications,* Novosibirsk, 106–110.

Rouche N., Habets P., Laloy M. (1977). *Stability Theory by Liapunov's Direct Method*. Springer. New York.

Rumyantsev V.V. (1957). On stability of motions with respect to part of variables. *Vestnik of Moscow University, Series in Mathematics and Mechanics* **4**, 9–16.

Rumyantsev V.V., Oziraner A.S. (1987). *Stability and stabilization with respect to part of variables*. Nauka. Moscow.

Savchenko A.Ya., Ignatyev A.O. (1989). *Some stability problems of nonautonomous dynamic systems*. Naukova Dumka. Kiev.

Vorotnikov V.I. (1998). *Partial stability and control*. Birkhauser. New York.